PROPER HOLOMORPHIC IMAGES OF
STRICTLY PSEUDOCONVEX DOMAINS.

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0. Introduction.

H. Poincaré showed for the first time that the ball in $\mathbb{C}^2$ and the bidisc are not biholomorphically equivalent. Later R. Remmert and K. Stein [11], G.M. Henkin [8] and A.T. Huckleberry [9] generalized this result more and more by considering larger classes of domains and also proper holomorphic mappings. In all their results the existence of local complex analytic foliations of parts of the boundary for one domain and some strict pseudoconvexity of the other boundary play an essential role.

On the other hand, there are well-known examples of proper holomorphic mappings $f$ with non-empty branching locus from certain bounded, $C^\infty$-smooth pseudoconvex domains $\Omega_1$ onto strictly pseudoconvex domains $\Omega_2$. But it has been conjectured that any proper holomorphic mapping $f: \Omega_1 \to \Omega_2$ is necessarily unbranched if $\Omega_1$ is strictly pseudoconvex and $\Omega_2$ is weakly pseudoconvex and $C^\infty$-smooth.

For both $\Omega_1$ and $\Omega_2$ being strictly pseudoconvex the conjecture has first been fully verified by S. Pinčuk [10] building on work of H. Alexander [1] (see also D. Burns and St. Shnider [5] and W. Rudin [12]). If $\Omega_1$ and $\Omega_2$ as in the conjecture are in addition known to be complete Reinhardt domains, St. Bell [3] has confirmed the claim. In the case of real-analytic boundaries the result is contained in Bell [4].
In this paper we prove now

**Theorem 1.** Let $\Omega_1, \Omega_2 \subseteq \mathbb{C}^n$ be domains with $C^\infty$-boundaries and $\Omega_1$ strictly pseudoconvex. Then any proper holomorphic mapping $f : \Omega_1 \to \Omega_2$ is unbranched and, therefore, extends to a $C^\infty$-covering $\hat{f} : \tilde{\Omega}_1 \to \tilde{\Omega}_2$. In particular, $f$ extends to an unbranched $C^\infty$-covering $\hat{f} : \tilde{\Omega}_1 \to \tilde{\Omega}_2$ (because of [7]) and $\Omega_2$ is also strictly pseudoconvex.

In section 1 we explain the notations and the relevant results of St. Bell [2] which are basic for our proof. In section 2 we find a generic branching point $z_0$ of $f$ on $\partial \Omega_1$ where the branching locus of $f$ hits $\partial \Omega_1$ at $z_0$ as a transverse manifold, and we show that $f$ extends in a $C^\infty$ way to $\partial \Omega_1$ near $z_0$.

For section 3 we use this to show that the branching locus has to be empty. In section 4 we mention some more general results than the theorem above that can be derived with the same methods.

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1. Notations and tools.

For a $C^\infty$-smooth domain $\Omega \subseteq \mathbb{C}^n$ we denote by $A^\infty(\Omega)$ the algebra of functions in $C^\infty(\overline{\Omega})$ holomorphic on $\Omega$ and we always will write $u$ for the Jacobian determinant of the given mapping $f$. The following statement is a special case of theorem 2 (and it's proof) of St. Bell [2] and will be the basic tool in our proof of the theorem:
Proposition. In the situation of the theorem the function $u \circ (h \cdot f) \in A^\infty(\Omega_1)$ if $h \in A^\infty(\Omega_2)$. In particular, $u \in A^\infty(\Omega_1)$.

2. $C^\infty$-extension at generic branching points on $b\Omega_1$.

We assume that the branching locus $\hat{X} = \{ z \in \Omega_1 : u(z) = 0 \}$ of the mapping $f$ is non-empty.

2.1. At first, we want to find sufficiently generic points on $b\Omega_1$ where, in particular, $\hat{X}$ hits $b\Omega_1$ transversally.

Along each connected component of the regular locus of $\hat{X}$ the function $u$ has a well-defined constant order of vanishing.

Let $X_1$ be one such component on which this order is minimal, say $k$. The set $X = X_1 \cap \Omega_1$ is an irreducible branch of $\hat{X}$.

