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A PEAK SET OF HAUSDORFF DIMENSION 2n-1 FOR THE ALGEBRA $A(\mathcal{D})$ IN THE BOUNDARY OF A DOMAIN $\mathfrak{D}$ WITH C^{∞}-BOUNDARY IN C^n.

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A peak set of Hausdorff dimension 2n-1 for the algebra $A(\mathcal{D})$ in the boundary of a domain $\mathcal{D}$ with C^{∞}-boundary in $\mathbb{C}^n$.

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In this paper $\mathcal{D}$ is a domain in $\mathbb{C}^n$ and $A(\mathcal{D})$ the algebra of functions which are holomorphic in $\mathcal{D}$ and continuous on $\bar{\mathcal{D}}$. A compact set $F \subset \partial\mathcal{D}$ is a peak set for $A(\mathcal{D})$ if there exists a function $g \in A(\mathcal{D})$ with the properties: $g|_F = 1$ and $|g(p)| < 1$ whenever $p \in \bar{\mathcal{D}} \setminus F$.

We are interested in finding a holomorphic function f in $\mathcal{D}$ continuous in $\bar{\mathcal{D}} \setminus F$ such that $\operatorname{Re} f(p) \to \infty$ when $p \to p_0 \in F$. Adding a large constant to f such that $\operatorname{Re} f > 0$ in $\bar{\mathcal{D}}$ and letting $\varphi = \frac{1}{f}$ we get a peak-function for F by defining $g = \frac{1-\varphi}{1+\varphi}$.

The following theorem is the main result of this paper:

<u>THEOREM:</u>

When $\mathcal{D}$ is a domain with C^{∞}-boundary in $\mathbb{C}^n$ there exists a peak set for $A(\mathcal{D})$ of Hausdorff dimension 2n-1.

<u>DEFINITION:</u>

$\mathcal{H}^k$ is the Hausdorff measure with respect to the induced Euclidean metric on $\partial\mathcal{D}$. A set $B \subset \partial\mathcal{D}$ has Hausdorff dimension k if $\mathcal{H}^{k-\epsilon}(B) = \infty$ and $\mathcal{H}^{k+\delta}(B) = 0$ whenever $\epsilon, \delta > 0$.

We shall show this theorem when $\mathcal{D}$ is strictly pseudoconvex. Generally lemma 2 in [5] gives the existence of a point $p \in \partial\mathcal{D}$ and a strictly convex set $C \supset \mathcal{D}$ such that $\partial C \cap \partial\mathcal{D}$ contains a neighbourhood of p in $\partial\mathcal{D}$. Since the construction of F is local the result for strictly pseudoconvex domains gives the general result.

First we find peak-sets $F^m \subset \partial\mathcal{D}$ where $\dim F^m \geq 2n-1-\frac{n}{m}$ for each integer $m \geq 4$. Then we let $F = \bigcup_{m=4}^{\infty} F^m$ and then compactify F.

When $\mathcal{D}$ is strictly pseudoconvex Darboux' theorem [1] gives the existence of real local coordinates $\varphi = (x^1,\ldots,x^{n-1},y^1,\ldots,y^{n-1},z)$ on $\partial\mathcal{D}$ such that the vector-fields $\{\frac{\partial}{\partial x^i}\}_{i=1}^{n-1}$ and $\{\frac{\partial}{\partial y^i} + x^i \frac{\partial}{\partial z}\}_{i=1}^{n-1}$ $= \{\eta_i\}_{i=1}^{n-1}$ generate $T_{\mathbb{C}}\partial\mathcal{D}$. Let J be the complex structure tensor.

