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AN EXPLICIT HOLOMORPHIC MAP OF A BOUNDED DOMAIN IN $\mathbb{C}^n$ WITH C^2-BOUNDARY ONTO THE POLYDISC.

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§1. Introduction.

We consider in this paper the problem of mapping a domain $\Omega_1 \subset \mathbb{C}^n$ onto a domain $\Omega_2 \subset \mathbb{C}^n$ by a holomorphic mapping. In [1] and [2] Fornæss and Stout proved the following result.

Theorem. Let D_n and B_n denote the unit polydisc and ball in $\mathbb{C}^n$, and let Ω be as connected, paracompact n-dimensional complex manifold. Then there exist regular holomorphic mappings from D_n and B_n onto Ω, both with finite fibers.

Hence the problem reduces to mapping a given domain Ω_1 onto the polydisc or the ball. In § 2 we give an example of a domain $\Omega \subset B_n$ $(n \geq 2)$ which cannot be mapped onto the ball. This example is a domain with a Hartogs phenomenon. I do not know any example of a bounded domain of holomorphy which cannot be mapped onto the ball. The strictly pseudoconvex case, however, is covered in § 3, where we prove that any bounded domain in $\mathbb{C}^n$ with C^2-boundary can be mapped onto the polydisc. This is an easy consequence of the main theorem, which gives an explicit mapping of the ball B_n onto the polydisc which is surjective on any ball $B \subset B_n$ tangent to B_n at a given boundary point.

§2. A counterexample.

According to the classical Schwarz lemma a holomorphic function $f: D \to D$ satisfies

$$|f'(z)| \leq \frac{1-|f(z)|^2}{1-|z|^2} \leq [d(z,\partial D)]^{-1},$$

where $d(z,\partial D)$ denotes the boundary distance.

If $f: B_n \to B_n$ is holomorphic, $z = (z_1,\ldots,z_n) \in B_n$ and $D_i = \{w \in \mathbb{C};\ (z_1,\ldots,z_{i-1},w,z_{i+1},\ldots,z_n) \in B_n\}$, the Schwarz lemma applied to the components of f on D_i gives

$$\left|\frac{\partial f_j}{\partial z_i}\right| \leq [d(z_i,\partial D_i)]^{-1} \leq [d(z,\partial B_n)]^{-1}$$

Hence the complex Jacobian Jf of f, a polynomial of degree n in $\frac{\partial f_j}{\partial z_i}$, satisfies $|Jf| \leq C[d(z,\partial B_n)]^{-n}$, and the real Jacobian $J_R f = |Jf|^2 \leq C[d(z,\partial B_n)]^{-2n}$.

(The sharper estimate $J_R f \leq C[d(z,\partial B_n)]^{-(n+1)}$ follows from theorems 2.2.2 and 8.1.2 in [3]).

The counterexample is now found by removing a sequence of disjoint spherical shells $K_i = \{z \in \mathbb{C}^n;\ r_i \leq |z| \leq s_i\}$, with small holes punched in them, from the ball. The holes will ensure that holomorphic mappings extend over the removed sets. If we remove such sets from the ball with a high enough density near the boundary, the above estimate on $J_R f$ will give that there is not enough volume left to map the set onto the ball.

§3. The main theorem.

Given a bounded domain $\Omega \subset \mathbb{C}^n$ with C^2-boundary, we can find balls B and B_R with a common boundary point such that $B_R \subset \Omega \subset B$. Hence the following theorem implies that there is a holomorphic mapping from Ω onto the polydisc.

Theorem. Let B be a ball in $\mathbb{C}^n$ and $p \in \partial B$. Then there is a holomorphic mapping $G: B \to D_n$ which is surjective on any ball $B_R \subset B$ such that $p \in \partial B_R$.

Proof: We may assume that $B = \{z;\ |z| < 1\}$, $p = e_1 = (1,0,\ldots,0)$ and $B_R = \{z = (z_1,\ldots,z_n);\ |z_1 - (1-R)|^2 + \sum_{i=2}^{n} |z_i|^2 < R^2\}$, a ball of radius R and center $(1-R,0,0,\ldots,0)$.