There is a multiindex $\alpha$, $|\alpha| = k-1$, such that the function

$$v := \frac{\partial^\alpha u}{\partial z^\alpha} \in A^\infty(\Omega_1)$$

vanishes along some non-empty relatively open set $U_1 \subset X_1$ with order 1 and we can find an index $\beta$, $1 \leq \beta \leq n$ such that

$$\frac{\partial v}{\partial z^\beta}(p) \neq 0$$

for some $p \in U_1$. Notice that $v|\hat{X} = 0$. We put

$$S := \bar{X} \cap b\Omega.$$

Because of the maximum principle applied to $X$ there is a $q \in S$ with

$$\frac{\partial v}{\partial z^\beta}(q) \neq 0$$

(1)
and we can extend \( \nu \) to a \( C^\infty \)-function \( \tilde{\nu} \) on an open neighborhood \( U \) of \( q \) such that \( \tilde{\nu} \) vanishes to infinite order along \( b\Omega_1 \cap U \). Because of (1) the set

\[
Y := \{ z \in U : \tilde{\nu}(z) = 0 \}
\]

is a smooth \( C^\infty \)-submanifold of \( U \), if \( U \) was chosen small enough. Notice that \( \tilde{\nu} \cap U \subset Y \). Furthermore, after shrinking \( U \) again, we can write \( Y \) as a graph over its tangent space \( T_qY \) at \( q \), which is complex, in the following way:

After a linear coordinate change we may assume that

\[
T_qY = \{ t = (t_1, \ldots, t_n) = (t', t_n) \in \mathbb{R}^n : t_n = 0 \} = \mathbb{R}^{n-1}.
\]

Let \( \pi : \mathbb{R}^n \to T_qY \), \( \pi((t', t_n)) = t' \), be the projection and \( U' := \pi(U) \).

Then there is a \( C^\infty \)-function \( g : U' \to \mathbb{C} \) which is holomorphic on \( \pi(Y \cap \Omega_1) \) and whose differential \( \partial \bar{\partial} g \) vanishes to infinite order along \( \pi(Y \cap b\Omega_1) \) such that

\[
Y = \{ (z', g(z')) : z' \in U' \}.
\]

Let now \( \rho \) be a strictly plurisubharmonic defining function of \( \Omega_1 \) defined in a neighborhood of \( \bar{\Omega}_1 \) and put

\[
\sigma(z') := \rho((z', g(z'))) \text{ for } z' \in U'.
\]

Then, after shrinking \( U \) again, \( \sigma \) becomes a strictly plurisubharmonic function on \( U' \).

We call \( S' := \pi(S \cap U) \) such that \( \sigma|_{S'} = 0 \).

Claim: \( d\sigma|_{S'} \neq 0 \). (2)

Suppose \( d\sigma|_{S'} = 0 \). The Taylor expansion of \( \sigma \) around \( q' := \pi(q) \) in real coordinates \( z' = x' + iy' \) after a suitable linear change
of coordinates has the form
\[ \sigma = \sum_{j=1}^{n-1} \left( x_j^2 + \alpha_j y_j^2 \right) + \text{higher order terms} \]
with \( \alpha_j > -1 \). Therefore, the set
\[ \Sigma = \{ z' : \frac{\partial \sigma}{\partial x_1}(z') = 0 \} \]
is a real hypersurface in \( U' \) which can be supposed to divide \( U' \) into exactly two connected components. We choose a component intersecting \( \pi(X \cap U) \) and call it \( \Sigma^- \). Notice that \( X \cap U \subset Y \cap \Omega_1 \) is a closed subvariety and \( \pi|_Y \) is proper. Therefore, \( \pi(X \cap U) \) is a closed subvariety of \( \pi(Y \cap \Omega_1) \) of full dimension. The boundary of \( \pi(X \cap U) \) in \( U' \) is \( S' \) and \( S' \subset \Sigma \). Hence \( \pi(X \cap U) \supset \Sigma^- \). This shows that \( \sigma|\Sigma^- < 0 \) such that the Hopf-lemma applied to \( \sigma \) at \( q' \) gives
\[ d\sigma(q') \neq 0 \]
contradicting the assumption \( d\sigma|_S' = 0 \).

