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SOME CONSEQUENCES OF AD FOR
KIEENE RECURSION IN ${ }^{3}$ E
by
E。R. Griffor ${ }^{1)}$
Inst. of Math., University of Oslo

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> Some Consequences of $A D$ for Kleene Recursion in ${ }^{3} \mathbb{E}$. E.R. Griffor Oslo -1981

## § 0 Introduction

In this paper we derive some consequences of $A D$（axiom of Determinacy）for Kleene recursion in ${ }^{3} \mathbb{E}$（the type three func－ tional giving the equality predicate for sets of reals）．In § 1 we state and sketch the proof of a result of Moschovakis which is key to many of the results in subsequent sections．

A．S．Kechris asked：Does AD imply any＂large cardinal properties＂for $K_{1}^{3} \mathbb{E}$ in $L[\mathbb{R}] ?$ In § 2 we show that $K_{1}{ }^{3} \mathbb{E}$ is weakly inaccessible under AD，＂larger＂than measurability need imply（ $\mathbb{L}_{1}$ is measurable under $\left.A D\right)$ 。

A recurrent theme in recursion in higher types and E－recur－ sion is that of selection theorems，i。e。for which sets $Z$ are the classes semirecursive（ $R E$ ）in ${ }^{3} \mathbb{E}$ and a real closed under the quantifier $\exists a \in Z$ ？Under $A D$ we show that these classes are closed under the quantifier $\exists a<\gamma$ for every $\gamma<\mathbb{K}_{1}^{{ }^{3}} \mathbb{E}$ ：This should be contrasted with the situation under $V=I$ where $K_{1}^{3} \mathbb{E}>\lambda_{1}{ }_{1}$ ，but the $R E$ classes are not closed under $\exists a<\gamma$ for any $\gamma$ such that：$\left\langle\sum_{1} \leq y<K_{1}^{3} \mathbb{E}\right.$ ．We also sketch the proof of a part of the folklore using $A D$ that we can select an element from a set of reals recursive in ${ }^{3} \mathbb{E}$ ，a uniformly in a for $a \in 2^{\omega}$ 。 These results on selection appear in §3．

It was known that $A D$ implied that the structure of the $R E$ degrees was trivial，i．e．an $R E$ class is either complete RE or REC（recursive）．In § 4 we strengthen this result to show that $A D$
implies that any regular $R E$ class is $R E C$ ，hence there is no regular complete $R E$ class．This adds some force to the conjecture that there is a model of ZF （assuming ZF is consistent）in which any regular $R E$ set is REC（and hence that the degree structure is trivial）。 Also in § 4 we employ the tools of $\S 2$ to describe the degrees of subsets of $\mathbb{K}_{1}^{3} \mathbb{E}$ under $A D$ ．The result here is that any $A \subseteq K_{1}^{3} \mathbb{E}$ is recursive in ${ }^{3} \mathbb{E}$ and a real．

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§ 1．Prewellorderings and the Moschovakis Lemma．

Following Moschovakis［1980］，a prewellordering of $\mathbb{R}$（with field $\mathbb{R}$ ）is a relation $\leq$ on $\mathbb{R}$ such that for some ordinal $\delta$ ， some surjection

$$
\begin{aligned}
\varphi: \mathbb{R} \longrightarrow & >\delta \text { and all } x, y \in \mathbb{R}, \\
x & \leq y<=>\varphi(x) \leq \varphi(y), \text { where }
\end{aligned}
$$

the $\leq$ on the right side is the order relation on OR（＝ordinals）． It is immediate that such $\varphi$ and $\delta$ are unique when they exist； we call $\delta$ the length of $\leq$ and $\varphi$ the canonical surjection （of s）。

A relation is a prewellordering iff it is reflexive，transi－ tive，connected and well－founded（equivalently（DC）has no un－ finite descending chains。）．Let $\Lambda$ be aclass of subsets of $\mathbb{R}$ and assume $\Lambda$ contains all singletons and is closed under con－ tinuous substitution and 7 。 An ordinal $\delta$ is realized in $\Lambda$
if there is a prewellordering $\leq$ of some subset of $\mathbb{R}$ in $\Lambda$ with length $\delta$ ．These closure properties of $\Lambda$ imply：
（i）$\delta$ realized in $\Lambda$ and $\zeta<\delta \Rightarrow$ $\zeta$ is realized in $\Lambda$ ；
（ii）$\delta$ realized in $\Lambda \Rightarrow \delta+1$ is realized in $\Lambda$ ；and
（iii）$\delta>0 \wedge \delta$ realized in $\Lambda \Rightarrow \delta$ is the length of some prewellordering of $\mathbb{R}$ in $\Lambda$ 。

