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REGULARIZATIONS OF PLURISUBHARMONIC FUNCTIONS

John Erik Fornæss
Inst. of Math., University of Oslo

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1. Introduction. Plurisubharmonic functions are useful tools in the theory of several complex variables. They are easier to construct than holomorphic functions, but properties of plurisubharmonic functions on a space often carry over to properties of holomorphic functions. In this process it is usually at first necessary to approximate a given plurisubharmonic function with one which is more regular. Richberg proved in 1968 the following regularization-result:

Theorem ([3]). Let ρ be a continuous strongly plurisubharmonic function on a complex manifold M. Then there exists a sequence $\{\rho_n\}_{n=1}^{\infty}$ of $\mathcal{C}^{\infty}$ plurisubharmonic functions on M such that $\rho_n \searrow \rho$.

For domains of holomorphy M in $\mathbb{C}^n$, the conclusion of the theorem holds with the only hypothesis that ρ is plurisubharmonic. However, there exists also a domain Ω in $\mathbb{C}^2$ and a discontinuous plurisubharmonic function $\rho: \Omega \to \mathbb{R}$ for which there does not exist a sequence $\{\rho_n\}$ of continuous plurisubharmonic functions such that $\rho_n \searrow \rho$ (see [1]).

In this paper we show that for each positive integer $k = 0,1,\dots$ (or $k = \infty$) there exists a complex manifold M_k with a $\mathcal{C}^k$ (or $\mathcal{C}^{\infty}$) plurisubharmonic function ρ_k and two points p_k, q_k such that $\rho_k(p_k) \neq \rho_k(q_k)$ while all $\mathcal{C}^{k+1}$

(or $\mathcal{C}^\omega$) plurisubharmonic functions σ on M_k satisfy the equation $\sigma(p_k) = \sigma(q_k)$.

2. The Examples.

We will consider three cases, I: $k = 0$, II: $1 \leq k < \infty$ and III: $k = \infty$.

I: Let $\{r_n\}_{n=1}^{\infty}$ be a counting of the rational points in the open unit interval $(0,1)$. If we define $p_n = r_n + \frac{1}{2^n} + \frac{i}{2^n}$ and $q_n = r_n + \frac{2}{2^n} + \frac{i}{2^n}$ we obtain two sequences clustering at all points in $[0,1] \subset \mathbb{R} \subset \mathbb{C}$. The discs $\Delta_n = \{z \in \mathbb{C}; |z-p_n| < \frac{1}{2^{n+2}}\}$ and $D_n = \{z \in \mathbb{C}; |z-q_n| < \frac{1}{2^{n+2}}\}$ have pairwise disjoint closures which do not intersect the real axis. Let χ be a $\mathcal{C}^\infty$ function with compact support in the unit disc, $0 \leq \chi \leq 1$ and $\chi \equiv 1$ in a neighborhood of zero.

Lemma 1. There exist concentric discs $\Delta_n' \subset \Delta_n$, $D_n' \subset D_n$, Δ_n', D_n' have the same radius, $n = 1,2,\ldots$ and a continuous subharmonic function ρ on $\mathbb{C}$ such that $\rho(z) = z\bar{z}$ for $z \in \mathbb{R}$ and $\rho|_{\Delta_n'} \equiv r_n^2 \equiv \rho|_{D_n'}$ $\forall n$.

Proof. We define ρ by $\rho(z) = z\bar{z}$ on $\mathbb{C} - \cup\Delta_n \cup D_n$. On Δ_n, we let $\rho(z) = \max\{z\bar{z} + \epsilon_n\chi(\frac{z-p_n}{2^{n+2}})\log|z-p_n|, r_n^2\}$ and similarly on $D_n, \rho(z) = \max\{z\bar{z} + \epsilon_n\chi(\frac{z-q_n}{2^{n+2}})\log|z-q_n|, r_n^2\}$ for $\epsilon_n > 0$ small enough. $\square$

To define M_o, let at first Ω be the open set in $\mathbb{C}^2$ defined by:

$$\Omega = \{(z,\omega);\ |\omega| < \tfrac{1}{2}\} \cup \{z \in \cup\Delta_n' \cup D_n' \text{ and } |\omega| < 2\}.$$

The complex manifold M_o is obtained by making for each n the identification

$$z \in \Delta_n', \ \tfrac{1}{2} < |\omega| < 2 \Rightarrow (z + \frac{1}{2^n}, \frac{1}{\omega}) \in \{z \in D_n' \text{ and } \tfrac{1}{2} < |\omega| < 2\}.$$