We consider first the case

$n = 1$

We use the notations

$$H = \{z;\ \mathrm{Re}\, z > 0\}$$

$$-H = \{z;\ \mathrm{Re}\, z < 0\}$$

$$S = \{z;\ |\mathrm{Im}\, z| < \tfrac{\pi}{2}\}\,.$$

The mapping $g_1(z) = \frac{1+z}{1-z}$ is a biholomorphic mapping from D to H. It maps the disc B_R onto the set $H_R = \{z;\ \mathrm{Re}\, z > \frac{1-R}{R}\}$ and circles through -1 and 1 making an angle α at -1 with the real axis onto rays from 0, also making an angle α with the real axis. Consider now the mapping $g_2 = h_3 \circ h_2 \circ h_1$, where $h_1(z) = \log z$, $h_2(z) = -iz - \pi/2$ and $h_3(z) = \exp z$.

$$H \xrightarrow{h_1} S \xrightarrow{h_2} (-H) \xrightarrow{h_3} D$$

g_2 is a function mapping H into the annulus $A_1 = \{z\colon \exp(-\pi) < |z| < 1\}$ h_2 actually maps S into $\{z\colon -\pi < \mathrm{Re}\, z < 0\}$.

The set H_R contains a tail of any ray from the origin in H, so the image of H_R in S will contain a tail of any line $\mathrm{Im}\, z = \alpha$. Hence the image in $(-H)$ will contain a tail (in the downward direction) of any line $\mathrm{Re}\, z = \beta$, $-\pi < \beta < 0$, which means that its image in D will cover any circle $|z| = \exp \beta$. This proves that the function $g_2 \circ g_1$, restricted to any B_R is surjective on the annulus A_1. $A_1 \subset D$ can be mapped onto the disc by moving the hole $D \setminus A_1$ away from origin by a linear automorphism of D and then taking the square.

We now consider the case

<u>n = 2</u>

Let $z = (z_1, z_2)$. We get

$$\operatorname{Re} g_1(z) - \left|\frac{z_2}{1 - z_1}\right|^2 = \frac{1 - |z|^2}{|1 - z_1|^2} > 0 \quad \text{whenever} \quad z \in B_2 .$$

The function $(g_1(z_1), \frac{z_2}{1 - z_1}$ is in fact a biholomorphic mapping from B_2 to

$$\Omega = \{w \in \mathbb{C}^2 ; \operatorname{Re} w_1 > |w_2|^2\}$$

We have $\operatorname{Re}(w_1 - w_2^2) > 0$, so we get a mapping G_1 from B_2 to H^2 defined by

$$G_1(z) = (g_1(z_1), \; g_1(z_1) - (\frac{z_2}{1-z_1})^2)$$

Define g_2' from H to D as g_2, except that we multiply h_2 by $\sqrt{2}$. The image of g_2' will then be $A_2 = \{z \in C; \exp(-\sqrt{2}\pi) < |z| < 1\}$.

Define G_2 from H^2 to D_2 by

$$G_2(z) = (g_2(z_1), g_2'(z_2))$$

and G_3 from D_2 to H^2 by

$$G_3(z) = (g_1(z_1), g_1(z_2)) .$$

We then get a mapping $G = G_2 \circ G_3 \circ G_2 \circ G_1$ from B_2 to D_2. Its image will be $A_1 \times A_2$. We claim that G restricted to any B_R is surjective on $A_1 \times A_2$. By composing with surjective mappings from these annuli onto the disc, we obtain the theorem.

To prove the claim, we first investigate the image V_1 of B_R under the mapping G_1. For fixed z_1 the points $(z_1, z_2) \in B_R$ satisfy

