ISBN 82 553-0468-1

Mathematics
No 19 November 16, 1981

CHARACTERIZATION OF GLOBAL PEAK SETS FOR $A^{\infty}(D)$.

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PREPRINT SERIES - Matematisk institutt, Universitetet i Oslo

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1. Introduction.

If K is a compact subset of the boundary ∂D of a domain D in $\mathbb{C}^n$, we call K a peak set for $A^{\infty}(D)$ if there exists a $\mathcal{C}^{\infty}$-function f on $\bar{D}$ holomorphic on D such that $f|K \equiv 1$ and $|f| < 1$ in $\bar{D} \setminus K$. We will be interested in the case when D is strictly pseudoconvex with $\mathcal{C}^{\infty}$-boundary.

Chaumat and Chollet proved in [2] that K is a local peak set for $A^{\infty}(D)$ if and only if K is locally contained in integral manifolds for the complex structure of the boundary of D. They also proved [1] that K is a peak set for $A^{\infty}(D)$ if K is globally contained in an integral manifold.

The purpose of this paper is to discuss the following two questions ([1]):

<u>Question 1</u>: If K is locally contained in ($\mathcal{C}^{\infty}$) integral manifolds, does there always exist an integral manifold containing all of K?

<u>Question 2</u>: Are local peak sets for $A^{\infty}(D)$ always (global) peak sets for $A^{\infty}(D)$?

Chaumat and Chollet, [3], have shown that the answer to question 1 is no for arbitrary strongly pseudoconvex domains in $\mathbb{C}^n$, $n \geq 4$. In section 3 we show that the answer is yes if $n = 3$ (if $n = 2$ the answer is trivially yes).

By introducing a suitable concept of dimension of K and using techniques from [3] we prove (in section 4) that the answer to question 2 is yes.

2. Preliminary remarks.

If D is a strongly pseudoconvex domain with $\mathcal{C}^\infty$ boundary in $\mathbb{C}^n$ the Darboux theorem gives the existence of local real coordinates $(x_1,\dots,x_{n-1}, y_1,\dots,y_{n-1},t)$ on ∂D such that $T_{\mathbb{C}}\partial D = \{\xi \in T\partial D : \omega(\xi) = 0\}$ where $\omega = dt + \sum_{i=1}^{n-1} x_i dy_i$ and $T_{\mathbb{C}}\partial D$ is the complex tangentspace of ∂D.

DEFINITION 2.1:

A $\mathcal{C}^\infty$ submanifold of ∂D is an integral manifold if $TN_p \subset T_{\mathbb{C}}\partial D_p$ whenever $p \in N$.

It is well known that integral manifolds are totally real and therefore have dimension at most $n-1$.

LEMMA 2.2 ([2], [6]).

An integral manifold is locally a graph over $\{x_{i_1},\dots,x_{i_k}, y_{j_1},\dots,y_{j_l}\}$ where $\{i_1,\dots,i_k\} \cap \{j_1,\dots,j_l\} = \emptyset$.

LEMMA 2.3 ([1], [6]):

If K is a compact subset of an integral manifold N, there exists a neighborhood ω of K in $\mathbb{C}^n$ and a function $u \in C^\infty(\omega)$ with the following properties:

(1) $D^\alpha \bar\partial u|_N \equiv 0$ for each multi index α,

(2) $\{p \in \omega : u(p) = 0\} \cap \bar{D} = K$,

(3) $\operatorname{Re} u(p) \geq d^2(p,N)$ when $p \in \omega \cap \bar{D}$ and

(4) $u \geq 0$ on $N \cap \omega$.

In order to construct such u for a K which is locally contained in integral manifolds it is necessary to introduce a concept of dimension.

DEFINITION 2.4.

Let K be a subset of $\mathbb{C}^n$ and $p \in K$. Then $\dim_p K = \min\{\dim M : M$ is a C^∞-manifold and there exists a neighborhood ω_p of p in $\mathbb{C}^n$ such that $\omega_p \cap K \subset M\}$.

If $K \subset \partial D$ is locally contained in integral manifolds we define $\dim \operatorname{int}_p K = \min\{\dim N : N$ is an integral manifold containing a neighborhood of p in $K\}$.

LEMMA 2.5:

If K is locally contained in integral manifolds, then $\dim \operatorname{int}_p K = \dim_p K$.