As a consequence of (2) we can now move \( q \) on \( S \) such that \( Y \) intersects \( b\Omega_1 \) at \( q \) transversally. This implies in particular because of the choice of \( v \) that

\[ \hat{X} \cap U = Y \cap \Omega_1 \] (shrink \( U \) if necessary) (3)

2.2. Next we want to show that the mapping \( f \) can be extended in a \( C^\infty \) way to \( b\Omega_1 \) near \( q \). Because of the proposition of Bell from section 1 applied to the coordinate functions \( w_j \in A^\infty(\Omega_2) \) it is enough for this purpose to prove that the functions \( u \cdot f_j \in A^\infty(\Omega_1) \) can be divided near \( q \) by \( u \) without destroying the differentiability.
For this we choose suitable coordinates near \( q \) in the following way: we may assume that

\[
\frac{\partial \tilde{v}}{\partial z^1}(q) \neq 0
\]

such that

\[
z^*_1 = \tilde{v}(z)
\]

\[
z^*_j = z_j - z_j(q), \quad j = 2, \ldots, n
\]

is a \( C^\infty \)-coordinate change holomorphic on \( \Omega_1 \cap U \). It, therefore, does not destroy the strict pseudoconvexity of \( \Omega_1 \cap U \) at \( b\Omega_1 \).

We call the new coordinates again \( z \) and now have

\[
Y = \{ z \in U : z_1 = 0 \}
\]

Let now \( g \in \mathcal{A}^\infty(\Omega_1) \) with \( g|\bar{X} = 0 \) be arbitrary and let \( \tilde{g} \) be any \( C^\infty \)-extension of \( g \) to \( U \). We want to normalize this extension along \( Y \) in a suitable way by showing

**Lemma 1.** For any given integer \( l > 0 \) the extension \( \tilde{g} \) can always be chosen in such a way that it's Taylor expansion at the points of \( Y \) in \( z_1, z^*_1 \) has the form

\[
\tilde{g} = \sum_{i=1}^{l} g_i z^i_1 + R_1
\]

where \( g_j \) are \( C^\infty \)-functions on \( Y \) and

\[
R_1 = O(|z_1|^{1+l})
\]

**Proof.** We choose a \( C^\infty \)-retraction

\[
\pi : U \to Y
\]

with \( \pi(U \cap \Omega_1) = \bar{X} \cap U \).

Let \( \tilde{g} \) be an arbitrary \( C^\infty \)-extension of \( g \) to \( U \) and define
If \( \mathcal{E}_r \) has been defined and

\[
\mathcal{E}_r = \sum_{i+j \leq r} \mathcal{E}_{i,j} z_1^i \overline{z}_1^j + R_r
\]

is its Taylor series along \( Y \) with

\[
R_r = O(|z_1|^{r+1}),
\]

we put inductively:

\[
\mathcal{E}_{r+1} = \mathcal{E}_r - \sum_{i+j \leq r} \mathcal{E}_{i,j} (\alpha_i \cdot \alpha_j) z_1^i \overline{z}_1^j.
\]

Then the Taylor expansion of \( \mathcal{E}_{r+1} \) along \( Y \) has the shape \( f \) \( (4) \) for \( r+1 \) and \( \mathcal{E}_{1+1} \) satisfies the requirements of the lemma.

A simple consequence of this is

Lemma 2. Let \( g \in \mathcal{A}^\infty(\Omega_1) \) be a function with \( g|\hat{\Omega} = 0 \). Then the function \( \hat{g} = g/z_1 \) extends from \( \Omega_1 \cap U \) to \( \Omega_1 \cap U \) in a \( C^\infty \)-way.

Proof. Let \( \alpha \) be any multiindex and \( 1 > |\alpha| \) a positive integer. Choose an extension \( \mathcal{E} \) of \( g \) according to lemma 1 with this \( l \). Then one obviously has near \( Y \) for \( z_1 \neq 0 \)

\[
\frac{\partial^\alpha (\mathcal{E}/z_1)}{\partial z_1^\alpha} = O(|z_1|^{1-|\alpha|}).
\]

This proves the lemma.

We now can easily prove
Lemma 3. If $q \in \hat{X}$ has been chosen as in the beginning of this section, the mapping $f$ extends in a $C^\infty$-way to $b\Omega_1$ near $q$.

Proof. 1) Since $u$ vanishes along $Y \cap \Omega_1$ to the order $k$ exactly, lemma 2 gives that

$$u|\Omega_1 \cap U = z_1^k \tilde{u}$$

(5)

with a holomorphic function $\tilde{u}$ which extends to $b\Omega_1 \cap U$ in a $C^\infty$-way and such that

$$\tilde{u}(q) \neq 0.$$  

(6)

2) The proposition of Bell from section 1 says

$$u \cdot f_j \in A^\infty(\Omega_1) \text{ for } j = 1, \ldots, n.$$  

(5) and (6) therefore imply that

$$g_j := z_1^k f_j$$

extends to $b\Omega_1 \cap U$ in a $C^\infty$-way. Because of lemma 2 this must therefore also be true for $f_j$.