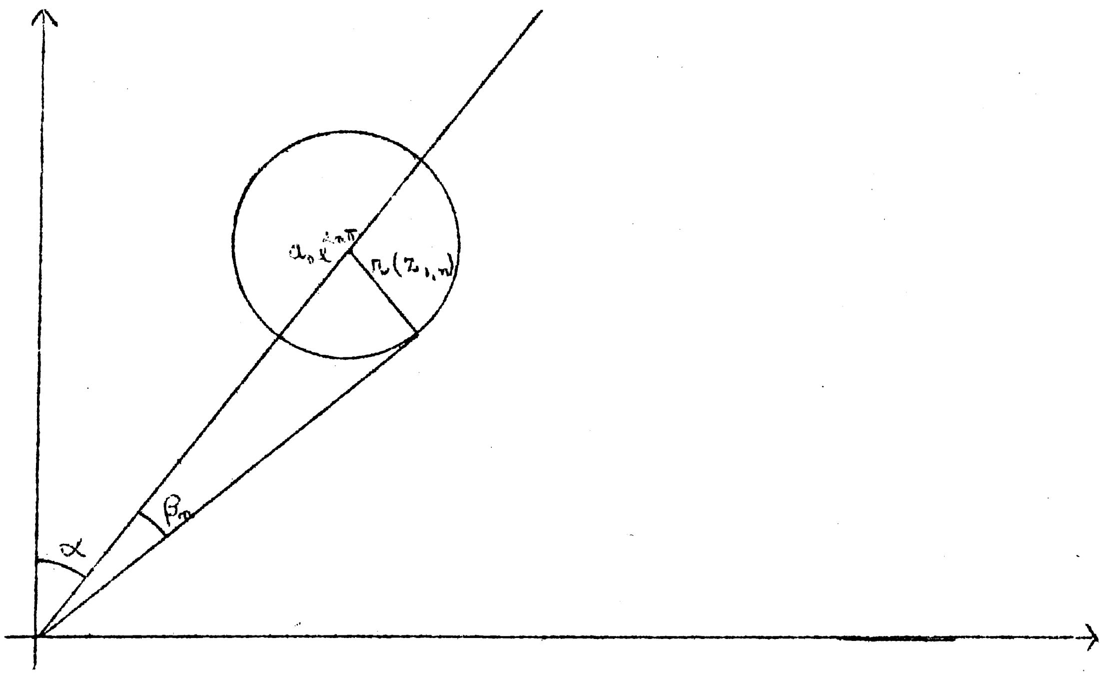
$$|z_2|^2 < R^2 - |z_1 - (1-R)|^2 = 2R\ \mathrm{Re}(1-z_1) - |z_1 - 1|^2$$

Hence

$$\left|\frac{z_2}{1 - z_1}\right|^2 < \frac{2R\ \mathrm{Re}(1-z_1)}{|1-z_1|^2} - 1 = r(z_1).$$

So V_1 consists of a disc with centre $g_1(z_1)$ and radius $r(z_1)$ in the second factor. In case $R = 1$, this disc extends up to the boundary of H.

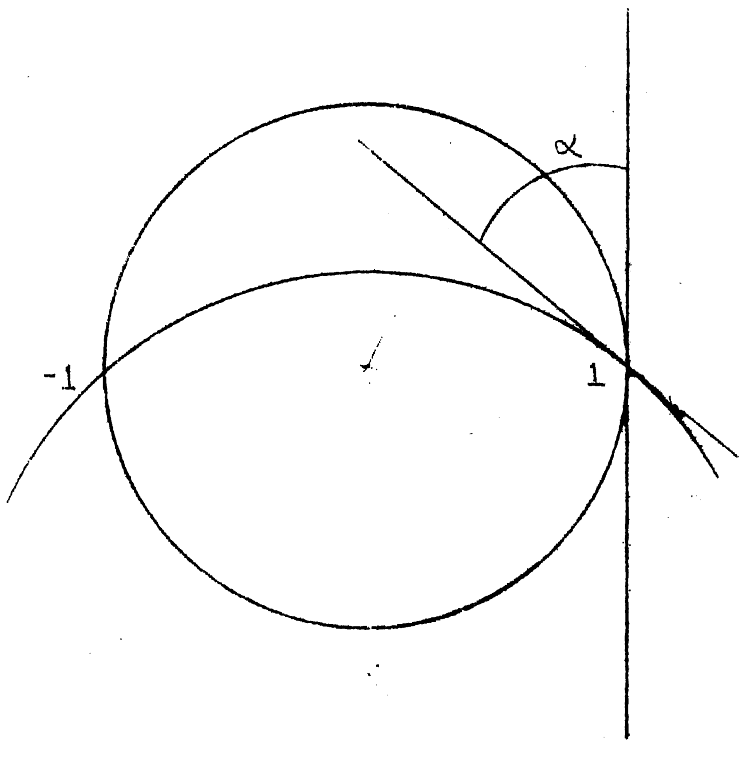
Let V_2 be the image of B_R in $A_1 \times A_2$ under the map $G_2 \circ G_1$, and let $w_1 \in A_1$. For convenience we consider only points in the outer part of A_1, $|w_1| = \exp(-\alpha)$ where $0 < \alpha \leq \frac{\pi}{2}$. The situation is symmetric in the inner part. w_1 is the image of all points $-\alpha + i(\arg w_1 + 2n\pi)$ in $-H$. These points come from the points $i(\frac{\pi}{2} - \alpha) - (\arg w_1 + 2n\pi)$ in S, and in H they come from points on the ray from the origin making an angle α with the positive imaginary axis, having modulus $a_o e^{2n\pi}$ with $a_o = \exp(-\arg w_1)$. Hence the image of V_1 will contain the image of all the discs at these points in the second factor. Suppose the points in H come from points $z_{1,n}$ in D. Hence the discs have radii $\dfrac{2R\ \mathrm{Re}(1-z_n)}{|1 - z_{1,n}|^2} - 1$ and we draw the following picture