Proof:

Obviously $\dim_p K \leq \dim \operatorname{int}_p K$ so we only have to show the reverse inequality. We choose an M of minimal dimension such that $K \cap \omega_p \subset M$. Suppose ω_p is chosen so small that $K \cap \omega_p$ is contained in an integral manifold N. Since M is minimal, $TM_p \subset TN_p$. Therefore the orthogonal projection M' of M into N is a submanifold of N and $\dim M' < \dim M$. A submanifold of an integral manifold is an integral manifold and $K \cap \omega_p \subset M'$.

3. Integral manifolds.

In this section we at first find a "stratification" by integral manifolds whose union contains K. Secondly we apply this to show that the answer to question 1 (section 1) is yes when $n = 3$.

THEOREM 3.1:

If K is a compact subset of ∂D and is locally contained in integral manifolds, there exist integral manifolds $N_1, \ldots, N_m$ with the following properties:

1) $\dim N_i < \dim N_j$ when $i < j$
2) $\bigcup_{i=1}^{m} N_i \supset K$
3) $K \cap N_i$ is open in K
4) $N_i \cap N_j$ is open in N_i when $i < j$.

Proof:

Assume that $r = \max_{p \in K} \dim_p K$. Observe that the set S of r-dimensional points of K is compact. Let $U_1, \ldots, U_k$ be integral manifolds such that:

a) A neighborhood of S in K is contained in $\bigcup_{j=1}^{k} U_j$.

b) Each U_j is a graph as in lemma 2.2.

c) Either $\bar{U}_i \cap \bar{U}_j = \emptyset$ or $U_i \cap U_j$ contains r-dimensional points and a neighborhood of them in K.

If $\bar{U}_1 \cap \bar{U}_2 = \emptyset$, we let $U_{1,2} = U_1 \cup U_2$. If not, let p be an r-dimensional point of K in $U_1 \cap U_2$. **Then** $TU_{2|p} = TU_{1|p}$ **which implies** that U_1 is a graph over the same coordinates as U_2 in a neighborhood (in U_1) of the r-dimensional points of K in $U_1 \cap U_2$. Let F_1, F_2 parametrize U_1, U_2 around these points. We may assume that F_1, F_2 have the same domain of definition V.

Choose a C^∞ function $\chi : V \to [0,1]$ such that $\chi(p_k) = 1$ for all sufficiently large k if $F_1(p_k)$ converges to a point in $U_1 \setminus F_1(V)$ and $\chi(q_k) = 0$ for sufficiently large k when

$F_2(q_k)$ converges to a point in $U_2 \setminus F_2(V)$. Then $F = \chi F_1 + (1-\chi)F_2$ parametrizes a manifold whose tangent space at the r-dimensional points of K lies in the complex tangent space of ∂D.

There exist neighborhoods $\tilde{U}_i$ in U_i of the r-dimensional points in $U_i \setminus F_i(V)$ $i = 1,2$ such that $U_{1,2}' := \tilde{U}_1 \cup \tilde{U}_2 \cup F(V)$ is a C^∞ manifold containing a neighborhood relative to K of the r-dimensional points of K in $U_1 \cup U_2$.

If $\omega_{|U_{1,2}'}$ vanishes on $K \cap U_{1,2}'$, Theorem 7 of [2] gives the existence of an r-dimensional integral manifold $U_{1,2}$ containing $U_{1,2}' \cap K$.

We know that $\omega|_{\tilde{U}_i} \equiv 0$ $i = 1,2$ so it suffices to show that $\omega_{|F(V)}$ vanishes on $K \cap F(V)$. But $\omega(F) = \chi\omega(F_1) + (1-\chi)\omega(F_2)$ on K and therefore equals zero. Doing the same with $U_{1,2}$ and U_3 we get $U_{1,2,3}$. Continuing inductively we obtain an integral manifold $N_r = U_{1,2,\ldots,k}$ containing a neighborhood in K of the r-dimensional points.

Let $N_r' \subset \tilde{N}_r' \subset N_r$ be another integral manifold containing all r-dimensional points of K. Then the set of $(r-1)$-dimensional points in $K \setminus N_r'$ is compact. (If this set is empty, consider instead the $(r-2)$-dimensional points etc.)