Furthermore each submanifold $N \subset \partial\mathcal{D}$ where $TN_p \subset T_{\mathbb{C}}\partial\mathcal{D}_p$ when $p \in N$ has the property: $TN_p \cap J\,TN_p = \{0\}$ for each $p \in N$. This implies that η_i is a linear combination of $\{J\frac{\partial}{\partial x^i}\}_{i=1}^{n-1}$ and $\{\frac{\partial}{\partial x^i}\}_{i=1}^{n-1}$. If (φ,U) is such a chart on $\partial\mathcal{D}$ it is possible to find a neighbourhood ω of U in $\mathbb{C}^n$ and a $v \in C^\infty(\omega,\mathbb{R})$ where:

<u>(0.1)</u>:

1) $\omega \cap \partial\mathcal{D} = U$
2) $\omega \cap \mathcal{D} = \{p : v(p) < 0\}$
3) (x,y,z,v) are coordinates on ω
4) $\frac{\partial}{\partial v}\big|_{\partial\mathcal{D}} = J\frac{\partial}{\partial z}\big|_{\partial\mathcal{D}}$

This is possible because $\frac{\partial}{\partial z} \notin T_{\mathbb{C}}\partial\mathcal{D}$ and therefore $J\frac{\partial}{\partial z} \notin T\partial\mathcal{D}$, so if we choose z carefully $J\frac{\partial}{\partial z}$ is an outward pointing vector.

Let $M = \{p \in \omega : v(p) = 0$ and $y^i(p) = a^i,\ i = 1,\ldots,n-1\}$

where the a^i's are constants and

Q a compact subset of $N = M \cap \{p : z(p) = b\}$.

If m is an integer, $m \geq 4$, $\epsilon > 0$, we can find a function $\psi \in C^{\infty}(\mathbb{R}^{n-1})$, $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ with the properties:

(0.2): ψ vanishes to infinite order on $x(Q)$

(0.3): $|\frac{\partial}{\partial \xi^i} \psi(\xi)| \leq \epsilon^{1/2}$ for each $\xi \in \mathbb{R}^{n-1}$

(0.4): $\psi(\xi) \geq \epsilon^{1-1/m}$ whenever $d(\xi, x(Q)) \geq \epsilon^{1/10}$

Define $\tilde{u} \in C^{\infty}(\omega)$ by

$$\tilde{u}(p) = \psi(x(p)) + i(z(p) - b) + \tfrac{1}{2}(z(p) - b)^2 + \tfrac{1}{2}\sum_{i=1}^{n-1}(y^i(p) - a^i)^2$$

Then $\tilde{u}|Q = 0.$

Using the same method as Wermer in 17.4 [7] we find a neighbourhood ω_0 of M, $\omega_0 \subset \omega$, and a function $u \in C^{\infty}(\omega_0)$ where:

(0.5): $u|_M = \tilde{u}|_M$

(0.6): $D^{\alpha}\bar{\partial}u|_M = 0$ for each multiindex α.

(0.7): If we let $(a^1, \ldots, a^{n-1}, b)$ vary over a compact set in $\mathbb{R}^n$, we can find a constant $C > 0$ independent of ϵ, $(a^1, \ldots, a^{n-1}, b)$ and Q such that:

$$\mathrm{Re}\, u(p) \geq c\, d^2(p, N) + \psi(x(p))$$

in $\omega_0 \cap \bar{\mathscr{D}}$ [2].

(0.6) implies that

(0.8): $X(\mathrm{Re}\, u)|_M = Y(\mathrm{Im}\, u)|_M$ when X and Y are vectorfields where $JX = Y$

When Taylorexpanding u around N and using the fact that $\frac{\partial}{\partial v}|_{\partial\mathscr{D}} = J\frac{\partial}{\partial z}|_{\partial\mathscr{D}}$ and $\frac{\partial}{\partial y^i} = \eta_i - x^i \frac{\partial}{\partial z}$ we get:

$$(0.9)\quad u(p) = \psi(x(q)) + i[(z(p)-b) - \sum_{i=1}^{n-1} x(q)(y^i(p)-a^i)] + \sum_{i=1}^{n-1} \eta_i(u)(q)(y^i(p)-a^i) - v(p) + O(d^2(p,N))$$

where $q \in N$ and $x(q) = x(p)$.