3. Elimination of the branching.

In section 2 we worked under the assumption that the branching locus $\hat{X}$ of $f$ is non-empty and we found the point $q$ of lemma 3 in $\hat{X} \cap b\Omega_1$. Hence we will have obtained a contradiction and, therefore, proved the theorem if we will have shown:

Lemma 4. Let $f : U_1 \to U_2$, $U_1 \subset C^R$ open, be a proper holomorphic mapping. Suppose, there are relatively open sets $M_1 \subset bU_1$ where $bU_1$ is a $C^\infty$-smooth pseudo-convex hypersurface and let $M_1$ even
be strictly pseudoconvex. Furthermore, suppose that \( f \) extends in a \( C^\infty \)-way to \( U_1 \cup M_1 \) and that \( f(M_1) \subseteq M_2 \). Then \( f \) is unbranched near \( M_1 \) (and \( M_2 \) is strictly pseudoconvex at \( f(M_1) \)).

**Remark.** The statement is purely local at points of \( M_1 \). We, therefore, will shrink the \( U_1 \) during the proof suitably without mentioning it explicitly.

**Proof.** We will use the transformation formula for a complex Monge-Ampere-equation in a way which is due to N. Kerzman, J.J. Kohn and L. Nirenberg. We call \( p := f(q) \in M_2 \). According to [6] we can choose a (local) \( C^\infty \)-defining function \( \rho_2 \) of \( M_2 \) on \( U_2 \) such that
\[
\psi_2 := (-\rho_2)^{2/3}
\]
is (strictly) plurisubharmonic on \( U_2 \). Define
\[
\rho_1 := \rho_2 \circ f \in C^\infty(U_1 \cup M_1).
\]
Then \( \psi_1 := (-\rho_1)^{2/3} = \psi_2 \circ f \) is negative and plurisubharmonic on \( U_1 \) and
\[
\lim_{z \to M_1} \psi_1(z) = 0.
\]
Therefore, by the Hopf lemma there is a constant \( C > 0 \) such that
\[
\psi_1(z) \leq -C \text{ dist}(z, M_1).
\]
This means that
\[
\rho_1(z) \leq -C^{3/2} \text{ dist}^{3/2}(z, M_1) \quad \text{for } z \in U_1
\]
such that \( d\rho_1(z) \neq 0 \) for \( z \in M_1 \). Hence, \( \rho_1 \) is a defining function of \( U_1 \) along \( M_1 \). Because \( M_1 \) is strictly pseudoconvex we can find a constant \( L > 0 \) such that
\[ \varphi_1 := \rho_1 e^{-L \rho_1} \]

is even a strongly plurisubharmonic defining function of \( U_1 \) along \( M_1 \). Notice that \( \varphi_1 = \varphi_2 f \) with \( \varphi_2 := e^{-L \rho_2} \) being a defining function of \( U_2 \) along \( M_2 \). Since \( \delta f \) vanishes to infinite order at \( M_1 \) we have

\[ 0 \leq \det(\frac{\partial^2 \varphi_1}{\partial z_i \partial \bar{z}_j})(z) = |u(z)|^2 \det(\frac{\partial^2 \varphi_2}{\partial \omega_i \partial \bar{\omega}_j})(f(z)) \]

for all \( z \in M_1 \). Therefore, \( u(z) \neq 0 \) for all \( z \in M_1 \) and \( M_2 \) is also strictly pseudoconvex at all points in \( f(M_1) \).

4. Remarks.

1) Our proof shows, in fact, that the following statement holds:

Theorem 2. Let \( \Omega_1, \Omega_2 \subset \mathbb{C}^n \) be pseudoconvex domains with \( C^\infty \)-smooth boundaries. Suppose \( \Omega_1 \) satisfies condition \( R \) (in the sense of Bell [2]) for the Bergman projection operator on \( \Omega_1 \). Let \( f : \Omega_1 \rightarrow \Omega_2 \) be a proper holomorphic mapping. Then \( f \) does not have any branching points near the strictly pseudoconvex boundary points of \( \Omega_1 \).