By the same process as above we get an $(r-1)$-dimensional integral manifold $\tilde{N}_{r-1}$ containing a neighborhood of the $(r-1)$-dimensional points of $K \setminus N_r'$ in $K \setminus N_r'$. If there are no $(r-1)$-dimensional points in $N_r \setminus N_r'$ we shrink N_r and $\tilde{N}_{r-1}$, so that their closures are disjoint. Otherwise let $N_r' \subset \tilde{N}_r' \subset N_r'' \subset \tilde{N}_r'' \subset N_r''' \subset \tilde{N}''' \subset N_r$ be integral manifolds, and M the orthogonal projection π to N_r of a neighborhood in $\tilde{N}_{r-1}$ of the $(r-1)$-dimensional points of $K \setminus N_r'$ in $N_r \setminus N_r'$.

We can cover $M \cap (\bar{N}_r''' \setminus N_r'')$ by a finite number of coordinate neighborhoods given as graphs (as in lemma 2.2). Patching these inductively as above to $(M \cap N_r'') \cup (\tilde{N}_{r-1} \setminus \pi^{-1}(\bar{N}_r'''))$ we obtain an integral manifold $\tilde{\tilde{N}}_{r-1}$. Replacing N_r by a small neighborhood of $\bar{N}_r'$ and letting $N_{r-1} = \tilde{\tilde{N}}_{r-1} \setminus \bar{N}_r'$ we obtain integral manifolds such that:

i) N_r contains all r-dimensional points of K

ii) N_{r-1} contains all the (r-1)-dimensional points in $K \setminus N_r$

iii) $N_{r-1} \cap N_r$ is open in N_{r-1}

iv) $K \cap N_i$ is open in K, $i = r, r-1$.

Continuing inductively we choose N_r' and N_{r-1}' as earlier. Then there exists an integral manifold N_{r-2} containing all (r-2)-dimensjonal points in $K \setminus (N_r' \cup N_{r-1}')$.

By the same process as above we may assume that $N_{r-2} \cap N_{r-1}$ is open in N_{r-2} and by repeating it for N_{r-2} and N_r we may assume that $N_{r-2} \cap N_r$ is open in N_{r-2}.

Finally we obtain $N_1, \ldots, N_m$ as required in the theorem.

THE CASE $D \subset \mathbb{C}^3$.

In the rest of this section let D be a strongly pseudoconvex domain with $\mathcal{C}^\infty$ boundary in $\mathbb{C}^3$.

THEOREM 3.2:

If K is a compact set in ∂D which is locally contained in integral manifolds, there exists an integral manifold N containing all of K.

Proof:

Let N_1 and N_2 be as in theorem 3.1. We may assume that $\dim N_i = i$ since the 0-dimensional points are isolated in K.

There are two cases

(1) When $N_1 \cap N_2$ contains no one-dimensional points, we can shrink N_1 and N_2 such that $\bar{N}_1 \cap \bar{N}_2 = \emptyset$ and then we can let N be $N_1 \cup N_2$

(2) If $N_1 \cap N_2$ contains one-dimensional points we shrink N_1 and N_2 such that there exist two-dimensional integral manifolds $N_3, \ldots, N_k$ with the properties:

a) $N_1 \subset \bigcup_{i=3}^{k} N_i$,

b) each N_i is a graph over a couple of coordinates when $i \geq 3$,

c) $\bar{N}_i \cap \bar{N}_j \cap \bar{N}_s = \emptyset$, $2 \leq i < j < s$,

d) $K \cap N_j$ is open in K and

e) if $\bar{N}_i \cap \bar{N}_j \neq \emptyset$, then there exists a one to one curve $\gamma_{ij}[a,b] \to N$, such that $\gamma_{ij}(a,b) = N_1 \cap N_i \cap N_j$ when $i \geq 2$ and $j \geq 3$ and $\gamma(a) \in N_i \setminus N_j$ and $\gamma(b) \in N_j \setminus N_i$ if $i \neq j$.

Fix $2 \leq i < j$ so that $\bar{N}_i \cap \bar{N}_j \neq \emptyset$. If there exists a point on $\gamma_{ij} \cap N_i \cap N_j$ such that both can be parametrized by the same coordinates in a neighborhood of p, we can patch N_i and N_j as in theorem 2.1 preserving a), c), d) and e). If not, we can parametrize over pairs of coordinates which have one in common since there is a curve in the intersection. Without loss of generality we may assume that $N_i(N_j)$ is parametrized over $(x_1,x_2)((x_1,y_2))$.