DEFINITION:

$B(Q,\delta) = \{p \in \mathfrak{D} : |y^i(p) - a^i| \le \delta^{\frac{1}{2}},\ \ i = 1,\dots,n-1,\ |v(p)| \le \delta$ and $|z(p) - b - \sum_{i=1}^{n-1} x^i(p)(y^i(p)-a^i)| \le \delta\} \cap \{q \in \mathfrak{D} : d(x(q),x(Q)) \le \delta\}$.

LEMMA 1:

There exists a constant K, independent of $(a^1,\dots,a^{n-1},b)$, such that $|u(p)| \le K\delta$ whenever $p \in B(Q,\delta)$.

Proof:

This follows immediately from the fact that $\psi \circ x \in C^\infty(N)$ vanishes on Q and η_i is spanned by $\{J\frac{\partial}{\partial x^i}\}_{i=1}^{n-1}$ and $\{\frac{\partial}{\partial x^i}\}_{i=1}^{n-1}$.

If we let $\tilde{Q} = \{p \in N : d(x(p),x(Q)) \le \epsilon^{1/10}$, $\alpha = 1 - \frac{(n-1)/2 + 3}{2nm}$ and assume that $C < 1$ we get:

LEMMA 2:

$|u(p)| \ge \frac{C}{2}\epsilon^\alpha$ whenever $p \in \mathfrak{D} \setminus B(\tilde{Q}, \frac{2\epsilon^\alpha}{C}) \cap \omega_0$.

Proof:

Point (0.9) together with (0.3), (0.4), (0.7) and (0.8) implies lemma 2.

Let $\beta \in C_0^\infty(\omega_0;\mathbb{R})$ be equal to one in a neighbourhood of Q, $\beta \ge 0$.

If $H_\epsilon = \begin{cases} \bar\partial \left(\dfrac{\beta}{u+\epsilon^{1-(1/nm)}}\right) & \text{in } \omega_o \setminus Q \\ 0 & \text{elsewhere} \end{cases}$

$H_\epsilon = \dfrac{\bar\partial u}{(u+\epsilon^{1-(1/nm)})^2}$ in a neighbourhood of Q which implies that $H_\epsilon \in C^\infty_{(o,1)}(\bar{\mathcal{D}})$ and $\bar\partial H_\epsilon = 0$. $\|H_\epsilon\|_\infty \leq \|H_o\|_\infty$ so by [4] there exists a constant independent of ϵ and a $g_\epsilon \in C^\infty(\bar{\mathcal{D}})$ such that $\bar\partial g_\epsilon = H_\epsilon$ and $\|g_\epsilon\|_\infty \leq B$. Add a constant to g_ϵ such that $\operatorname{Re} g_\epsilon < -1$ and let

$$s_\epsilon = \frac{\beta}{u+\epsilon^{1-(1/nm)}} - g_\epsilon$$

Then $\bar\partial s_\epsilon = 0$ in $\mathcal{D}$ and $\operatorname{Re} s_\epsilon \geq 1$ which implies that $\dfrac{1}{s_\epsilon} \in A(\mathcal{D}) \cap C^\infty(\bar{\mathcal{D}})$.

(2.1) $$\frac{1}{s_\epsilon(p)} = \begin{cases} \dfrac{u(p)+\epsilon^{1-(1/nm)}}{\beta(p) - g_\epsilon(p)(u(p)+\epsilon^{1-(1/nm)})} & \text{when } p \in \omega_o \cap \bar{\mathcal{D}} \\ \dfrac{1}{g_\epsilon(p)} & \text{when } p \in \bar{\mathcal{D}} \setminus \omega_o . \end{cases}$$

(2.1) implies: (2.2) $\left|\dfrac{1}{s_\epsilon(p)}\right| \geq \dfrac{\epsilon^\alpha}{4}$ when $p \in \bar{\mathcal{D}} \setminus B(\tilde Q, \dfrac{2\epsilon^\alpha}{C})$.