We define a continuous plurisubharmonic function ρ_o on M_o by $\rho_o(z,\omega) = \rho(z)$. By Lemma 1 this is invariant under the above identifications and hence is well defined. We let $p_o = (0,0)$ and $q_o = (1,0)$. Then $\rho_o(p_o) = 0$ and $\rho_o(q_o) = 1$. Assume that there exists a $\mathcal{E}^1$ plurisubharmonic σ such that $\sigma(p_o) \neq \sigma(q_o)$. Writing $z = x + iy$, it follows that $\frac{\partial\sigma}{\partial x}(x_o,0) \neq 0$ for some $x_o \in (0,1)$. This implies that there exists an n so that $\sigma(p_n) \neq \sigma(q_n)$. However, there exists a compact complex submanifold - a $\mathbb{P}^1$ - of M_o containing both p_n and q_n. Hence $\sigma(p_n) = \sigma(q_n)$, a contradiction.

<u>II:</u> Let $\{r_n\}_{n=1}^{\infty}$ be a counting of the rational points in the open unit interval $(0,1)$. The points $p_n = r_n + \frac{i}{2^n}$ cluster all over $[0,1]$. Each p_n is the limit of a sequence $\{p_{n,m}\}_{m=n}^{\infty}$, $p_{n,m} = p_n + \frac{1}{2^m}$. Let $\rho(z)$ be the subharmonic function $z\bar{z}$ on $\mathbb{C}(z)$.

We will set up a perturbed version of this situation. To avoid confusion we will use ''s. As before let $p_n' = r_n + \frac{i}{2^n}$. Let $\epsilon \in (0,1)$ be given. We define $p_{n,2m}' = p_n' + \frac{1}{2^{2m}}$ and $p_{n,2m+1}' = p_n' + \frac{1}{2^{2m+1}} + \epsilon(\frac{1}{2^{2m+3}})^{k+1}$ if $2m \geq n$ and $2m + 1 \geq n$ respectively.

Let $\chi : \mathbb{C} \to [0,1]$ be a $\mathcal{E}^\infty$ function, $\chi(z') \equiv 1$ in a neighborhood of 0, χ has support in $\{|z'| < 1\}$. The discs $\Delta_{n,m} = \{z' \in \mathbb{C};\ |z' - p_n' - \frac{1}{2^m}| < \frac{1}{2^{m+2}}$ have disjoint closures. We define $\rho'(z')$ by $\rho' = z'\bar{z}'$ on $\mathbb{C} - \cup\Delta_{n,2m+1}$. On $\Delta_{n,m}$ when m

is odd, let

$$\rho'(z') = \left[1-\chi\left(\frac{z'-p_n'-1/2^m}{1/2^{m+2}}\right)\right]z'\bar{z}'$$

$$+\ \chi\left(\frac{z'-p_n'-1/2^m}{1/2^{m+2}}\right)\left|z'-\varepsilon\left(\frac{1}{2^{m+2}}\right)^{k+1}\right|^2.$$

Observe that if ε is small enough then there exists a neighborhood of each $p'_{n,2m+1}$ on which $\rho'(z') \equiv \left|z'-\varepsilon\left(\frac{1}{2^{2m+3}}\right)^{k+1}\right|^2$.

<u>Lemma 2.</u> <u>If</u> ε <u>is small enough, then</u> ρ' <u>is a</u> $\mathcal{E}^k$ <u>subharmonic function.</u>

<u>Proof.</u> It suffices to show that ρ' is $\mathcal{E}^k$ and that if ε is small enough then $\rho'|_{\Delta_{n,2m+1}}$ is subharmonic for all n,m.

On $\Delta_{n,m}$ - when m is odd - ,

$$\rho'(z') = z'\bar{z}' + \chi\left(\frac{z'-p_n'-1/2^m}{1/2^{m+2}}\right)\left(\left|z'-\varepsilon\left(\frac{1}{2^{m+2}}\right)^{k+1}\right|^2 - z'\bar{z}'\right).$$

Differentiating the χ at most k times gives an expression like $\mathcal{O}((2^{m+2})^k)$ while any derivative of the function in ( ) is $\mathcal{O}(\varepsilon(\frac{1}{2^{m+2}})^{k+1})$. Hence if α is any multiindex of order at most k, then $D^\alpha\rho' = D^\alpha z'\bar{z}' + \varepsilon\,\mathcal{O}(\frac{1}{2^{m+2}})$. This proves that ρ is $\mathcal{E}^k$. Since also

$$\frac{\partial^2\rho'}{\partial z'\partial\bar{z}'} = 1 + \mathcal{O}\left((2^{m+2})^2\cdot\varepsilon\left(\frac{1}{2^{m+2}}\right)^{k+1}\right) \quad \text{on } \Delta_{n,m},\quad m \text{ odd},$$

it follows that ρ' is subharmonic on all $\Delta_{n,m}$ if ε is small enough (recall that $k \geq 1$).