Choose an interval $(c,d) \subset\subset (a,b)$. Say N_j is given by (x_1, X_2, Y_1, y_2, T) in the strip over $\gamma_{ij}((c,d))$. If $\frac{\partial X_2}{\partial y_2} \neq 0$ at a point on $\gamma_{ij}((c,d))$ we can reparametrize over (x_1, x_2) in a neighborhood and then patch N_i, N_j there as before. Otherwise we twist N_j around γ_{ij} in the following way: Let $p \in \gamma_{ij}(c,d)$ and choose $\eta_2 = \eta_2(x_1, y_2)$ such that $\frac{\partial \eta_2}{\partial y_2} \neq 0$ in a neighborhood of p, $\eta_{2|N_1} = 0$ and $\eta_2 \equiv 0$ outside a small neighborhood U of p.

We are interested in finding η, and θ such that

$$d(T+\theta) + x_1 d(Y_1+\eta_1) + (X_2+\eta_2)dy_2$$

$$= d\theta + x_1 d\eta_1 + \eta_2 dy_2 = 0$$

which is possible if $dx_1 \wedge d\eta_1 + d\eta_2 \wedge dy_2 = 0$. Furthermore we want θ and η, to equal zero on N_1 and outside U.

Solving the equation $\frac{\partial \eta_1}{\partial y_2} = \frac{\partial \eta_2}{\partial x_1}$ with initial condition $\eta_{1|N_1} = 0$ we obtain a function η_1 vanishing outside a small neighborhood of p. Next we solve the equations $\frac{\partial \theta}{\partial \eta_1} = -x_1 \frac{\partial \eta_1}{\partial x_1}$ and $\frac{\partial \theta}{\partial y^2} = -(x_1 \frac{\partial \eta_1}{\partial y_2} + \eta_2)$. Since $d\theta|_{N_1} \| d\eta_1|_{N_1} \equiv 0$ we can choose θ such that $\theta|_{N_1} \equiv 0$. These equations also imply that $\theta = 0$ outside a small neighborhood of p.

4. Global peak functions.

We shall show that the answer to question 2 is yes for a general $n \geq 2$.

LEMMA 4.1:

Let D be a strongly pseudoconvex domain in $\mathbb{C}^n$ with $\mathcal{C}^\infty$ boundary. If $K \subset \partial D$ is compact and contained in $N_1 \cup N_2$ where N_1, N_2 are integral manifolds and $\dim N_1 < \dim N_2$, $N_1 \cap N_2$ is open in N_1 and $K \cap N_i$ is open in K, then K is a peak set for $A^\infty(D)$.

Proof:

Choose $N_2^1 \subset\subset N_2^2 \subset\subset N_2^3 \subset\subset N_2^4 \subset\subset N_2$ such that $K \setminus N_2^1 \subset N_1$ and let $K_2 = \overline{N}_2^4 \cap K$ and $K_1 = K \setminus N_2^1$.

Choose $\mathcal{C}^\infty$ cut-off functions χ_0 and χ with the properties:

$\operatorname{supp} \chi_0 \subset N_2^4 \setminus N_2^1$ and $\chi_0 \equiv 1$ on $N_2^3 \setminus N_2^2$, $\chi|_{\overline{N}_2^2} \equiv 1$ and $\operatorname{supp} \chi \subset \overline{N}_2^3$.

We can find a function $g \in \mathcal{C}^\infty(N_2, \mathbb{R})$ which equals $d^2(p, N_1)$ near $N_1 \cap N_2$.

From [5] we have the existence of functions $\tilde{\chi}_0, \tilde{\chi}$ and $\tilde{g}$ where:

a) $\tilde{\chi}_0|_{N_2} = \chi_0$, $\tilde{\chi}|_{N_2} = \chi$ and $\tilde{g}|_{N_2} = g$.

b) $D^\alpha \bar{\partial} \tilde{\chi}_0|_{N_2} = D^\alpha \bar{\partial} \tilde{\chi}|_{N_2} = D^\alpha \bar{\partial} \tilde{g}|_{N_2} = 0$ for each multiindex α.

c) $\tilde{\chi}_0 (\tilde{\chi})$ is locally constant in $\mathbb{C}^n$ near where $\chi_0|_{N_2} (\chi|_{N_2})$ is locally constant.

d) First derivatives of $\tilde{\chi}_0, \tilde{\chi}$ and $\tilde{g}$ vanish on N_2 in directions perpendicular to $TN_2 + iTN_2$.