(2.3) Let $f_\epsilon = \left(\dfrac{1}{s_\epsilon}\right)^{2nm} \cdot \dfrac{1}{\epsilon^{2nm-3}}$, then $f_\epsilon \in A(\mathcal{D}) \cap C^\infty(\bar{\mathcal{D}})$.

LEMMA 3:

(3.1) $\operatorname{Re} f_\epsilon(p) \geq \dfrac{\epsilon}{2^{4nm}}$ when $p \in B(Q,\epsilon)$

(3.2) $|f_\epsilon(p)| \leq 2^{4nm}\epsilon$ when $p \in B(Q,\epsilon)$

(3.3) $|f_\epsilon(p)| \geq \epsilon^{-(n-1)/2}$ when $p \in \bar{\mathcal{D}} \setminus B(\tilde Q, \dfrac{2\epsilon^\alpha}{C})$

(3.4) $|f_\epsilon(p)| \geq C_o\epsilon$ where $C_o > 0$ is a constant independent of ϵ.

Proof:

This is a consequence of Lemma 1, Lemma 2 and (2.2)

Choose a sequence of positive real numbers $\{\epsilon_j\}_{j=0}^{\infty}$ converging to zero and $\lim_{j\to\infty} \epsilon_{j-1}\,\epsilon_j^{-\delta} = \infty$ for each $\delta > 0$.

Define:

$$b(j,m) = \left[\frac{j}{j-1}\left(\frac{\epsilon_{j-1}}{\epsilon_j}\right)^{\frac{1}{2}(1-(1/m))}\right]$$

when $j > 1$; if $j = 1$ put 2 into the equations instead of $\frac{j}{j-1}$. Finally we let $c(j,m) = (b(j,m))^2$.

Let $\Gamma_o^m = \{0\} \subset \mathbb{R}$ and $\Gamma_j^m = \bigcup_{f\in\Gamma_{j-1}^m} \Gamma_j^m(t)$

where: $\Gamma_j^m(t) = \{t + k_1 \cdot \frac{\epsilon_{j-1}^{\frac{1}{2}}}{b(j,m)},\ k_1$ is an integer $0 \leq k_1 \leq b(j,m)-1\} \subset \mathbb{R}$

and $S_j^m \subset \mathbb{R}$:

$S_o^m = \{0\}$, $S_j^m = \{k_2 \frac{\epsilon_{j-1}}{3c(j,m)},\ k_2$ is an integer, $0 \leq k_2 \leq 6\,\frac{\epsilon_o e(j,m)}{\epsilon_{j-1}}\}$

Let $Q_o = \varphi^{-1}([0,\epsilon_o^{\frac{1}{2}}]^{n-1} \times \{(0,\ldots,0)\} \times \{0\})$.

Let $\tilde{R}_j^m = \{\Gamma_j^m\}^{n-1} \times S_j^m \subset \mathbb{R}^n$, if $(a^1,\ldots,a^{n-1},b) = (a,b) \in \tilde{R}_j^m$ we let

$$Q(a,b),\quad j = \varphi^{-1}([0,\epsilon_o^{\frac{1}{2}}]^{n-1} \times \{(a,b)\} \cap G_{j-1}$$

where

$$G_j = \bigcup_{(a,b)\in\hat{R}_j^m} V_{(a,b),j} \cap \{p \in \partial\mathcal{D} : \tfrac{2}{3}\epsilon_j > z(p) - b - \sum_{i=1}^{n-1} x^i(p)(y^i(p) - a^i) > \tfrac{1}{3}\epsilon_j\}$$

and

$V_{(a,b),j} = \{p \in \partial\mathcal{D} : \varphi^{-1}(x(p),a,b) \in Q_{(a,b),j},\quad \epsilon_j^{\frac{1}{2}} \geq y^i(p) - a^i \geq 0$

and $\epsilon_j \geq z(p) - b - \sum_{i=1}^{n-1} x^i(p)(y^i(p) - a^i) \geq 0\}$.