The angle β_n defined by this drawing satisfies

$$\sin \beta_n = \frac{r(z_{1,n})}{a_o e^{2n\pi}} = \frac{2R\ \mathrm{Re}(1-z_{1,n})}{|1+z_{1,n}|\cdot|1-z_{1,n}|} - \left|\frac{1-z_{1,n}}{1+z_{1,n}}\right|$$

The points $z_{1,n}$ lie on a circle as indicated by the figure below and converge to 1 when n increases.



This implies that $\lim \sin \beta_n = R \sin \alpha$. Hence β_n converge to an angle β and $\sin \beta = R \sin \alpha$, so we have at least $\beta > \frac{1}{2} R \alpha$. This means that any ray from the origin in H between the angles $\alpha + \frac{1}{2} R \alpha$ and $\alpha - \frac{1}{2} R \alpha$ will eventually cut the discs in infinitely many intervals $I_0 \cdot e^{2n\pi}$, $n \geq 0$. The disc where we will find I_0 may depend on the angle. Mapping this picture into $-H$ will give us infinitely many vertical intervals $J_0 - \sqrt{2} \cdot 2n\pi i$ for any real value between $-\sqrt{2}\alpha(1 + \frac{1}{2}R)$ and $-\sqrt{2}\alpha(1 - \frac{1}{2}R)$. Since $\sqrt{2}$ is irrational the centres of these intervals will be mapped onto a dense subset of a circle by the exponential, and the intervals themselves will cover the entire circle. This means that we cover an annular region

$$A(r,\alpha) = \{w_2;\ \exp(-\sqrt{2}\alpha\,(1+\tfrac{1}{2}R)) < |w_2| < \exp(-\sqrt{2}\alpha\,(1-\tfrac{1}{2}R))\}$$

We have proved:

(3.1) For any B_R the image V_2 of B_R in D_2 contains the set

$$\{(w_1,w_2);\ |w_1| = \exp(-\alpha),\ 0 < \alpha \leq \tfrac{\pi}{2},\ w_2 \in A(R,\alpha)\}$$

