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REGULARIZATIONS OF PLURISUBHARMONIC FUNCTIONS

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1. Introduction. Plurisubharmonic functions are useful tools in the theory of several complex variables. They are easier to construct than holomorphic functions, but properties of plurisubharmonic functions on a space often carry over to properties of holomorphic functions. In this process it is usually at first necessary to approximate a given plurisubharmonic function with one which is more regular. Richberg proved in 1968 the following regularization-result:

Theorem ([3]). Let \( p \) be a continuous strongly plurisubharmonic function on a complex manifold \( M \). Then there exists a sequence \( \{ p_n \}_{n=1}^{\infty} \) of \( \mathcal{C}^\infty \) plurisubharmonic functions on \( M \) such that \( p_n \rightarrow p \).

For domains of holomorphy \( M \) in \( \mathcal{C}^n \), the conclusion of the theorem holds with the only hypothesis that \( p \) is plurisubharmonic. However, there exists also a domain \( \Omega \) in \( \mathcal{C}^2 \) and a discontinuous plurisubharmonic function \( \rho: \Omega \rightarrow \mathbb{R} \) for which there does not exist a sequence \( \{ p_n \} \) of continuous plurisubharmonic functions such that \( p_n \rightarrow \rho \) (see [1]).

In this paper we show that for each positive integer \( k = 0, 1, \ldots \) (or \( k = \infty \)) there exists a complex manifold \( M_k \) with a \( \mathcal{C}^k \) (or \( \mathcal{C}^\infty \)) plurisubharmonic function \( p_k \) and two points \( p_k, q_k \) such that \( p_k(p_k) \neq p_k(q_k) \) while all \( \mathcal{C}^{k+1} \).
(or \( C^\infty \)) plurisubharmonic functions \( \sigma \) on \( M_k \) satisfy the equation \( \sigma(p_k) = \sigma(q_k) \).

2. The Examples.

We will consider three cases, I: \( k = 0 \), II: \( 1 \leq k < \infty \) and III: \( k = \infty \).

I: Let \( \{r_n\}_{n=1}^{\infty} \) be a counting of the rational points in the open unit interval \((0,1)\). If we define \( p_n = r_n + \frac{1}{2^n} + \frac{i}{2^n} \) and \( q_n = r_n + \frac{2}{2^n} + \frac{i}{2^n} \), we obtain two sequences clustering at all points in \([0,1] \subset \mathbb{R} \subset \mathbb{C}\). The discs \( \Delta_n = \{z \in \mathbb{C} ; |z-p_n| < \frac{1}{2n+2}\} \) and \( D_n = \{z \in \mathbb{C} ; |z-q_n| < \frac{1}{2n+2}\} \) have pairwise disjoint closures which do not intersect the real axis. Let \( \chi \) be a \( C^\infty \) function with compact support in the unit disc, \( 0 \leq \chi \leq 1 \) and \( \chi = 1 \) in a neighborhood of zero.

Lemma 1. There exist concentric discs \( \Delta_n' \subset \Delta_n, D_n' \subset D_n, \Delta_n', D_n' \) have the same radius, \( n = 1,2,\ldots \) and a continuous subharmonic function \( \rho \) on \( \mathbb{C} \) such that \( \rho(z) = z\bar{z} \) for \( z \in \mathbb{R} \) and \( \rho|_{\Delta_n'} = r_n^2 = \rho|_{D_n'} \forall n \).

Proof. We define \( \rho \) by \( \rho(z) = z\bar{z} \) on \( \mathbb{C} - U_{\Delta_n} \cup D_n \).

On \( \Delta_n' \), we let \( \rho(z) = \max\{z\bar{z} + \epsilon_n \chi(z-p_n) \log|z-p_n|, r_n^2\} \) and

similarly on \( D_n \), \( \rho(z) = \max\{z\bar{z} + \epsilon_n \chi(z-q_n) \log|z-q_n|, r_n^2\} \) for \( \epsilon_n > 0 \) small enough.

To define \( M_0 \), let at first \( \Omega \) be the open set in \( \mathbb{C}^2 \) defined by:

\[
\Omega = \{(z,w); |w| < \frac{1}{4} \} \cup \{z \in \cup_{\Delta_n} \cup D_n \text{ and } |w| < 2\}.
\]
The complex manifold $M_0$ is obtained by making for each $n$ the identification

$$z \in \Delta_n', \frac{1}{2} < |w| < 2 \rightarrow (z + \frac{1}{2^n}, \frac{1}{w}) \in \{z \in D_n' \text{ and } \frac{1}{2} < |w| < 2\}.$$