Finally we let $R_j^m = \{(a,b) \in \hat{R}_j^m$ where $Q_{(a,b),j} \neq \emptyset\}$.

Define the desired set $F^m = \bigcap_{j=o}^{\infty} F_j^m$ where $F_j^m = \bigcup_{(a,b)\in R_j^m} V_{(a,b),j}$.

LEMMA 4:

F^m is a peak set for the algebra $A(\mathfrak{D})$.

Proof:

For each $\bar{Q}_{(a,b),j}$ there exists a function $f_{(a,b),j}$ as in (2.3).

We define $f(p) = \sum_{j=o}^{\infty} \sum_{(a,b)\in R_j^m} \frac{\epsilon_j}{f_{(a,b),j}(p)}$ and have to show that: 1) f is holomorphic in $\mathfrak{D}$ and continuous in $\bar{\mathfrak{D}} \setminus F^m$

2) $\operatorname{Re} f(p) \to \infty$ when $p \to p_o \in F^m$.

(1): When K is a compact subset of $\bar{\mathfrak{D}} \setminus F^m$ there exists an integer J_K such that $p \notin B(\tilde{Q}_{(a,b),j}, \frac{2\epsilon_j^{\alpha}}{C})$ for each $(a,b) \in R_j^m$ when $j \geq J_K$, and since:

Since $|f(p)| \leq \sum_{j=o}^{\infty} \sum_{(a,b)\in R_j^m} \frac{\epsilon_j}{|f_{(a,b),j}(p)|}$

lemma 3 implies that

$$|f(p)| \leq \sum_{j=o}^{J_{K-1}} \sum_{(a,b)\in R_j^m} \frac{1}{C_o} + \sum_{j=J_K}^{\infty} \sum_{(a,b)\in R_j^m} \epsilon_j^{1+(n-1)/2} < \infty .$$

The uniform convergence of the series on each compact subset of $\bar{\mathfrak{D}} \setminus F^m$ gives (1).

(2):

We observe that $B(\tilde{Q}_{(a_1,b_1),j}, \frac{2\epsilon_j^{\alpha}}{C}) \cap B(\tilde{Q}_{(a_2,b_2),j}, \frac{2\epsilon_j^{\alpha}}{C}) = \emptyset$ each time $(a_1,b_1) \neq (a_2,b_2)$ and

$B(\tilde{Q}_{(a,b),j}, \frac{2\epsilon_j^{\alpha}}{C}) \subset B(Q_{(a_o,b_o),j-1}, \epsilon_{j-1})$ whenever $(a,b) \in R_j^m$ and $Q_{(a,b),j} \subset V_{(a_o,b_o),j-1}$.

If p is near F^m but $p \notin F^m$ there exists an integer $J_p \to \infty$ when $p \to p_o \in F^m$ and $(a_j,b_j) \in R_j^m$, $j \leq J_p$, such that $p \in B(Q_{(a_j,b_j),j}, \epsilon_j)$ but $p \notin B(Q_{(a,b),j}, \epsilon_j)$ for any $(a,b) \in R_j^m$ when $j > J_p$. This implies that $p \notin B(\tilde{Q}_{(a,b),j}, \frac{2\epsilon^{\alpha}}{C})$ if $(a,b) \neq (a_j,b_j)$ when $j \leq J_p$ and for each $(a,b) \in R_j^m$ when $j \geq J_p + 2$.

If $j = J_p + 1$ $p \in B(\tilde{Q}_{(a_o,b_o),j}, \frac{2\epsilon^{\alpha}}{C})$ for at most one choice of $(a_o,b_o) \in R_j^m$.