In order to reduce this to what has been done in the proof of theorem 1 it is enough to show: if there is a strictly pseudoconvex boundary point \( p \) of \( \partial \Omega \) with \( p \in \partial \tilde{X} \), then \( q \) as in section 2 can be chosen arbitrarily close to \( p \). For this, we define \( \tilde{X} \) as an irreducible branch of \( \hat{X} \) on which \( u \) vanishes to minimal order among all branches of \( \hat{X} \) clustering on \( \partial \Omega \) in a given neighborhood \( U \) of \( p \). We may assume that \( p \in \tilde{X} \) and
define $v$ and $S$ with respect to $X$ as in section 2. We claim:

There is a $q \in S \cap U$ with

$$\frac{\partial v}{\partial z}(q) \neq 0.$$ 

Suppose $v' = \frac{\partial v}{\partial z}$ vanishes identically on $S \cap U$. Notice that there is a function $f \in A^\infty(\Omega_1)$ with

$$f(p) = 1 \quad \text{and} \quad |f|_{(1)} < 1.$$ 

Therefore, it is easy to find an $\varepsilon > 0$, a $z_0 \in X \cap U$ and an $N \in \mathbb{N}$ such that

$$|(f+\varepsilon)^N v'(z_0)| > 1 \quad \text{and} \quad |(f+\varepsilon)^N v'|_{b\Omega \cap U} < 1.$$ 

Because $v'|S \cap U = 0$ this contradicts the maximum principle for $v'$ on $X$. - We now can apply the proof in section 2 to the situation near $q$.

2) Theorem 2 shows that in the situation as given there the branching locus of $f$ hits $b\Omega_1$ only at weakly pseudoconvex points. One might, therefore, ask whether this excludes all branching of $f$ if the set of weakly pseudoconvex points on $b\Omega_1$ is small enough. This is, indeed, the case. More precisely we have

**Theorem 3.** Let $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ be pseudoconvex domains with $C^\infty$-smooth boundaries. Suppose that $\Omega_1$ satisfies condition R and that the set $E$ of weakly pseudoconvex boundary points of $\Omega_1$ has Hausdorff-measure

$$\Lambda_{2n-3}(E) = 0.$$ 

Then any proper holomorphic mapping $f: \Omega_1 \to \Omega_2$ is unbranched,
and, therefore, extends to a covering map

\[ \hat{f} : \tilde{\Omega}_1 \to \tilde{\Omega}_2 . \]

**Proof.** Because of theorem 2 we only have to show that the branching locus of \( f \) has to hit \( b\Omega_1 \) at a strictly pseudoconvex point if it is non-empty. For this using the notations of section 2 we have to observe that a point \( q \in b\Omega_1 \) where \( \tilde{X} \)

intersects \( b\Omega_1 \) as a transversal \( C^\infty \) real manifold of real codimension 2 (in \( \mathbb{C}^n \)) can be found without using strict pseudoconvexity of \( b\Omega_1 \) at \( q \). Namely, to achieve this one replaces the strict plurisubharmonic defining function \( \rho \) by a local defining function \( \rho \) of \( \Omega_1 \) near \( q \in b\Omega_1 \) as chosen in (1) such that

\[ \varphi : = -(-\rho)^{2/3} \]

is strictly plurisubharmonic on \( \Omega_1 \) near \( q \), thereby getting a \( C^\infty \)-function \( \sigma \) on \( U' \) with

\[ \psi : = -(-\sigma)^{2/3} \]

being plurisubharmonic on \( \pi(Y \cap \Omega_1) \) and \( \sigma(z') = 0 \) for \( z' \in \pi(Y \cap b\Omega_1) \). Now we choose a point \( z_0' \in \pi(Y \cap \Omega_1) \) very close to \( q' \) and let \( B' \subset U' \cap \pi(Y \cap \Omega_1) \) be the largest ball around \( z_0' \). Then there is a point \( \bar{q}' \in bB' \cap S' \). Applying Hopf lemma to \( \psi|B' \)

at \( \bar{q}' \) gives

\[ d\sigma(\bar{q}') \neq 0 \]

such that at \( \bar{q} = (\bar{q}', g(\bar{q}')) \in S \)

\[ d(\rho|Y)(\bar{q}) \neq 0. \]

This shows that \( Y \) intersects \( b\Omega_1 \) at \( \bar{q} \) transversally. Therefore, \( S \) has to be near \( \bar{q} \) a real manifold of real codimension 3 (in \( \mathbb{C}^n \)) and cannot be contained in \( E \) since \( \Lambda_{2n-3}(E) = 0. \)
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