In the rest of the construction we fix an $\varepsilon > 0$ small enough. We now choose small discs $\tilde{\Delta}_{n,m}$ centered at $p_{n,m}$ and $\tilde{\Delta}'_{n,m}$

centered at $p'_{n,m}$ such that

(i) $\Delta_{n,m}$ and $\Delta'_{n,m}$ have the same radius,

(ii) the $\Delta_{n,m}$'s ($\Delta'_{n,m}$'s) have pairwise disjoint closures which do not intersect the real axis,

(iii) if m is even, then $\rho' = z'\bar{z}'$ on $\Delta'_{n,m}$ and

(iv) if m is odd, then $\rho' = |z' - \varepsilon(\frac{1}{2^{m+2}})^{k+1}|^2$ on $\tilde{\Delta}'_{n,m}$.

Let $\Omega_1^k \subset \mathbb{C}^2(z,\omega)$, $\Omega_2^k \subset \mathbb{C}^2(z',\omega')$ be open sets,

$$\Omega_1^k = \{|\omega| < \tfrac{1}{2}\} \cup \{z \in \cup\Delta_{n,m} \text{ and } |\omega| < 2\} ,$$

$$\Omega_2^k = \{|\omega'| < \tfrac{1}{2}\} \cup \{z' \in \cup\tilde{\Delta}'_{n,m} \text{ and } |\omega'| < 2\} .$$

We define a complex manifold M_k by patching Ω_1^k and Ω_2^k where $\frac{1}{2} < |\omega| < 2$ and $\frac{1}{2} < |\omega'| < 2$: If $z \in \Delta_{n,2m}$, $\frac{1}{2} < |\omega| < 2$ and $z' \in \tilde{\Delta}'_{n,2m}$, $\frac{1}{2} < |\omega'| < 2$ use the coordinate transformation $z' = z$, $\omega' = \frac{1}{\omega}$. If $z \in \tilde{\Delta}_{n,2m+1}$, $\frac{1}{2} < |\omega| < 2$ and $z' \in \tilde{\Delta}'_{n,2m+1}$, $\frac{1}{2} < |\omega'| < 2$ let $z' = z + \varepsilon(\frac{1}{2^{2m+3}})^{k+1}$, $\omega' = \frac{1}{\omega}$. Then ρ_k, given by $\rho_k(z,\omega) = \rho(z)$ on Ω_1^k and $\rho_k(z',\omega') = \rho'(z')$ on Ω_2^k is a $\mathcal{C}^k$ plurisubharmonic function. Let $P_k = 0 \in \Omega_1^k$ and $Q_k = (1,0) \in \Omega_1^k$. Then $\rho_k(P_k) = 0$ and $\rho_k(Q_k) = 1$.

Assume that there exists a $\mathcal{C}^{k+1}$ plurisubharmonic function σ on M_k such that $\sigma(P_k) \neq \sigma(Q_k)$. Then there exists an n such that $\frac{\partial\sigma}{\partial x}(p_n,0) \neq 0$. We compare the Taylor expansions of order $k+1$ of σ about p_n and p'_n in the x and x' direction respectively:

$$\sigma(x+\frac{i}{2^n},0) = \sigma(p_n,0) + \sum_{j=1}^{k+1} A_j(x-r_n)^j + o(|x-r_n|^{k+1}) ,$$

$$\sigma(x'+\frac{i}{2^n},0) = \sigma(p_n',0) + \sum_{j=1}^{k+1} A_j'(x'-r_n)^j + o(|x'-r_n|^{k+1})$$

where $A_1 \neq 0$.

Now $\sigma(p_n + \frac{1}{2^{2m}},0) = \sigma(p_n' + \frac{1}{2^{2m}},0)$, $m \geq n/2$,

it follows that $\sigma(p_n,0) = \sigma(p_n',0)$ and $A_j' = A_j$, $j = 1,\dots,k+1$. We also have that $\sigma(p_n + \frac{1}{2^{2m+1}}, 0) = \sigma(p_n' + \frac{1}{2^{2m+1}} + \varepsilon(\frac{1}{2^{2m+3}})^{k+1},0)$ $2m+1 \geq n$. Comparing the Taylor expansions we obtain that $A_1\varepsilon(\frac{1}{2^{2m+3}})^{k+1} = o((\frac{1}{2^{2m+1}})^{k+1})$, which is a contradiction.