Since $\operatorname{Re} f(p) = \sum_{j=o}^{\infty} \sum_{R_j^m} \frac{\epsilon_j \operatorname{Re} f_{(a,b),j}(p)}{|f_{(a,b),j}(p)|^2}$ lemma 3 implies that

$$\operatorname{Re} f(p) \geq \sum_{j=o}^{J_p} \frac{\epsilon_j(\epsilon_j/2^{4nm})}{2^{8nm}\epsilon_j^2} - \sum_{j=o}^{J_p} \sum_{(a,b)\neq(a_j,b_j)} \epsilon_j^{1+(n-1)/2}$$

$$- \sum_{\substack{(a,b)\in R^m_{J_p+1} \\ (a,b)\neq(a_o,b_o)}} \epsilon_j^{1+(n-1)/2} - \frac{1}{C_o} - \sum_{j=J_{p+2}}^{\infty} \sum_{R_j^m} \epsilon_j^{1+(n-1)/2}$$

$$\geq \frac{J_p}{2^{12nm}} - \frac{1}{C_o} - \sum_{j=o}^{\infty} \sum_{R_j^m} \epsilon_j^{1+(n-1)/2}.$$

C_o is a constant, $\sum_{j=o}^{\infty} \sum_{R_j^m} \epsilon_j^{1+(n-1)/2} < \infty$ and $J_p \to \infty$ whenever $p \to p_o \in F^m$ so $\operatorname{Re} f(p) \to \infty$ whenever $p \to p_o \in F^m$.

LEMMA 5:

The Hausdorff dimension of F^m is $\geq 2n-1-\frac{n}{m}$.

Proof:

This follows from Lemma 3 in [6]:

Let $E \subset \mathbb{R}$ be the intersection of a decreasing sequence of sets $\{E_j\}_o^{\infty}$ with E_o of length 1 and E_j the union of a finite set of intervals of length δ_j, where $\{\delta_j\}_o^{\infty}$ is a sequence of positive numbers monotonously converging to zero. Assume that for $j > 1$ the following conditions hold:

1) The distance between any two of the segments in E_j is not smaller than
$$\rho_j = C\,\delta_{j-1}\left(\frac{\delta_j}{\delta_{j-1}}\right)^{\alpha}, \quad C > 0.$$

2) In each segment in E_{j-1} there is not contained less than $(\delta_{j-1}/\delta_j)^{\alpha}$ of the segments in E_j.

Then $\mathcal{H}^{\alpha}(E) > 0$.

And the following theorem which is a consequence of 2.10.4 and 2.10.7 in [3]

THEOREM:

Let $E \subset \mathbb{R}^m \times \mathbb{R}^n$, E compact, $E_x = \{y: (x,y) \in E\}$ and $\pi(E)$ be the projection into $\mathbb{R}^m$.

If $\dim E_x \geq \beta$ whenever $x \in \pi(E)$ and $\dim \pi(E) > \gamma$ then $\dim E \geq \beta + \gamma$.

Choose $\{\epsilon^m\}_{m+4}^{\infty}$ as a decreasing sequence of positive real numbers converging to zero. And let $\{a_m\}_{m=4}^{\infty}$ be a sequence in $\mathbb{R}$ converging to a point a such that $a_{m+1} - a_m > (\epsilon^m)^{\frac{1}{2}}$.

For each m let ϵ^m be the above ϵ_0 and replace F^m by $F^m + p_m$ where the n'th coordinate of $\varphi(p_m)$ is a_m and the rest is zero.

Finally we let $F = (\bigcup_{m=4}^{\infty} F^m) \cup p$ where $p = \lim_{m \to \infty} p_m$. Then F is compact.

PROPOSITION:

F is a peak-set for the algebra $A(\mathcal{D})$ and the Hausdorff dimension of F is $2n-1$.

Proof:

Since every point $p \in \partial\mathcal{D}$ is a peak point for $A(\mathcal{D})$ when $\mathcal{D}$ is strictly pseudoconvex with C^{∞}-boundary F is a countable union of peak sets and F is compact. Bishop's theorem implies that F is a peak-set.

If $\delta > 0$ there exists an m such that $\frac{n}{m} < \delta$ and $\mathcal{H}^{2n-1-\delta}(F) \geq \mathcal{H}^{2n-1-\delta}(F^m) > 0$ so $\dim F = 2n-1$.

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