Let $\theta(\Lambda)=\sup \{\delta \mid \delta$ is realized in $\Lambda\}$ 。
Remark Let $\Lambda^{3} \mathbb{E}=\{x \mid x$ is a prewellordering of $\mathbb{R} \wedge x$ is re－ cursive in ${ }^{3} \mathbb{E}$ and a real\}, then $\theta(\Lambda)=K_{1}^{3} \mathbb{E}$ and $\Lambda$ clearly satisfies the closure properties above．

If the length of $\leq$ is $\delta$ and $\varphi:$ fld $(\leq) \rightarrow>\delta$ is the canonical surjection，then $\varphi$ gives a＇coding＇of $\delta$ in the space $X$（a product space on $\mathbb{R}$ ），i。e．we can think of each $x \in \operatorname{fld}(\leq)$ such that $\varphi(x)=\eta<\delta$ as a code or name for $\eta_{0}$ If $Y$ is such a product space and $f: \delta \rightarrow Y_{2}$ is a function on $\delta$ to subsets of $Y$ ，we can represent it by a subset of $X \times Y$ as follows：

$$
\operatorname{Cod}(f ; \leq)=\operatorname{df}\{(x, y): x \leq x \wedge y \in f(\varphi(x))\}
$$

Definition．Suppose $f: \delta \rightarrow Y_{2}$ is a function．A choice sub－ function of $f$ is any $g: \delta \rightarrow Y_{2}$ such that for all $\eta<\delta$ ：

$$
\begin{aligned}
g(\eta) & \subseteq f(\eta) \\
f(\eta) \neq \varnothing & \Rightarrow g(\eta) \neq \varnothing, \quad \text { i.e. }
\end{aligned}
$$

$g(\eta)$ chooses a non－empty subset of $f(\eta)$（provided $f(\eta) \neq \varnothing$ ）．

The following lemma is the main tool of this note and is due to Moschovakis [1970].

## The Moschovakis Lemma (ML) assume AD.

Let $\leq$ be a prewellording with field a subset of some $X$ and length $\delta$, let $f: \delta \rightarrow Y_{2}$ be a function. Then there exists a choice subfunction $g$ of $f$ such that

$$
\operatorname{Cod}(\mathrm{g} ; \leq) \text { is a } \sum_{\sim}^{1}(\leq) \text { subset of } X \times Y \text {. }
$$

pf we give the main idea of the proof - the case we shall be interested in is $\mathbb{R}=X=Y$. For each $\zeta<\delta$ let $f_{\zeta}$ be defined by:

$$
\begin{aligned}
f_{\zeta}(\eta)= & f(\eta), \\
\varnothing & \text { if } \eta<\zeta \\
\varnothing, & \text { if } \zeta \leq \eta<\delta
\end{aligned}
$$

Suppose there is some $\zeta \leq \delta$ such that $f_{\zeta}$ does not have a choice subfunction with cod in $\underset{\sim}{1} 1(\leq)$, let $\lambda$ be the last such $S$. ${\underset{\sim}{1}}_{1}^{1}(\leq)$ is parameterized, hence the class of ${\underset{\sim}{1}}_{1}^{1}(\leq)$ subsets of $X \times Y$ is parameterized let

$$
G \subseteq \mathbb{R} \times X \times Y \text { be fixed universal set for it. As usual, }
$$

let

$$
G_{\alpha}=\{(x, y) \mid(\alpha, x, y) \in G\} \text { and }
$$

consider the game: I plays $\alpha$ and II plays $\beta$, then

II wins $<>7 \exists \eta\left[g_{\eta}\right.$ is a choice subfunction

$$
\begin{aligned}
& \text { of } f_{\eta} \wedge G_{\alpha}=\operatorname{Cod}\left(g_{\eta} ; \leq\right) \vee(\exists \eta<\lambda) \\
& (\exists \zeta<\lambda)\left(\exists g_{\eta}\right)\left(\exists g_{\zeta}\right)\left[g_{\eta}\right. \text { is a choice } \\
& \text { subfn of } f_{\eta} \wedge g_{\zeta} \text { is a choice subfn } \\
& \text { of } \left.f_{\zeta} \wedge \eta<\zeta \wedge G_{\alpha}=\operatorname{Cod}\left(g_{\eta} ; \leq\right) \wedge G_{\beta}=\operatorname{Cod}\left(g_{\zeta} ; \leq\right)\right]
\end{aligned}
$$