<u>III</u>: We use r_n, p_n, $p_{n,m}$, p_n', $p_{n,2m}'$, ρ, χ and $\Delta_{n,m}$ as in II. However choose $p_{n,2m+1}' = p_n' + \frac{1}{2^{2m+1}} + \varepsilon(\frac{1}{2^{2m+3}})^{2m+3}$ whenever $2m+1 \geq n$. We define $\rho'(z') = z'\bar{z}'$ on $\mathbb{C} - \cup\Delta_{n,2m+1}$. When m is odd, define ρ' on $\Delta_{n,m}$ by

$$\rho'(z') = \left[1 - \chi(\frac{z'-p_n'-1/2^m}{1/2^{m+2}})\right]z'\bar{z}' + \chi(\frac{z'-p_n'-1/2^m}{1/2^{m+2}})|z' - \varepsilon(\frac{1}{2^{m+2}})^{2m+3}|^2.$$

Then, if $\varepsilon > 0$ is small enough, there exist neighborhoods of each $p_{n,2m+1}'$ on which $p'(z') = |z' - \varepsilon(\frac{1}{2^{m+2}})^{2m+3}|^2$ and ρ' is a $\mathcal{C}^\infty$ subharmonic function on $\mathbb{C}$, $\rho'|_{\mathbb{R}} = z'\bar{z}'$.

It is possible to choose discs $\tilde{\Delta}_{n,m}$ and $\tilde{\Delta}_{n,m}'$ as in II except that (iv) is replaced by (iv)' if m is odd, then $\rho' = |z' - \varepsilon(\frac{1}{2^{m+2}})^{2m+3}|^2$ on $\tilde{\Delta}_{n,m}'$.

The open sets Ω_1^∞, Ω_2^∞ and the manifold M_∞ is defined as in II except that if $z \in \tilde{\Delta}_{n,2m+1}$, $\frac{1}{2} < |\omega| < 2$ and $z' \in \tilde{\Delta}'_{n,2m+1}$, $\frac{1}{2} < |\omega'| < 2$, then $z' = z + \epsilon(\frac{1}{2^{2m+3}})^{2m+3}$, $\omega' = \frac{1}{\omega}$. Furthermore, the $\mathcal{C}^\infty$ plurisubharmonic function ρ_∞ on M_∞ and P_∞, Q_∞ are defined as in II. Again, we have that $\rho_\infty(P_\infty) = 0$ and $\rho_\infty(Q_\infty) = 1$.

If there exists a real analytic plurisubharmonic function σ on M_∞ such that $\sigma(P_\infty) \neq \sigma(Q_\infty)$, then there exist power series expansions in the $x(x')$ direction about some $p_n(p'_n)$,

$$\sigma(x + \frac{i}{2^n}, 0) = \sum_{j=o}^{\infty} A_j (x - r_n)^j ,$$

$$\sigma(x' + \frac{i}{2^n}, 0) = \sum_{j=o}^{\infty} A'_j (x' - r_n)^j$$

with $A_1 \neq 0$.

Since $\sigma(p_{n,2m}, 0) = \sigma(p'_{n,2m}, 0)$ it follows that $A'_j = A_j$ for all j, and hence that $\sigma(x + \frac{1}{2^n}, 0) = \sigma(x' + \frac{i}{2^n}, 0)$ whenever $x = x'$. The fact that $A_1 \neq 0$ implies also that $x = x'$ whenever $\sigma(x + \frac{i}{2^n}, 0) = \sigma(x' + \frac{i}{2^n}, 0)$ and x, x' are close enough to r_n. This contradicts the fact that $\sigma(p_{n,2m+1}, 0) = \sigma(p'_{n,2m+1}, 0)$ for all $2m+1 \geq n$.

<u>Remark</u>. All the complex manifolds M_k contain many compact complex subvarieties ($\mathbb{P}_1$'s). Because of removable singularity theorems for plurisubharmonic functions, ([2]) they can all be punctured by removing a suitable family of two dimensional totally real submanifolds. These new M_k's will still have the same properties as above but will contain no positive dimensional compact complex subvarieties.

References

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