ISBN 82-553-0461-4
Mathematics 1981
October 2 No 14

RULES OF DEFINITION

by

Nils Aall Barricelli
Institute of Mathematics
University of Oslo, Blindern, Norway

PREPRINT SERIES - Matematisk institutt, Universitetet i Oslo

ABSTRACT

In this paper we call attention to an obvious but often ignored condition which must be fulfilled by all definitions in order to avoid inconsistencies, namely that a definition is not allowed, neither explicitly nor implicitly , to make statements not involving the object (or objects) it is defining, except such statements which beforehand are assumed to be true as a condition for its applicapability (basic legality requirement). If a definition implies or includes statements not involving the objects being defined it can be used legaly only in those cases in which it is beforehand established that the statements in question (legality conditions) are true, or in other words when the legality conditions are fulfilled. When they are, the definition is called "conditionally legal"; when they are not, the definition is "conditionally illegal". If there are no legality conditions (implied statements not involving the concepts being defined) the definition is "unconditionally legal". If there is an absurd legality condition the definition is "illegal", and the defined concept (or concepts) is (are) meaningless under all circumstances.

A number of applications of this concept in several different fields are shown in the paper. It is also shown that the legality concept presented above to a large extent overlaps with the existence requirement. The very meaning of the concept of existence of a defined entity is discussed and it is shown that by an appropriate definition of "existence" based on the above legality concept, fulfilment of the existence requirement can be used as a legality criterium, or as a means to establish the legality of a definition in the sense specified above, and under the same conditions. However the use of our legality concept makes it possible to give a clear and rigorous analysis of the legality problems involved in various types of definitions and axiomatic systems, which in many cases can reveal the choice possibilities available, and help selecting a system of mutually consistent postulates for an analytical science. A most important observation illustrated by several examples is that in many cases a definition-legality problem can have more than one solution, and it is a matter of choise which solution is prefered.

RULES OF DEFINITION

1. Introduction

This paper deals with some of the main problems related to the use of definitions in analytical sciences.

It is costumary to consider the axioms and postulates of several mathematical or logical diciplines as a definition or a set of definitions of the concepts to be used in the field introduced, rather than a set of self-evident statements. For example the axioms and postulates of euclidean geometry are considered a definition of the euclidean space and the concepts which are used in euclidean geometry. The purpose of this paper is to inquire what kind of implications, if any, the use of definitions instead of self evident statements as a basis for an analytical dicipline will have. As will be shown below, the fact that the basic propositions on which a science can be built are definitions, has indeed an obvious implication whose consequences are in many cases well known by other means, and may seem trivial. There are however cases in which the implications of the definition rules are not commonly known, a fact which in some cases can lead to staggering consequences (cfr. Barricelli 1981, last two sections dealing with Gödel's proof).

We shall start by presenting the definition rules and their more obvious implications. Even if many of the implications are facts which are well known in different ways, and may seem trivial, the reader may not have been aware that they are consequences of the definition rules, and that these rules give a common interpretation of many appearently unrelated discrepancies and paradoxes of which the commonly given interpretations are not always satisfactory.

2. The function and purpose of definitions

A science based on a series of statements (axioms and/or postulates) which can be considered as definitions, or as a single combined definition, of the concepts used shall be called an "autonomous analytical science". Two basic requirements which one attempts to fulfill by a definition are:

1. The definition must (explicitly or implicitly) identify all the properties of the object (or objects) which is (are) being defined.
2. The definition must not change the meaning of any concept which is not the one (or the ones) being defined and must therefore not make (neither implicitly nor explicitly) any statement not involving the object (or objects) of the definition.

The first requirement is designated as "completeness" requirement. The second one as "legality" requirement.

The completeness requirement is designed to make it possible to prove all the true statements or theorems which apply to the defined object (or objects) without requiring the introduction of new assumptions. The question whether and when this completeness requirement can be fulfilled by a set of definitions (or postulates) designed to be the basis for an autunomous analytical science, is not the main object of this paper, but some aspects of the problem will be discussed in the last two sections of the next paper (Barricelli 1981).