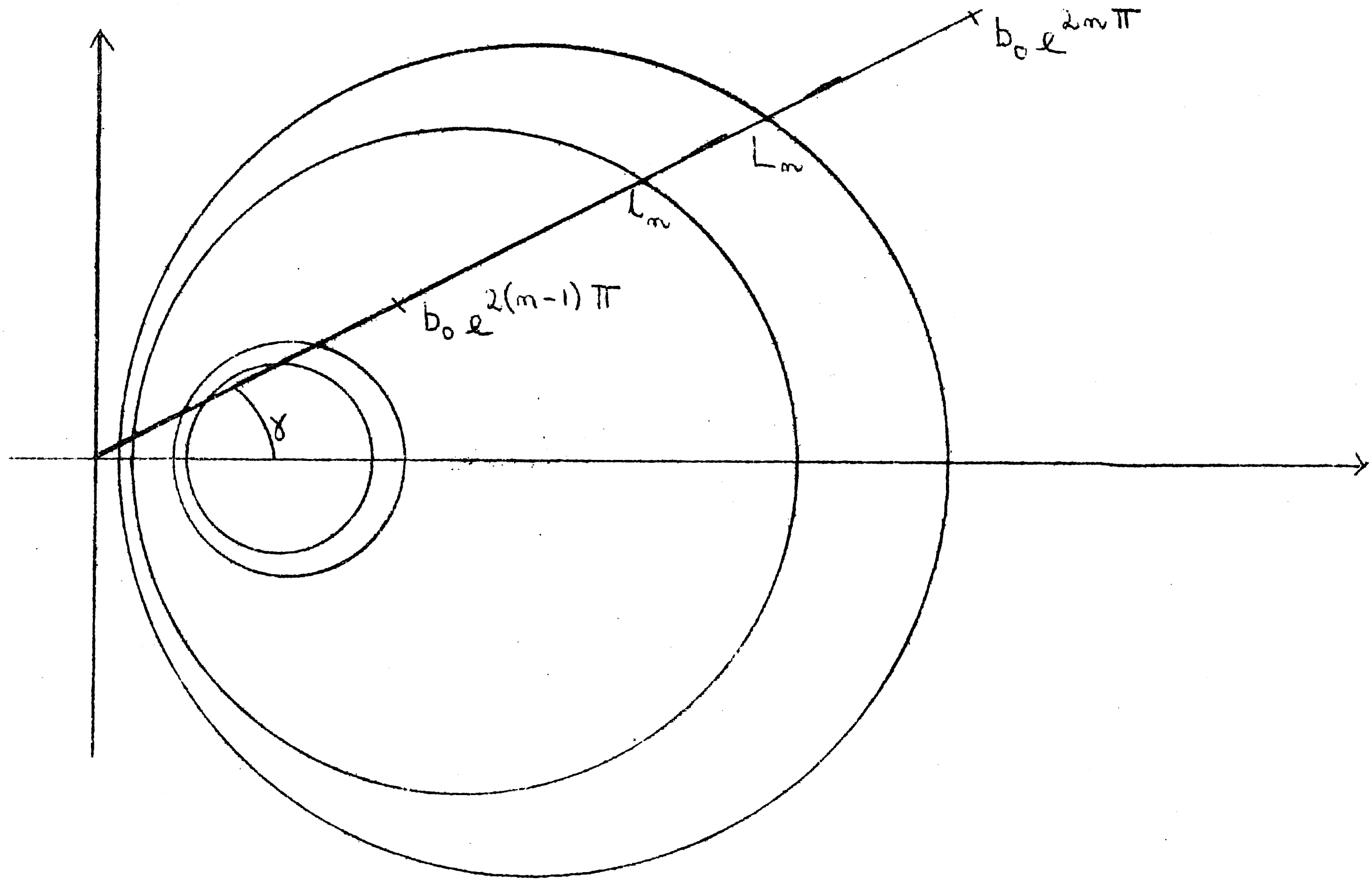
We now consider the final image of B_R, i.e. the image of V_2 under the mapping $G_2 \circ G_3 : D_2 \to D_2$. We want to prove it is the entire $A_1 \times A_2$. Let therefore $p_1 \in A_1$. The inverse image of p_1 under the mapping G_3 consists of points on a ray in H. We now use the angle γ between this ray and the positive real axis, so $-\frac{\pi}{2} < \gamma < \frac{\pi}{2}$. These points have modulus $b_0 e^{2n\pi}$ for some b_0, and they come from points on the circle through -1 and 1 in D, making an angle γ with the positively oriented real axis at -1 (and hence the angle $-\gamma$ at 1). Denote these points by $w_{1,n}$.

When n increases $w_{1,n}$ will approach 1 on this circle. Hence for n large enough V_2 will contain the set

$$\{w_{1,n}\} \times A(R,\alpha_n)$$

where $|w_{1,n}| = \exp(-\alpha_n)$.

The annular region $A(R,\alpha_n)$ will be mapped onto a region between two circles in H



When n increases these circles will cut the real axis at points converging to 0 on the left and to infinity on the right. A ray in the angle δ will cut the inner and outer circles in lengths l_n and L_n, respectively. (We consider the intersection points of greatest length). We want to show that $l_n/e^{2n\pi}$ and $L_n/e^{2n\pi}$ both approach limiting values when n increases, and we therefore make approximations which are asymptotically equal to the quantities considered.