If we think of $\alpha$ as a code of a function $g$ when $G_{\alpha}=\operatorname{Cod}(g ; \leq)$ then II wins if either I does not code a choice subjunction of an initial segment of $f$ on $I$ does and II codes a choice subfn of a longer initial segment of $f$ 。

Moschovakis shows that the existence of a winning strategy for either $I$ or II (AD) yields a choice subjunction of $f_{\lambda}, g_{\lambda}$ with $\operatorname{Cod}\left(g_{\lambda} ; \leq\right) \in \sum_{\mathcal{N}}^{1}(\leq)$, contradicting the choice of $\lambda_{0}$ For details of the proof see Moschovakis [1970].
$\frac{\text { Remark }}{3^{3} \mathbb{E}}$ Note that if $\leq$ is a prewellordering on $\mathbb{R}$ sot.
$\leq \in \Lambda^{2}$ and

$$
\begin{aligned}
& \operatorname{cod}(\mathrm{g} ; \leq) \in \underset{\sim}{\Sigma_{1}^{1}}(\leq) \text { then } \\
& \operatorname{Cod}(\mathrm{g} ; \leq) \in \Lambda^{3} \mathbb{E}
\end{aligned}
$$

As a corollary of the Moschovakis Lemma we have:

Corollary 1 Assume $A D$. Let $\leq$ be a prewellordering on $\mathbb{R}$ with length $\delta$ and let $A \subseteq \delta$. Then

$$
\operatorname{Cod}(A ; \leq) \in \underset{\sim}{\sum_{1}^{1}}(\leq)
$$

pf $f i x \quad \alpha_{0}, \alpha_{1} \in \mathbb{R}$ and set

$$
f(\eta)=\left\{\begin{array}{lll}
\left\{\alpha_{0}\right\}, & \text { if } & A(\eta) \\
\left\{\alpha_{1}\right\}, & \text { if } & A(\eta)
\end{array}\right.
$$

The only choice subfunction of $f$ is $f$ itself and hence by $M I$,

$$
\begin{aligned}
& \operatorname{Cod}(f ; \leq) \in \underset{\sim}{\Sigma_{1}^{1}}(\leq) \text { and } \\
& x \in \operatorname{Cod}(A ; \leq)<=>\left(x, a_{0}\right) \in \operatorname{Cod}(f ; \leq)
\end{aligned}
$$

$\S 2\left[\mathbb{K}_{1}^{3} \mathbb{E}\right]^{\mathrm{L}[\mathbb{R}]}$ under AD.
We will now use the tools of § 1 to answer the question of Kechris stated in the introduction positively.

Theorem 2 Assume $A D$. In $L[\mathbb{R}]$ we have that $K_{1}{ }^{3} \mathbb{E}$ is weakly inaccessible.
pf it suffices to show that
$\mathbb{K}_{1}^{3} \mathbb{E}$ is a regular limit cardinal in $L[\mathbb{R}]$. We use the notation $\delta^{+}(\delta \in \mathrm{OR})$ to denote the least cardinal greater than $\delta$ and $\delta_{2}=\{f: \delta \rightarrow\{0,1\}\}$.
We first show that $\mathbb{K}_{1}{ }^{3} \mathbb{E}$ is a cardinal. Suppose not and let $\eta<K_{1}{ }^{3} \mathbb{E}$ and

$$
\mathrm{f}: \mathbb{K}_{1}^{3} \mathbb{E} 1-1 \text {. П. Also }
$$

let $\leq \in \Lambda^{3}{ }^{3}$ with length $\eta_{0}$ (we can suppose $\eta$ is a cardinal). The function $f$ induces a relation on $\eta \times \eta$ ), (a subset of $\eta \times \eta$ ), $\mathrm{R}_{\mathrm{f}}$, given by

$$
\left.\langle\sigma, \delta\rangle \in R_{f}<=\right\rangle f^{-1}(\sigma) \leq f^{-1}(\delta) .
$$