The legality requirement to be discussed here is designed to avoid modifying the meaning of concepts which are not objects of the definition, thus implicitly introducing statements which could not be proved before and which do not involve the defined objects. The basic philosophy behind the legality requirement is that if a statement or "would be" theorem can not be proved, it is not legal to include it implicitly or explicitly in the definition of a new concept which is not mentioned in the statement or would be theorem whose proof is desired. This procedure would be illegal unless it is specifically stated that the new definition is also to be considered as a redefinition or a part of the definition of some of the concepts used in the theorem.

The legality requirement will, in our presentation, be used as a substitute for the existence requirement often applied as a criterion of definition legality. This criterion can still be applied, and the meaning of "existence" is specified in more precise terms (see below) by our legality concept.

An extreme case of illegal definition is the case in which a definition has absurd or self-contradictory implications. It will be instructive to study also other possible cases which will be designated as cases of "conditionally" illegal definitions.

3. Examples of definitions.

One of the most common types of definitions are equations defining unknown quantities or unknown functions. The completeness and legality criteria presented above will appear trivial when applied to equations, and other methods of defining quantities, or classes of quantities. Nevertheless, we shall use this as the first example, because no other class of definitions is so familiar and so appropriate to make our point clear.

Suppose we want to define a certain number of unknowns x, y and z by a certain number of equations of the type $F(x,y,z)=0$ and possibly some other conditions such as for example $x>y$ and $z<10$. The equations and the conditions are the statements which define the unknowns x, y and z. This definition is legal if the only conclusions or theorems which can be derived from these statements and could not have been proved before these statements were made, are theorems involving one or several of the unknowns x, y or z. The definition is illegal or conditionally illegal if some of the new conclusions which can be derived from the definition do not involve the defined entities x, y or z. As a result, a system in which the number of independent equations is larger than the number of unknowns would be illegal. Likewise, to take a familiar example, the following system of two linear equations

$$a_1x + b_1y = c_1 \qquad (3a)$$
$$a_2x + b_2y = c_2$$

has a solution and is conditionally legal under certain conditions, while it has no solution and is conditionally illegal under other conditions. Eventhough the conditions for the existence of solutions are well known, it may nevertheless be instructive in view of subsequent applications, to analyse this problem on the basis of the definition legality concept. In order to be under all conditions a legal definition of the unknowns x and y, the system (3 a) should neither explicitly nor implicitly contain any implications concerning the coeffecients a_1, b_1, c_1, a_2, b_2, c_2 or any expression used in the system (3 a), other than the two unknowns x and y defined by it. However it is easy to show that the system (3 a) actually has some implications concerning the coeffecients which can be derived in the following manner: According to (3 a) by using well known properties of determinants we find:

$$\begin{vmatrix} c_1 b_1 \\ c_2 b_2 \end{vmatrix} = \begin{vmatrix} a_1 x + b_1 y & b_1 \\ a_2 x + b_2 y & b_2 \end{vmatrix} = \begin{vmatrix} a_1 x & b_1 \\ a_2 x & b_2 \end{vmatrix}$$

or

(3b) $$\begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix} x = \begin{vmatrix} c_1 b_1 \\ c_2 b_2 \end{vmatrix} \quad \text{and likewise} \quad \begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix} y = \begin{vmatrix} a_1 c_1 \\ a_2 c_2 \end{vmatrix}$$

These two relations imply that if

(3c) $$\begin{vmatrix} a_1 b_1 \\ a_2 b_2 \end{vmatrix} = 0$$

then both of the conditions

(3d) $$\begin{vmatrix} c_1 b_1 \\ c_2 b_2 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} a_1 c_1 \\ a_2 c_2 \end{vmatrix} = 0$$

must be fulfilled: evidently an implication concerning the coeffecients a_1, b_1, c_1, a_2, b_2, c_2, which does not involve the defined unknowns x and y. This implication will be called "a legality condition" for the definition (3 a). If this legality condition is not fulfilled, the system (3 a) is an illegal definition of x and y. Only if this legality condition is fulfilled, meaning if either the determinant of the system (left side of formula (3 c)) is different from zero, or both of the determinants (3 d) are zero, the system may be a legal definition of x and y.