We define a continuous plurisubharmonic function $\rho_0$ on $M_0$ by $\rho_0(z,w) = \rho(z)$. By Lemma 1 this is invariant under the above identifications and hence is well defined. We let $p_0 = (0,0)$ and $q_0 = (1,0)$. Then $\rho_0(p_0) = 0$ and $\rho_0(q_0) = 1$. Assume that there exists a $C^1$ plurisubharmonic $\sigma$ such that $\sigma(p_0) \neq \sigma(q_0)$. Writing $z = x + iy$, it follows that $\frac{\partial \sigma}{\partial x}(x_0,0) \neq 0$ for some $x_0 \in (0,1)$. This implies that there exists an $n$ so that $\sigma(p_n) \neq \sigma(q_n)$. However, there exists a compact complex submanifold - a $\mathbb{P}^1$ - of $M_0$ containing both $p_n$ and $q_n$. Hence $\sigma(p_n) = \sigma(q_n)$, a contradiction.

II: Let $\{r_n\}_{n=1}^\infty$ be a counting of the rational points in the open unit interval $(0,1)$. The points $p_n = r_n + \frac{i}{2^n}$ cluster all over $[0,1]$. Each $p_n$ is the limit of a sequence $\{p_{n,m}\}_{m=n}^\infty$, $p_{n,m} = p_n + \frac{1}{2^m}$. Let $\rho(z)$ be the subharmonic function $z\bar{z}$ on $\mathbb{C}(z)$.

We will set up a perturbed version of this situation. To avoid confusion we will use 's. As before let $p_n' = r_n + \frac{i}{2^n}$. Let $\epsilon \in (0,1)$ be given. We define $p_{n,2m} = p_n' + \frac{1}{2^{2m}}$ and $p_{n,2m+1} = p_n' + \frac{1}{2^{2m+1}} + \epsilon(\frac{1}{2^{2m+3}})^{k+1}$ if $2m \geq n$ and $2m+1 \geq n$ respectively.

Let $\chi : \mathbb{C} - [0,1]$ be a $C^\infty$ function, $\chi(z') = 1$ in a neighborhood of $0$, $\chi$ has support in $\{|z'| < 1\}$. The discs $\Delta_{n,m} = \{z' \in \mathbb{C} ; |z' - p_n' - \frac{1}{2^m}| < \frac{1}{2^{m+2}}\}$ have disjoint closures. We define $\rho'(z')$ by $\rho' = z'\bar{z}'$ on $\mathbb{C} - \cup \Delta_{n,2m+1}$. On $\Delta_{n,m}$ when $m$
is odd, let
\[ \rho'(z') = \left[ 1 - \chi(\frac{z' - p_n - 1/2^m}{1/2^m + 2}) \right] z' \overline{z}' 
+ \chi(\frac{z' - p_n - 1/2^m}{1/2^m + 2}) |z' - \epsilon(\frac{1}{2^m + 2})^{k+1}|^2. \]

Observe that if \( \epsilon \) is small enough then there exists a neighborhood of each \( p_n, 2m+1 \) on which \( \rho'(z') = |z' - \epsilon(\frac{1}{2^m + 2})^{k+1}|^2. \)

**Lemma 2.** If \( \epsilon \) is small enough, then \( \rho' \) is a \( \mathcal{C}_k \) subharmonic function.

**Proof.** It suffices to show that \( \rho' \) is \( \mathcal{C}_k \) and that if \( \epsilon \) is small enough then \( \rho'|_{\Delta_{n,2m+1}} \) is subharmonic for all \( n,m \).

On \( \Delta_{n,m} \) when \( m \) is odd -
\[ \rho'(z') = z' \overline{z}' + \chi(\frac{z' - p_n - 1/2^m}{1/2^m + 2})(|z' - \epsilon(\frac{1}{2^m + 2})^{k+1}|^2 - z' \overline{z}'). \]

Differentiating the \( \chi \) at most \( k \) times gives an expression like \( \mathcal{O}(\epsilon(2^m + 2)^k) \) while any derivative of the function in \( (\ ) \) is \( \mathcal{O}(\epsilon(\frac{1}{2^m + 2})^{k+1}) \). Hence if \( \alpha \) is any multiindex of order at most \( k \), then \( \mathcal{D}^\alpha \rho' = \mathcal{D}^\alpha z' \overline{z}' + \epsilon \mathcal{O}(\frac{1}{2^m + 2}). \) This proves that \( \rho \) is \( \mathcal{C}_k \). Since also
\[ \frac{\partial^2 \rho'}{\partial z' \partial \overline{z}'} = 1 + \mathcal{O}((2^m + 2)^2 \epsilon(\frac{1}{2^m + 2})^{k+1}) \] on \( \Delta_{n,m} \), \( m \) odd, it follows that \( \rho' \) is subharmonic on all \( \Delta_{n,m} \) if \( \epsilon \) is small enough (recall that \( k \geq 1 \)).