$R_{f}$ is a well-founded relation on $\eta^{2}$ of height $K_{1}^{3} \mathbb{E}$. By an easy generalization of the corollary to ML to $n$-arr relations

$$
\operatorname{Cod}\left(R_{f} ; \leq\right) \in \Sigma_{1}^{1}(\leq) \text { and }
$$

hence

$$
\operatorname{Hod}\left(R_{f} ; \leq\right) \in \Lambda^{3} \mathbb{E}_{1} \text { which is }
$$

absurd since the bounding principle then implies that the height of $R_{f}$ is less than $K_{1}{ }^{3} \mathbb{E}$ 。 Suppose now that $\mathbb{K}_{1}^{3 \mathbb{E}}$ is singular:
let $\eta<K_{1}{ }^{3} \mathbb{E}$ be a cardinal sot. there exists

$$
f: \eta \frac{\text { increase }}{\text { unbdd }} \cdot K_{1}^{3} \mathbb{E} .
$$

Let $\leq \in \Lambda^{3} \mathbb{E}$ of length $\eta$ and for each $\gamma<\mathbb{K}_{1}^{3} \mathbb{E}$ define

$$
I_{\gamma}=\{b \in \mathbb{R} \mid b \text { is an index for } \gamma\}
$$

(i.e. be codes a convergent computation in ${ }^{3} \mathbb{E}$ of height $\gamma$ ). Define $h: \eta \rightarrow \mathbb{R}_{2}$ by for $\delta<\eta$ :
$h(\delta)=I_{f(\delta)}$. By $M 山$ there exists a choice subfunction $g: \eta \rightarrow \mathbb{R}_{2}$ of $h$ such that

$$
\operatorname{Cod}(g ; \leq) \in \sum_{\sim}^{1}(\leq) \cdot \text { Note that }
$$

for $\gamma<K_{1}^{3} \mathbb{E}, \quad I_{\gamma} \neq \varnothing$ and hence for $\delta<\eta$

$$
\begin{aligned}
& g(\delta) \subseteq h(\delta)=I_{f(\delta)} \quad \text { and } \\
& g(\delta) \neq \varnothing
\end{aligned}
$$

Now define $\tau: \eta \rightarrow K_{1}^{3} \mathbb{E}$ by for $\delta<\eta$ :

$$
\tau(\delta)=\text { suprenum }\left\{\|b\|_{3_{\mathbb{E}}} \mid b \in g(\delta)\right\}
$$

Since $\operatorname{Cod}(\mathrm{g} ; \leq) \in{\underset{\sim}{1}}_{1}^{1}(\leq), \operatorname{Cod}(\mathrm{g} ; \leq)$ is recursive in ${ }^{3} \mathbb{E}, \leq$ and clearly $\forall \delta<\eta$

$$
\tau(\delta)=f(\delta), \text { contradicting }
$$

the bounding principle again.
It remains only to show that $\mathbb{K}_{1}^{3} \mathbb{E}$ is a limit cardinal. This will follow by an argument first noticed by H. Friedman [198?] for $\theta\left(\mathbb{R}_{2}\right)$. let $\delta<\mathbb{K}_{1}{ }^{\mathbb{E}}$, then we show that $\delta^{+}<\mathbb{K}_{1}{ }^{3} \mathbb{E}$ : We first show given $\delta<\mathbb{K}_{1}^{3} \mathbb{E}$ then there exists $\varphi^{*}: \mathbb{R} \rightarrow \delta_{2}$ such that $\varphi^{*}$
has

$$
\operatorname{cod}\left(\varphi^{*} ; \leq\right) \in \sum_{1}^{1}(\leq) \quad \text { where }
$$

$\leq \in \Lambda^{3} \mathbb{E}$ of length $\delta$. By the corollary to $M \mathbb{L}$ if $A \leq \delta$ then

$$
\operatorname{Cod}(A ; \leq) \in \sum_{\sim}^{1}(\leq), \text { thus }
$$

if $G$ is universal for ${\underset{\sim}{1}}_{1}^{1}(\leq)$ and hence recursive in ${ }^{3} \mathbb{E}_{1} \leq$ define for $\alpha \in \mathbb{R}$ :

$$
\varphi^{*}(\alpha)= \begin{cases}A & \text { if } G_{\alpha}=\operatorname{Cod}(A ; \leq), A \subseteq \delta \\ \varnothing & \text { otherwise }\end{cases}
$$

Then $\varphi^{*}$ is the desired surjection
Now consider $\delta<K_{1}^{3} \mathbb{E}$ and assume $\delta$ is a cardinal w.l.o.g. Then as above there exists $\psi: \mathbb{R} \rightarrow{ }^{\delta \times \delta}$ 2, but there is a map

$$
\begin{aligned}
x: & { }^{\delta x \delta_{2} \rightarrow} \delta^{+} \text {given by } \\
x(A) & = \begin{cases}0 . t .(A), & \text { if } A \subseteq \delta \times \delta \text { is a wellordering, } \\
0 & , \text { otherwise. }\end{cases}
\end{aligned}
$$