These are well known to be the same conditions commonly used in order to decide whether the system does or does not have solutions. The existence of the defined concept (or solution), not only when the definition is in the form of a system of equations, but in general, for any kind of definition, is used as a criterion of definition legality (see next section).

The system of equations (3a) is a typical example of a definition which is legal when a certain legality condition is fulfilled - in which case the definition is called "conditionally legal" - and illegal when the legality condition is not fulfilled - definition "conditionally illegal".

There are of course also definitions which are unconditionally legal, such as for example the equation $x = 5$ defining x

(see next section) and definitions which are unconditionally illegal, such as the definition $5 < x < 3$ implying an absurd legality condition $5 < 3$.

4. Definition legality criteria.

A procedure commonly used to verify the legality of a definition relies on the assumption that there are certain classes of familiar or "trustworthy" concepts (numbers, geometric entities, logic concepts etc.) which exist or have selfconsistent properties, a sort of way to imply that they are legally defined by the properties ascribed to them.

Once a certain number of trustworthy concepts are given, it is often possible to prove the legality of new definitions designed to introduce new concepts by the procedure of finding in a trustworthy field of knowledge examples of the entities we have defined. This is a very common procedure used in order to prove that a defined concept has a meaning by showing that there are examples of the defined concept (or concepts) in a trustworthy field of knowledge. This method of finding examples is considered a method of proving that the defined entities exist: The existence criterion of definition legality. Existence is however, a concept difficult to define, and perhaps the considerations presented above are as close as we can come to a definition of the concept of "existence of a defined object in a trustworthy field of knowledge". The value of this "existence criterion" is that it insures that the definition can not be used to prove any theorems which could not unconditionally be proved before about objects other than the defined ones, thus insuring conditional definition legality. In fact the existence of examples of objects obeying all the statements used in the definition, implies that any theorem which can be derived from those statements can already be proved, simply by substituting the names of the defined objects with the names or designations of the examples which have been found. If the theorem does not mention the defined objects such name substitutions would not be necessary, but the theorem would still have been provable before the definition was given. In other words the definition can not be unconditionally illegal.

In the case in which the definition is a system of equations defining a certain number of quantities, a way to apply the existence criterion for definition legality is to find a solution of the equations. The existence of a solution would be evidence that the equations do not contain self-contradictions. If there are not conditions for the existence of a solution, the equations (or definition) are unconditionally legal. Otherwise the definition would be only conditionally legal, and would be illegal when the conditions are not fulfilled.

Definitions given in the form of an identity between the defined concept and an entity or concept belonging to a trustworthy field of knowledge (such as $x = 5$) are always legal since they already contain an example, of the defined object (the number 5 is an example of an x).

An important implication of our legality concept is the following: Although a definition might be helpful in proving a theorem not involving the defined entity, it can not be strictly necessary. If it were necessary the theorem would not be provable by a legal procedure.

It should be mentioned however that a "new definition", in this connection, means a definition of a new concept given independently and not as part of the original definition (or set of postulates) applying to the concepts involved in the theorem to be proved. A simple illustration of this is the following example. If the variables x, y, z are defined by a set of conditions like

(4a) $$f(x, y, z) = 0$$

in order to prove a theorem such as $T(x,y) = 0$, we are not allowed to define a new variable u by one or several new conditions like

(4b) $$F(x, y, z, u) = 0$$

unless we can show that this new definition has no effect on the validity of the proof. In other words we have the choice between two alternatives:

(1) either we shall prove that the new definition (4b) is only used as a simplifying procedure for example in order to replace a complex expression of the variables x, y, z by a simpler expression u

(2) or else we will have to concede that the theorem T(x,y) is not proved by using the definition (4a) alone, and that also the conditions (4b) must be considered part of the definition of x, y, z, and not only a separate definition of u . Otherwise we would not have proved a theorem but only a legality condition for the new definition (4b).

While these considerations may seem trivial when they are applied to mathematical equations, they have not always received the attention they deserve in the definition of other logical and mathematical concepts. Notice moreover that the legality of the procedure does not depend on any considerations other than the question whether the theorem T(x,y) = 0 could be proved or could not be proved before the new definition was given.