In the rest of the construction we fix an \( \epsilon > 0 \) small enough.

We now choose small discs \( \mathcal{K}_{n,m} \) centered at \( p_n, m \) and \( \Delta_{n,m} \).
centered at \( p'_{n,m} \) such that

(i) \( \tilde{\mathcal{X}}_{n,m} \) and \( \tilde{\mathcal{X}}'_{n,m} \) have the same radius,

(ii) the \( \tilde{\mathcal{X}}_{n,m} \)'s (\( \tilde{\mathcal{X}}'_{n,m} \)'s) have pairwise disjoint closures which do not intersect the real axis,

(iii) if \( m \) is even, then \( \rho' = z' \bar{z}' \) on \( \tilde{\mathcal{X}}'_{n,m} \) and

(iv) if \( m \) is odd, then \( \rho' = |z' - \varepsilon \left( \frac{1}{2^{m+2}} \right)^{k+1}|^2 \) on \( \tilde{\mathcal{X}}'_n,m \).

Let \( \Omega^k_1 \subset \mathcal{C}^2(z, \omega) \), \( \Omega^k_2 \subset \mathcal{C}^2(z', \omega') \) be open sets,

\[
\Omega^k_1 = \{ |\omega| < \frac{1}{2} \} \cup \{ z \in \tilde{\mathcal{X}}_{n,m} \text{ and } |\omega| < 2 \},
\]

\[
\Omega^k_2 = \{ |\omega'| < \frac{1}{2} \} \cup \{ z' \in \tilde{\mathcal{X}}'_{n,m} \text{ and } |\omega'| < 2 \}.
\]

We define a complex manifold \( M_k \) by patching \( \Omega^k_1 \) and \( \Omega^k_2 \) where \( \frac{1}{2} < |\omega| < 2 \) and \( \frac{1}{2} < |\omega'| < 2 \): If \( z \in \tilde{\mathcal{X}}_{n,2m}, \frac{1}{2} < |\omega| < 2 \) and \( z' \in \tilde{\mathcal{X}}'_{n,2m}, \frac{1}{2} < |\omega'| < 2 \) use the coordinate transformation

\[
z' = z, \quad \omega' = \frac{1}{\omega}. \]

If \( z \in \tilde{\mathcal{X}}_{n,2m+1}, \frac{1}{2} < |\omega| < 2 \) and \( z' \in \tilde{\mathcal{X}}'_{n,2m+1}, \frac{1}{2} < |\omega'| < 2 \) let \( z' = z + \varepsilon \left( \frac{1}{2^{2m+2}} \right)^{k+1}, \omega' = \frac{1}{\omega} \). Then \( \rho_k \), given by \( \rho_k(z, \omega) = \rho(z) \) on \( \Omega^k_1 \) and \( \rho_k(z', \omega') = \rho'(z') \) on \( \Omega^k_2 \) is a \( \mathcal{C}^k \) plurisubharmonic function. Let \( \mathcal{P}_k = 0 \in \Omega^k_1 \) and \( \mathcal{Q}_k = (1,0) \in \Omega^k_2 \). Then \( \rho_k(\mathcal{P}_k) = 0 \) and \( \rho_k(\mathcal{Q}_k) = 1 \).

Assume that there exists a \( \mathcal{C}^{k+1} \) plurisubharmonic function \( \sigma \) on \( M_k \) such that \( \sigma(\mathcal{P}_k) \neq \sigma(\mathcal{Q}_k) \). Then there exists an \( n \) such that \( \frac{\partial \sigma}{\partial x}(p_{n,0}) \neq 0 \). We compare the Taylor expansions of order \( k+1 \) of \( \sigma \) about \( p_n \) and \( p'_n \) in the \( x \) and \( x' \) direction respectively:
\[ \sigma(x + \frac{i}{2^n}, 0) = \sigma(p_n, 0) + \sum_{j=1}^{k+1} A_j (x - r_n)^j + o(|x - r_n|^{k+1}), \]

\[ \sigma(x' + \frac{i}{2^n}, 0) = \sigma(p_n', 0) + \sum_{j=1}^{k+1} A_j' (x' - r_n)^j + o(|x' - r_n|^{k+1}) \]

where \( A_1 \neq 0 \).