So $x \circ \psi: \mathbb{R} \rightarrow \delta^{+}$and $\delta^{+}<K_{1}^{3} \mathbb{E}$. This completes the proof the $K_{1}{ }^{3} \mathbb{E}$ is weakly inaccessible

Under $A D$ (i.e. without the $A C$ ) the labyrinth, of so called 'large cardinal properties' becomes less manageable. For example, $A D \rightarrow\left\rangle_{1}\right.$, is measurable', but $\lambda_{1}$, is $\prod_{1} 1$ describable and a suchcessor。*) Nevertheless, assuming $A D K_{1}{ }^{3}$ 正 may well be measurable or satisfy $K \rightarrow(K)^{2}$ (weak compactness) etc..
*) Contrast this with the result under AC that every measurable is inaccessible and $\Pi_{1}^{2}$-indescribable.

## § 3 Selection under $A D$ 。

Section § 2 demonstrates the 'weakness' of the ordinals under $A D$. An example of previous evidence for this is the ordinal additivity of Lebesgue measure (i.e. $\lambda$ is $\gamma$-additive for every $\gamma \in O R$ ). Selection over ordinals shall be yet another. Definition 3: Let $Z \in L{ }_{K_{1}}[\mathbb{E}]$ and say that we have selection över $Z$, if for all $R(\vec{x}, b) \in \underset{\sim}{R E}$, let

$$
Q(\vec{x}) \equiv \exists b \in Z R(\vec{x}, b), \quad \text { then }
$$

$Q(\vec{x}) \in \underset{\sim}{R E}$.
Some known results are:

1) We do not have selection over $2^{\omega}$. (Moschovakis [1967]);
2) Selection over $\omega$ (Gandy [1962]).

An immediate result of 1) is,
Proposition 4: ( $V=I$ ) Let $\gamma \in O R$ s.t. $\left\langle_{1} \leq \gamma<K_{1}^{3} \mathbb{E}\right.$, then we do not have selection over $\gamma$.

and the $L$-wellordering $<{ }_{L} \prod_{\mathbb{R}^{I}} \times \mathbb{R}^{I}$ to show that we have selection over $2^{\omega}$ contradicting 1).

This 'weakness' of the ordinals under $A D$ is demonstrated by the following selection theorem

Theorem 5: Assume $A D$ and let $\gamma<K_{1}^{3} \mathbb{E}$. Then we have selection over $\gamma$.
pf let $\leq \in \Lambda^{3} \mathbb{E}$ with length $\gamma$ then the map $\varphi_{\gamma}^{*}: \mathbb{R} \rightarrow \gamma_{2}$ using the universal ${\underset{\sim}{1}}_{\mathcal{1}}(\leq)$ set $G$ given by：

$$
\varphi_{\gamma}^{*}(\alpha)= \begin{cases}A, & \text { if } G_{\alpha}=\operatorname{Cod}(A ; \leq), A \subseteq \gamma \\ \varnothing, & \text { otherwise }\end{cases}
$$

is recursive in ${ }^{3} \mathbb{E}, \leq$ ．With the power set of $\gamma$ we can now carry out a Harrington－MacQueen［1976］－style argument to show that we have selection over $\gamma$ 。

A selection result which is part of the folklore（it was noticed by A．S．Kechris and D．Normann）for ${ }^{3} \mathbb{E}$ under $A D$ is that ordinary uniformization holds（this actually only uses deter－ minacy for sets of reals recursive in ${ }^{3} \mathbb{E}$ and a real $-\Lambda^{{ }^{3} \mathbb{E}}$ ）。 If $Z \subseteq 2^{\omega}$ and $Z \leq{ }^{3} \mathbb{E}$ a for some real $a$ ，then uniformly in a we can choose an element of $Z$（assuming $Z \neq \varnothing$ ）．The proof uses a scale on $Z$ recursive in ${ }^{3} \mathbb{E}$ ，a．