5. The handling of conditionally legal definitions.

Conditionally legal definitions have been used many times in the history of mathematics and the methods of dealing with this kind of definitions open interesting perspectives which will be discussed below. We shall start by presenting some examples of such definitions. Only those aspects of the various definitions will be discussed which are relevant for the point we want to make.

The quotient between two numbers A and B is defined as a number $\frac{A}{B}$ which, when multiplied by B yields the number A. This definition is expressed by the following formula:

(Quotient definition) $\frac{A}{B}B = A$

This definition is conditionally illegal, because it implies the following statement "if B is zero than A must also be zero". This statement, which is a rigorous consequence of the above definition if $\frac{A}{B}$ is any finite number, can formally be expressed by the following symbolic expression:

$$B = o \rightarrow A = o$$

This statement does not involve the defined entity which is the quotient $\frac{A}{B}$ and does not appear in the above statement. Furthermore this statement could not be proved before the above definition was given. Evidently the definition is conditionally illegal if the above statement is untrue, namely if $B = o$ and $A \neq o$. The definition of the quotient $\frac{A}{B}$ is conditionally legal only in those cases in which either $B \neq o$ or $A = o$.

A method, which we may call "the conservative method", commonly used to deal with this definition is to avoid division by a zero (unless the numerator is also zero in which case the quotient is an undetermined number usually considered of little interest). By this method the use of this quotient concept is restricted to those cases in which its definition is conditionally legal. Another method, which we may call "the revision method" is to extend the concept of number to include so called "infinite numbers" for which the rule that multiplication by zero always gives a zero is not true. Without this rule the statement

$$B = o \rightarrow A = o$$

can not be derived from the definition of the quotient $\frac{A}{B}$. If $\frac{A}{B}$ is an infinite the product $\frac{A}{B}B$ does not have to be zero when B is zero. This revision method consists in redefining other concepts used in the definition (in this case the concept of number) so that the rule which permitted to derive an illegal statement from the definition does not apply any longer. We will see more examples both of conservative and revision methods of handling definitions which are conditionally illegal.

The square root $\sqrt{A}$ of a number A is defined by the following equation:

$$(\sqrt{A})^2 = A$$

This is another example of a definition which, before the introduction of imaginary numbers was conditionally illegal. In fact from this definition one could derive the statement

$$A \geq 0$$

Since A, being the square of a number $\sqrt{A}$, would have to be positive or 0 as long as only real numbers were being considered. This statement does not involve the defined concept which is $\sqrt{A}$ (not A, nor > nor 0) and could not be proved before the definition of $\sqrt{A}$ was given. Obviously the definition of $\sqrt{A}$ is conditionally illegal when $A<0$, and conditionally legal only when $A \geq 0$. The conservative method, which for a long time was the only method used to deal with this definition, was to avoid the use of square roots of negative numbers. A revision method, leading to new concepts which today are commonly applied, has been the introduction of imaginary and complex numbers. For these numbers the statement $n^2 \geq 0$ is not true, and the definition of square root is therefore unconditionally legal.

An other example of conditionally legal definitions was the definition of the difference between two numbers A and B before negative numbers and the number 0 had been introduced in the arithmetical calcules. The definition of the difference A - B can be expressed by the following statement:

$$(A - B) + B = A$$

Before the introduction of negative numbers and the number 0, the number A - B would have to be considered as a positive number, and an implication of the above definition would have been:

$$A > B$$

This is a statement not involving the defined concept (A - B) and is therefore a legality condition, not a theorem. A conservative method originally used in order to deal with this definition may have been to use the difference A - B only in those cases in which the above legality condition A > B was fulfilled. The method adopted today is a revision method based on the introduction of non-positive numbers which do not obey the rules implying the above legality condition.

Just as there are mathematical concepts which must be handled in a special way (for example by avoiding divisions by zero and square roots of negative numbers) in order to avoid problems related to definition illegality, there are also logical concepts which require careful handling for the same reason.