We have

$$(3.2)\qquad \begin{aligned} b_o e^{2n\pi} &= \left|\frac{1+w_{1,n}}{1-w_{1,n}}\right| \\ \text{so}\qquad e^{-2n\pi} &= \left|\frac{1-w_{1,n}}{1+w_{1,n}}\right| \cong \tfrac{1}{2} b_o |1-w_{1,n}| \end{aligned}$$

A circle $|w_2| = 1 - C$ will be mapped onto a circle intersecting the real axis at $2\text{-}C/C$ and $C/2\text{-}C$, hence its diameter will be

$$d(C) = \frac{2}{C} \cdot \frac{2-2C}{2-C} \cong \frac{2}{C} \quad \text{when } C \text{ is small.}$$

Since the left intersection with the real axis approaches 0 when C is small, the length of the intersection with the ray of angle δ will be

$$l(C,\delta) \cong \frac{2}{C} \cdot \cos\delta\,.$$

We have $|w_{1,n}| = \exp(-\alpha_n) \cong 1 - \alpha_n$, so $\alpha_n \cong 1 - |w_{1,n}|$.

The inner circle of $A(R,\alpha_n)$ has radius

$$\exp(-\sqrt{2}(1+\tfrac{1}{2}R)\alpha_n) \cong 1 - \sqrt{2}(1+\tfrac{1}{2}R)\alpha_n$$

Hence

$$(3.3)\qquad l_n \cong \frac{2\cos\delta}{\sqrt{2}(1+\frac{1}{2}R)\alpha_n} \cong \frac{2\cos\delta}{\sqrt{2}(1+\frac{1}{2}R)(1-|w_{1,n}|)}$$

(3.2) and (3.3) give

$$l_n e^{-2n\pi} \cong \frac{b_o \cos\delta}{\sqrt{2}(1+\frac{1}{2}R)} \cdot \frac{|1-w_{1,n}|}{1-|w_{1,n}|}$$

Since $w_{1,n}$ lie on the circle described above, we get

$$\lim l_n e^{-2n\pi} = \frac{b_o\cos\delta}{\sqrt{2}(1+\frac{1}{2}R)\cos\gamma}.$$

Similarly, we get

$$\lim L_n e^{-2n\pi} = \frac{b_o \cos\delta}{\sqrt{2}(1-\frac{1}{2}R)\cos\gamma} .$$

This means that the interval (l_n, L_n) approaches $I_o \cdot e^{2n\pi}$ where

$$I_o = \frac{b_o \cos\delta}{\sqrt{2}(1-\frac{1}{4}R^2)\cos\gamma}\,(1-\tfrac{1}{2}R,\ 1+\tfrac{1}{2}R) .$$

As before, we map this picture into $-H$, and get infinitely many vertical intervals $J_o - \sqrt{2}\, 2n\pi i$. This time we get such intervals for any real value **t between** $-\sqrt{2}\pi$ and 0. How far down we have to look for the first interval J_o will of course depend on t. By the same argument as before, these intervals will cover entire circles and this time for all radii in A_2. This proves that $G: B_R \to A_1 \times A_2$ is surjective for any R.

<u>The general case.</u>

The proof is a straightforward generalization of the case $n = 2$. The function

$$\left(\frac{1+z_1}{1-z_1}, \frac{z_2}{1-z_1}, \dots, \frac{z_n}{1-z_1}\right)$$

is a biholomorphic map from B_n to the set

$$\Omega = \{w \in \mathbb{C}^n; \operatorname{Re} w_1 > \sum_{i=2}^{n} |w_i|^2\}.$$

We therefore get a map G_1 from B_n to H^n defined by

$$G_1(z) = \left(\frac{1+z_1}{1-z_1}, \frac{1+z_1}{1-z_1} - \left(\frac{z_2}{1-z_1}\right)^2, \dots, \frac{1+z_1}{1-z_1} - \left(\frac{z_n}{1-z_1}\right)^2\right)$$