Theorem 6：$\left(\operatorname{Det}\left(\Lambda^{{ }^{3}} \mathbb{E}\right)\right)$ There exists a $S: \mathbb{R} \rightarrow \mathbb{R}$ recursive in ${ }^{3} \mathbb{E}$ such that if $B \subseteq \mathbb{R}$ and $b \leq{ }^{3} \mathbb{E}, a$ for some $a \in 2^{\omega}$ ，then

$$
\begin{gathered}
B \neq \varnothing \rightarrow S(a) \downarrow \text { and } \\
S(a) \in B .
\end{gathered}
$$

proof（sketch）For the theory of scales and their construc－ tion using determinacy the reader is directed to Moschovakis 「1980 T． Uniformly by transfinite recursion on the height of a computation construct a scale $\varphi=\left\{\varphi_{n}\right\}_{n} \epsilon_{\omega}$ on

$$
C=\left\{\langle e, \vec{a}, n\rangle \mid\{e\}\left({ }^{3} \mathbb{E}, \vec{a}\right) \downarrow \wedge\{e\}\left({ }^{3} \mathbb{E}, \vec{a}\right)=n\right\},
$$

the set of computation tuples, where a computation tuple $\sigma$ is of the form:

$$
\sigma=\langle e, \vec{a}, n\rangle \quad e, n \in \omega
$$

and $\vec{a}$ a finite vector of reals. For pairs $\sigma, \tau \in C$ we proceed by transfinite recursion on

$$
\begin{gathered}
\gamma=\max (\|\sigma\|,\|\tau\|), \text { where } \\
\|\sigma\|=\left\{\begin{array}{l}
\text { height of the computation given by } \\
\sigma, \text { if } \sigma \in C \\
\infty, \text { otherwise }
\end{array}\right.
\end{gathered}
$$

to define prewellorderings $\leq_{i}$. A standard argument will then show that a small change in the associated norms $\varphi_{i}$ gives a scale on $C$.

Computations of height 0 are given in a $\Sigma_{1}^{0}$ way and hence have a scale. Let $C_{\gamma}$ denote computations of height $\gamma$ and $C_{\leq \gamma}$ those of height less than or equal to $\gamma$ 。 If $\left\{\varphi_{i}^{\gamma}\right\}_{i \in \omega}$ are the norms given so far as $C_{\leq \gamma} \backslash C_{\gamma}$ we need only extend them to $C_{\gamma}$.

For $\sigma, \tau \in C_{\gamma}$ we order them $\sigma \leq i^{\top}$ by least index $\left((\sigma)_{0}\right)$ and value $(\sigma)_{l h(\sigma)}$ in that order, given by clopen sets and then proceed by induction on the schemata S1-S9 of Kleene. We consider only an application of S 8 :

$$
\begin{aligned}
&\{e\}(\vec{a})={ }^{3} \mathbb{E}(\lambda x\{e\}(\vec{a}, x)) \text { which satisfies: } \\
&\}_{\mathbb{E}}\left(\lambda x\left\{e^{\prime}\right\}(\vec{a}, x)\right)= \begin{cases}0, & \text { if } \forall b \in w_{w}\left\{e^{\prime}\right\}(\vec{a}, b)=0 \\
1, & \text { if } \exists b \forall c \exists n\end{cases} \\
& {\left[\{e\}(\vec{a}, c)=n \wedge\left\{e^{\prime}\right\}(\vec{a}, b) \neq 0\right] }
\end{aligned}
$$

corresponding to $\Pi_{1}^{1}\left(C_{<\gamma}\right)$ and $\Sigma_{2}^{1}\left(C_{<\gamma}\right)$ respectively. In both
cases the arguments in Moschovakis [1980, pp. 310-17] yield the necessary norms for defining $\leq_{i}$ here.

These norms can be shown to give a scale on $C$ using the recursion theorem $\bar{\varphi}=\left\{\varphi_{n}\right\}_{n} \in_{\omega}$.

Remark Notice that by the definition of $\left\{\leq_{i}\right\}_{i \in \omega}$ if $\left\{x_{i}\right\}_{i \in \omega} \subseteq C$ and

$$
\lim _{i} x_{i}=x \text { such that }
$$

for all $n$

$$
\varphi_{n}\left(x_{i}\right) \text { is constant for all large } i \text {, }
$$

then a tail if the sequence lies on the same level of each $\leq_{i}$ by definition of $s_{i}$. A straightforward induction on $K_{1}^{3} \mathbb{E}$ shows $\lim _{i} x_{i}=x \in C$.