One of these concepts is designated by the expression "itself". There are several examples of statements involving the expression "itself" which lead to paradoxes or absurdities. Some well known examples are "the Liar's paradox" and "Russels' paradox", both of which involve this expression. Logical statements involving "itself" are often of the type $x \leftrightarrow F(x)$ (expressing that a statement x is a Boolean function of itself) or the type $x \varepsilon f(x)$ (where f(x) is an attribute defined as a function of x which is ascribed to x itself). These cases have analogy with the equations of the type $x = f(x)$, which may or may not have solutions. Any user of concepts involving the expression "itself" or substitutes for it would be well advised to consider "itself" as a concept whose definition can be conditionally illegal. But, as opposed to some other conditionally illegal concepts (such as the concept of quotient or square root) the legality conditions for its definition are not fully explored and we have no safe prescription for making sure that they are fulfilled.

An important implication of the way autonomous analytical sciences are arranged, using definitions, instead of self- vident axioms, is the following:

Self-contradictory or absurd inferences, detected in autonomous analytical sciences, are always indicative of definition legality problems arising under the same conditions.

In fact as long as all the postulates and axioms are considered an integral part of the definitions of the concepts we are using, there is no possibility of deriving selfcontradictions without using such definitions, or theorems derived from them.

Any selfcontradiction discovered can therefore be blamed on a definition legality problem or an error.

6. The use of conditionally legal concepts.

When using a concept whose definition is known (or suspected) to imply some legality conditions one is well advised taking some precautions in order to avoid possible legality problems. Unless the legality conditions are well explored and known to yield a save alternative procedure, the precautions one would have to use are the same which are needed when giving a new definition in order to make sure that the definition is legal, or at least not illegally applied. Namely:

(1) Never use the suspected concept neither directly nor under other names in order to prove theorems in which the concept is not involved (unless one can show that its use is only a simplifying procedure which has no effect on the validity of the argument).

(2) Expressions in which the suspected concept is used and any substitute for such expressions should be treated as newly defined concepts, whose definition legality must be proved, and cannot be taken for granted.

These precautions can be dispensed of or substituted only if the legality condition is well explored. For example we can safely use quotients if we take good care of never dividing by zero, and we can safely use square roots in real number algebra if we take good care of never using the square root of a negative number. But the justification for this procedure is based on the fact that if a statement not involving the defined concept (quotient of two numbers or square root of a number) can be derived from theorems involving this concept, we always know how to derive it from other theorems not involving the said concept, if division by zero and square roots of negative numbers have been carefully avoided.

However we have no safe procedure for using statements involving the expression "it self" or substitutes for such statements, in the proof of theorems not involving this expression.

The safest procedure is to treat expressions involving the concept "it self" and substitutes for such expressions as representing new defined concepts and apply the same precautions required for new definitions. This safety procedure is worth applying whenever there can be the slightest possibility that definition legality problems may arise in connection with any of the concepts used.

7. A geometric implication of the definition rules.

In some areas of mathematics, such as equation theory where the questions relative to the existence or possible inexistence of solutions are routinely investigated, adequate definition legality verifications are commonly applied.

On the other hand there are also some areas of mathematics and logic in which precautions against definition illegality are inadequate. One of these areas is geometry, the very science in which the concept that the axiomes and postulates of an analytical science are to be considered as definitions were applied for the first time. One of our problems seem to be connected with the fact that we are used to live in a threedimensional space, and to us the question of definition legality in geometry seem to be identic to the question of definition legality in a threedimensional space. The fact that for a twodimensional being our threedimensional space could be physically inexistent is alien to us.

A well known theorem of projective geometry is the so called "homological trianglers theorem", which can be found in all projective geometry treatises. However, for the purpose of the argument we are going to make here, it is sufficient to know that the theorem can be formulated in the projective plane S_2 as well as in a three-dimensional space S_3. But no way has been found to prove the theorem in the projective plane S_2 without using constructions or concepts belonging to a threedimensional space S_3 of which S_2 is supposed to be a part.