Now \( \sigma(p_n + \frac{1}{2^{2m+1}}, 0) = \sigma(p_n + \frac{1}{2^{2m+1}}, 0), \) \( m \geq n/2, \)

it follows that \( \sigma(p_n, 0) = \sigma(p_n', 0) \) and \( A_j = A_j', \) \( j = 1, \ldots, k+1. \)

We also have that \( \sigma(p_n + \frac{1}{2^{2m+1}}, 0) = \sigma(p_n' + \frac{1}{2^{2m+1}} + \epsilon(\frac{1}{2^{2m+3}})^{k+1}, 0) \)

\( 2m + 1 \geq n. \) Comparing the Taylor expansions we obtain that

\[ A_1 \epsilon(\frac{1}{2^{2m+3}})^{k+1} = o((\frac{1}{2^{2m+1}})^{k+1}), \]

which is a contradiction.

**III:** We use \( r_n, p_n, p_n, m, p_n', p_n', 2m, p, \chi \) and \( \Delta_n, m \) as in II.

However choose \( p_n', 2m+1 = p_n' + \frac{1}{2^{2m+1}} + \epsilon(\frac{1}{2^{2m+3}})^{2m+3} \) whenever \( 2m + 1 \geq n. \) We define \( \rho'(z') = z' \bar{z}' \) on \( \mathbb{C} - \Delta_n, 2m+1. \) When \( m \) is odd, define \( \rho' \) on \( \Delta_{n,m} \) by

\[ \rho'(z') = \left[ 1 - \chi(\frac{z' - p_n' - 1/2^m}{1/2^m + 2}) \right] z' \bar{z}' + \chi(\frac{z' - p_n' - 1/2^m}{1/2^m + 2}) |z' - \epsilon(\frac{1}{2^{m+2}})^{2m+3}|^2. \]

Then, if \( \epsilon > 0 \) is small enough, there exist neighborhoods of each \( p_n, 2m+1 \) on which \( \rho'(z') = |z' - \epsilon(\frac{1}{2^{m+2}})^{2m+3}|^2 \) and \( \rho' \) is a \( C^\infty \) subharmonic function on \( \mathbb{C}, \rho'|_R = z' \bar{z}'. \)

It is possible to choose discs \( \tilde{\Delta}_n, m \) and \( \tilde{\Delta}_n', m \) as in II except that (iv) is replaced by (iv)' if \( m \) is odd, then \( \rho' = |z' - \epsilon(\frac{1}{2^{m+2}})^{2m+3}|^2 \) on \( \tilde{\Delta}_n, m. \)
The open sets $\Omega_1^\infty$, $\Omega_2^\infty$ and the manifold $M_\infty$ is defined as in II except that if $z \in \mathbb{C}_{n,2m+1}$, $\frac{1}{2} < |w| < 2$ and $z' \in \mathbb{C}'_{n,2m+1}$, $\frac{1}{2} < |w'| < 2$, then $z' = z + \varepsilon \left(\frac{1}{2n+3}\right)^{2m+3}$, $w' = \frac{1}{w}$. Furthermore, the $\mathcal{C}^\infty$ plurisubharmonic function $\rho_\infty$ on $M_\infty$ and $P_\infty, Q_\infty$ are defined as in II. Again, we have that $\rho_\infty(P_\infty) = 0$ and $\rho_\infty(Q_\infty) = 1$.

If there exists a real analytic plurisubharmonic function $\sigma$ on $M_\infty$ such that $\sigma(P_\infty) \neq \sigma(Q_\infty)$, then there exist power series expansions in the $x(x')$ direction about some $p_n(p'_n)$,

$$\sigma(x + \frac{i}{2n}, 0) = \sum_{j=0}^{\infty} A_j(x - r_n)^j,$$

$$\sigma(x' + \frac{i}{2n}, 0) = \sum_{j=0}^{\infty} A'_j(x' - r_n)^j$$

with $A_1 \neq 0$.

Since $\sigma(p_n, 2m, 0) = \sigma(p'_n, 2m, 0)$ it follows that $A'_j = A_j$ for all $j$, and hence that $\sigma(x + \frac{i}{2n}, 0) = \sigma(x' + \frac{i}{2n}, 0)$ whenever $x = x'$. The fact that $A_1 \neq 0$ implies also that $x = x'$ whenever $\sigma(x + \frac{i}{2n}, 0) = \sigma(x' + \frac{i}{2n}, 0)$ and $x, x'$ are close enough to $r_n$.

This contradicts the fact that $\sigma(p_n, 2m+1, 0) = \sigma(p'_n, 2m+1, 0)$ for all $2m+1 \geq n$.

**Remark.** All the complex manifolds $M_k$ contain many compact complex subvarieties ($\mathbb{P}_1$'s). Because of removable singularity theorems for plurisubharmonic functions, ([2]) they can all be punctured by removing a suitable family of two dimensional totally real submanifolds. These new $M_k$'s will still have the same properties as above but will contain no positive dimensional compact complex subvarieties.
References