To define the uniformizing function $S$ : take an index for $\bar{\varphi}$ and given $a \in 2^{\omega}$ such that

$$
\mathrm{B} \leq{ }^{3} \mathbb{E}, \mathrm{a} \text { and } \mathrm{B} \neq \varnothing \text {. }
$$

Recursively in a compute $B^{*} \subseteq B$ minimizing the height of computation, i。e. if

$$
\begin{aligned}
& \{e\}(a, b)=\left\{\begin{array}{l}
0, \text { if } b \in B \\
1, \text { if } b \notin B 。 \text { Now let }
\end{array}\right. \\
& \alpha=\mu_{\gamma} \in O R \exists b[\{e\}(a, b)=0 \wedge \\
& \quad\|\langle e, a, b, 0\rangle\|=\gamma] \quad \text { (recall } B \neq \varnothing),
\end{aligned}
$$

and let $B^{*}=\{b \in B \mid\|\langle e, a, b, 0\rangle\|=\alpha\}$. The index for $\bar{\varphi}$ gives a scale on computations of the same height i.e. $B^{*}$, call it $\bar{\varphi}_{\alpha}=\left\{\varphi_{\alpha, n}\right\}_{n \in \omega^{\circ}}$ If we now compute

$$
\left\{b \in B^{*} \mid \varphi_{a, o}(b) \text { is minimal }\right\} \text {, it }
$$

will be a singleton by the definition of $\bar{\varphi}$ and gives the value if $S(\langle e, a\rangle)$.

If $\Lambda$ is a class of subsets of $\mathbb{R}$ and we write:

Scale ( $\Lambda$ ) $\Leftrightarrow$ for all $Z \in \Lambda$ there is a scale on $Z$ defined by some $\omega \in \Lambda$,
then we have shown,

Corollary 7: Scale (RE)
pf use an index for $\bar{\varphi}$ on $C$ and an index for any $\underset{\sim}{R E}$ class to get an $\underset{\sim}{\mathrm{RE}}$ scale on that class.

Remark The proof of Theorem 6 is eased by the fact that $C$ is given by a positive monotone inductive definition. If one instead works with Harringtons [1973] representation of the sets of reals recursive in ${ }^{3} \mathbb{E}$ and a real i.e。

$$
\mathrm{I}_{K_{1}} \mathbb{E}^{\left(2^{(w)}\right), \quad \text { then the }}
$$

fact that levels here are given by first order definability makes the inherent positivity of the inductive definition less apparent.

The Moschovakis Lemma of $\S 1$ is a powerful tool for analyzing $\theta\left(\mathbb{R}_{2}\right)$ under $A D$. The obvious strengthening of $M L$, even in the absence of definability considerations, implies $\neg A D$ 。

Proposition 8: Let $\leq$ be a prewellordering of $\mathbb{R}$ of length $\eta \geq \lambda_{1}$, and let
$f: \eta \rightarrow \mathbb{R}_{2}$ sot. for uncountably many $\gamma<\eta$ $f(\gamma) \neq \varnothing$. If $\exists g \cdot \eta \rightarrow \mathbb{R}_{2}$ a choice subfunction of $f$ such that $\forall \gamma<\eta$

$$
f(\gamma) \neq \varnothing \rightarrow g(\gamma) \text { is a singleton, then } \sim A D \text {. }
$$

pf the existence of $g$ would yield an uncountable wellordered set of reals $W$. Now define from $W$ and uncountable set of reals with no perfect subset, contradicting AD.
§ 4 Degree Theory

$$
\begin{aligned}
& \text { If } A_{K_{1}}^{A_{\mathbb{E}}}\left(2^{\omega}\right) \text { is } \underset{\sim}{R E}, \text { then the degree of } A \text { is: } \\
& \left\{B \subseteq L_{K_{1}}{ }^{\left.\left(2^{(\omega)}\right) \mid \exists a, b \in 2^{\omega} \quad A \leq{ }^{3} \mathbb{E}, B, a \wedge B \leq{ }^{3} \mathbb{E}, A, b\right\} .}\right.
\end{aligned}
$$

The degrees under the induced ordering form an upper semi-lattice and we say that $A \in \underset{\sim}{R E}$ is complete, if for all $B \in R E$ there exists $b \in 2^{\omega}$ such that $B \leq^{3} \mathbb{E}, A, b$ 。 The degree structure is said to be trivial if every $A \in \underset{\sim}{R E}$ is either complete or $\underset{\sim}{R E C}$. We say that $A \subseteq I_{K_{1}} \mathbb{E}^{\left(2^{w}\right)}$ is regular (amenable) if $\forall Z \in I_{K_{1}}{ }^{\left({ }^{(2 w}\right)}$