Lemma 2.3 implies that there exists u_i satisfying (1) → (4) when $K = K_i$ and $N = N_i$, $i = 1,2$.

Let $\tilde{u} = \tilde{\chi}(u_2 + \epsilon \tilde{\chi}_0 \tilde{g}) + (1 - \tilde{\chi}) u_1$. Then $\tilde{u} \in \mathcal{C}^\infty(\omega)$ where ω is a neighborhood of $N_2 \cup N_1$ in $\mathbb{C}^n$ and:

i) $\tilde{u} = u_2$ when $\tilde{\chi} = 1$, $\tilde{\chi}_0 = 0$ and $\tilde{u} = u_1$ when $\tilde{\chi} = 0$.

ii) $D^\alpha \bar{\partial}\tilde{u}|_{N_1 \cup N_2} \equiv 0$ for each multiindex α.

iii) $\operatorname{Re}\tilde{u}(p) \geq \frac{\varepsilon}{2} d^2(p, N_2^2 \cup N_1) + O(\operatorname{Im}\tilde{\chi}\cdot \operatorname{Im} u) + O(\operatorname{Im}(1-\tilde{\chi})\cdot \operatorname{Im} u_1)$ if ε is sufficiently small.

Define $\tau(p) = Jn(p)$ where $n(p)$ is the outer normal to ∂D at p. Intergrate $\tau(p)$ from N_2 and let $\tilde{N}'$ be the union over N_2 of the integral curves. If U is a small neighborhood of N_2, $\tilde{N}' \cap U = \tilde{N}$ is totally real. When $p \in \tilde{N}$ there exists a unique $p_0 \in N_2$ and integral curve γ for τ such that $\gamma : [0,z] \to \tilde{N}$, $z = z(p)$, and $\gamma(0) = p_0$, $\gamma(z) = p$. The function $z : \tilde{N} \to \mathbb{R}$ is $\mathcal{C}^\infty$ and vanishes to first order on N_2.

Again we can find a $\mathcal{C}^\infty$-function $\tilde{z}$ where $\tilde{z}|_{\tilde{N}} = z$, first derivatives of $\tilde{z}$ in directions in $T_{\mathbb{C}}\partial D$ vanish on N_2 and $D^\alpha\bar{\partial}\tilde{z}|_{\tilde{N}} \equiv 0$ for each multiindex α. Let $\psi = \lambda\tilde{\chi}_0(\tilde{z})^2$ where $\lambda \gg 1$ is chosen sufficiently large. Then $u = \tilde{u} + \psi$ has the properties:

a) $\{p : u(p) = 0\} = K$

b) $D^\alpha\bar{\partial}u|_{N_2 \cup N_1} = 0$ for each α.

c) There exists a $C > 0$ such that $\operatorname{Re} u(p) \geq Cd^2(p, N_2^2 \cup N_1)$.

By the classical techniques described in [1], [2] and [4] we can now find a function in $A^\infty(D)$ which peaks at K.

<u>THEOREM 4.2</u>:

If a compact set $K \subset \partial D$ is locally contained in integral manifolds, then K is a peak set for $A^\infty(D)$.

<u>Proof</u>: This goes as in Lemma 4.1 inductively, so we will be very brief. Let $N_1, \ldots, N_m$ be as in theorem 3.1 and $N_i'' \subset\subset N_i' \subset\subset N_i$

such that the families $\{N_i''\}_{i=1}^m$ and $\{N_i'\}_{i=1}^m$, satisfy (1) → (4) in the theorem. If $K_i = K \cap \bar{N}_i'$ we choose u_i for the pair K_i, N_i. Modifying the u_i's inductively as in Lemma 4.1 we may assume that $\mathrm{Re}\, u_j \gtrsim d^2(p, N_i)$ in a neighborhood of $N_i \cap N_j$ whenever $i < j$. We can patch the u_i's as in lemma 4.1 and finally we get a function $u \in \mathcal{E}^\infty(\omega)$ (ω is a neighborhood of K in $\mathbb{C}^n$) such that:

(1) $\mathrm{Re}\, u(p) > 0$ when $p \in \bar{D} \setminus K$

(2) $u|_K = 0$

(3) $|\bar{\partial} u| \leq C_k\, (\mathrm{Re}\, u)^k$ for each k.

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