In this case, for a fixed $w_1 = \dfrac{1+z_1}{1-z_1}$, the image of a ball B_R

consists of the points $(w_2,\dots,w_n) \in H^{n-1}$ such that

$$\sum_{i=2}^{n} |w_i - w_1| < r(z_1).$$

This is a Hartogs polyhedron with centre at $(w_1,\dots,w_1) \in \mathbb{C}^{n-1}$ and contains a polydisc, the product of discs

$$D(w_1, \frac{1}{n-1} r(z_1)) = \{w \in \mathbb{C};\ |w-w_1| < \frac{1}{n-1} r(z_1)\}.$$

We now define G_2 by

$$G_2(z) = (g_2(z_1), g_{2,2}(z_2), \dots, g_{2,n}(z_n))$$

where $g_{2,i}$ $(i \geq 2)$ is defined as g_2, except that we multiply h_2 by a positive real number a_i, such that $a_2,\dots,a_n$ are linearly independent over $\mathbb{Q}$. This will mean that the angles

$$\{(a_2 \cdot 2k\pi, \dots, a_n \cdot 2k\pi) \in (S^1)^n;\ k \geq k_o\}$$

form a dense subset for any k_o. (3.1) immediately generalizes to

(3.4) For any B_R the image of B_R in D_2 contains the set

$$\{(w_1,w_2,\dots,w_n);\ |w_1| = \exp(-\alpha),\ 0 < \alpha \leq \tfrac{\pi}{2},\ w_i \in A_i(R,\alpha) \text{ for } i \geq 2\}$$

where

$$A_i(R,\alpha) = \{w_i; \exp(-a_i\alpha(1 + \frac{1}{2(n-1)}R)) < |w_i| < \exp(-a_i\alpha(1 - \frac{1}{2(n-1)}R)).$$

The function G_3 is defined as before in each factor. The image of the product of the annuli $A_i(R,\alpha)$ will be a product of regions between circles. Hence over a point $p_1 \in A_1$, coming from points on a ray of angle γ in H, we get an infinite union of such products. Each factor will cut a ray of angle δ_i asymptotically in intervals $I_o^i e^{2n\pi}$ with

$$I_o^i = \frac{b_o \cos \delta_i}{a_i \cos \gamma}\left(\frac{1}{1+\frac{1}{2(n-1)}R}, \frac{1}{1-\frac{1}{2(n-1)}R}\right)$$

Hence the function $G = G_2 \circ G_3 \circ G_2 \circ G_1$ will be surjective from B_R to $A_1 \times A_2 \times \ldots \times A_n$, where

$$A_i = \{z : \exp(-a_i \pi) < |z| < 1\}.$$

Mapping each annulus surjectively onto the disc concludes the proof of the theorem. $\square$

The function given in the theorem certainly has infinite fibers, and it is not regular, since the function mapping the annulus onto the disc described in the proof is not regular. I do not know any elementary regular function mapping the annulus onto the disc. Fornæss has shown me, however, the following existence proof of such a mapping: Let $D_n (n \geq 0)$ be the infinite sequence of unit discs at the points $3n$ and let

$$E = \left(\bigcup_{n=o}^{\infty} D_n\right) \cup (0, \infty)$$

i.e. we connect these discs by straight lines. Let U_n, $n \geq 0$, be discs contained in the unit disc D such that D is covered by the discs V_n $(n \geq 1)$, V_n having the same center as U_n and half its radius. If we connect these discs U_n by smooth curves, we can find a map φ mapping E onto this picture, by just putting D_n on the disc U_n and the connecting intervals on the connecting curves. Lemma II.2 of [1] now implies the existence of a regular holomorphic map f from a neighbourhood V of E

to the unit disc such that f is surjective on $\bigcup_{n=1}^{\infty} D_n$. Let $E_n \subset V$ be a neighbourhood basis for E, each E_n simply connected and define conformal equivalences $\varphi_n : D \to E_n$ such that $\varphi_n(0) = 0$. A subsequence of φ_n converges uniformly on compact sets to a limit φ. Since φ is open and E_n a neighbourhood basis, we must have $\varphi(D) \subset D_o$. Hence for any annular region A in the disc, the compact set $D \setminus A$ will be mapped into D_o by φ_n for n large enough. Hence φ_n maps A onto $\bigcup_{n=1}^{\infty} D_n$ and $f \circ \varphi_n$ is regular and surjective on A.

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