The question one may ask at this point is: "Can the theorem be considered as generally valid in any projective plane, without inquiring before hand whether or not the plane is part of a three-dimensional space?" Evidently, if the postulates of planar projective geometry are to be considered as the definition of the projective plane, the answer to this question is "definitely not". The so called "homological trianglers theorem" has not been proved valid in the projective plane on the basis of those postulates. The "theorem" has been proved by defining some additional entities external to the projective S_2-plane and belonging to an S_3-space of which S_2 is supposed to be part. In other words the only thing which has been proved is that if the so called "homological trianglers theorem" is not valid in a projective S_2 space, then the definition of an S_3 space containing that S_2 space would be illegal. The "theorem" is no theorem at all as far as the projective S_2-space is concerned. It is only a legality condition for the definition of an S_3 space containing S_2. If that legality condition is fulfilled, then there is a possibility that the definition of an S_3 space containing S_2 would be legal. Otherwise such an S_3 space would be illegally defined.

Notice that in an euclidean plane, obeying the metric conditions which apply in euclidean geometry, this legality condition would be fulfilled. A proof of the homological trianglers theorem, for example by using analytical geometry without resorting to three-dimensional considerations, would be messy, but certainly possible. The euclidean plane may without reservations be considered part of a threedimensional space. The same is not possible for every projective plane, unless a projective proof of the "homological trianglers theorem" can be found which does not resort to three-dimensional considerations.

It might be worth inquiring whether a planar projective geometry in which the homological trianglers theorem is not valid can be constructed, and if so what such a geometry would look like. Likewise it may be worth inquiring whether there are threedimensional projective geometry "theorems" (or more precisely "legality conditions") which would apply only if the projective S_3-space is part of an S_4 space but not otherwise.

Some readers may be inclined to consider a set of postulates defining a projective S_2-plane which does not fulfill all of the needed conditions for being part of an S_3-space as an incomplete definition of the projective plane. That is however a question of terminology. If your intention was to use the term "projective plane" in order to designate an S_2 which is part of an S_3, then the definition is obviously unsuccessful or incomplete, and would require some additional postulates in order to be made complete. If that was not the intention, then the definition might be quite acceptable for the concept it is intended to designate.

One may just apply two different terms to designate the two different concepts, such as the "autonomous projective plane", and the "S_3 embedded projective plane". To have proved a theorem by making use of S_3 considerations makes it valid in the S_3 embedded projective plane but not in the autonomous projective plane. That applies even if the theorem (like the homological trianglers theorem) can be formulated by using exclusively planar geometry concepts. The theorem will however not only be unprovable, but also impossible to disprove, otherwise planar projective geometry would be in conflict with space geometry. This may be an example of a proposition which is undecidable (neither provable nor disprovable) in the autonomous projective plane.

8. Autonomous and non-autonomous analytical sciences.

There is a lesson to be learned from this example:
If a theorem T(S) which has been formulated in a science or inference system S can be proved only by using arguments which are permitted in an other science or inference system S_o but not in the science S, one can not without reservations consider that theorem as valid in the science S. One will have to make a distinction between two cases or two versions of the science S. The theorem will be valid in a science S embedded in S_o(meaning including as extra postulates all statements about S-concepts which can be proved by using S_o) but not in an autonomous (meaning not embedded) science S.

The following general rule applies:

In every autonomous analytical science or demonstration system S, <u>the only propositions which are true are those which can be derived from the axioms and postulates defining the science S.</u> In such a system <u>provable is a synonym of true.</u>

This rule is important for questions concerning, for ex. , the applicability of Gödel's proof in PM or any other science without beforehand making sure by other means that it is not and can not be organized as an autonomous analytical science. If it can, "unprovable" could become a synonym of "untrue" and Gödel's well known proposition stating its own unprovability could be made equivalent to the liar's paradox stating its own untruthfulness (see Barricelli 1981, last two sections).

REFERENCES

Barricelli, N. Aall (1981) Analytical Logic The elementary B-function system for the solution of logical application problems.

Gödel, K. (1931) Über formal unentscheidbare Sätze der Prinzipia Mathematica und verwandter Systeme I. Monatshefte für Mathematik und Physik, 38, pp. 173 - 198.