$$
A \cap Z \in L_{K_{1}} \mathbb{E}^{\left(Z^{W}\right)}
$$

It had been remarked (cf. Normann [1979]) that AD implied that the degree structure for ${ }^{3} \mathbb{E}$ is trivial. We extend that result here to show any regular $\underset{\sim}{\mathrm{RE}} \mathrm{A}$ is $\underset{\sim}{\mathrm{REC}}$ under AD 。 Theorem 9: (AD) If $A \subseteq I_{K_{1}}\left(2^{\omega}, 3^{3} \mathbb{F}\right)$ is $R E$ and regular, then $A$
is REC. Corollary 10: (AD) Any RE subset of $\mathrm{I}_{\mathrm{K}_{1}}\left(2^{(\omega)},{ }^{3} \mathbb{F}\right)$ is either
REC or complete.
proof (Cor.) by the theorem $A$ is regular, then $A$ is REC so it suffices to show that if $A$ is not regular, then $A$ is complete. We show this by showing that there is a computation in $A$ with height $K_{1}^{3} \mathbb{F}$.


$$
A \cap I_{\gamma}\left(2^{(\omega)},{ }^{3} \mathbb{F}\right) \in L_{K_{1}}{ }_{\mathbb{F}}\left(2^{\omega},{ }^{3} \mathbb{F}\right)
$$

Remark In what follows we regard reduction procedures on the indexical set (Sacks [1980]) or computations uniform in indices for sets (Normand [1979]): where $X \in L_{K_{1}}\left({ }^{3} \mathbb{F}, 2^{(w)}\right.$ is indexical, if
$\exists I_{X} \subseteq 2^{\omega}$ sot.
(i) $I_{X} \neq \varnothing \wedge I_{X} \leq{ }_{{ }_{X}} X$ and
(ii) $\left(\forall a \in I_{X}\right)\left[\left\{(a)_{0}\right\}\left({ }^{3} \mathbb{F},(a)_{1}\right) \downarrow\right.$ and
$\left.\left\{\left(a_{0}\right)\right\}\left({ }^{3} \mathbb{F},(a)_{1}\right)=X\right]$ a set of
indices for the set $X$ 。

To complete the proof of the corollary let $\gamma_{0}$ be least witness to $A$ not regular, ide

$$
A \cap L_{Y_{0}}\left({ }^{3} \mathbb{F}, 2^{w}\right) \underset{K_{1}}{\mathcal{K}_{1}}\left({ }^{3} \mathbb{F}, 2^{(w)}\right)
$$

and define $f: I_{\gamma_{0}}\left({ }^{3} \mathbb{F}, 2^{w}\right) \rightarrow K_{1}{ }^{3} \mathbb{F}$ by

$$
f(X)= \begin{cases}\left|\{c\}\left({ }^{3} \mathbb{F}, X\right)\right|, & \text { if } X \in A \\ \varnothing & \text { otherwise, where }\end{cases}
$$

$c$ is the index witnessing $A$ RE．Then $f$ is REC in $3_{\mathbb{F}, A}$ and

$$
\sup _{X \in I_{\gamma_{0}}} \frac{f}{\left({ }^{3} \mathbb{F}, 2^{W}\right)}=K_{1}^{3} \mathbb{F} \text { by the }
$$

choice of $\gamma_{0}$ ，as desired．Cor。
proof（theorem）given $A$ RE via $c \in 2^{\omega}$ consider the game $G_{A}$＊where

$$
\begin{aligned}
& A^{*}=\left\{\langle a, b\rangle \in 2^{w} \times 2^{w} \mid a \text { is an index } \wedge\right. \\
& {[b \text { not index } \vee(b \text { is index } \wedge} \\
&\left.A \cap I_{|a|}\left({ }^{3} \mathbb{F}, 2^{(w)}\right) \neq A|b| \cap L_{|a|}\left({ }^{3} \mathbb{F}, 2^{w}\right)\right\},
\end{aligned}
$$

where $A_{\alpha}$ for $\alpha<K_{1}^{3} \mathbb{E}$ is
$\left\{X \in L_{\alpha}\left({ }^{3} \mathbb{F}, 2^{(\omega)}\right)\left|I_{\alpha}\left({ }^{3} \mathbb{F}, 2^{(\omega)}\right)\right|=\{c\}\left({ }^{3} \mathbb{F}, \mathbb{X}\right) \downarrow\right\}$ ，the portion of
A＇s enumeration complete by＇stage＇$\alpha$ 。 Hence Player I builds a and Player II builds b。

By $A D G_{A} *$ is determined：

Case 1：Player I has a winning strategy $\sigma$ ，then

$$
\sigma^{\prime \prime}\left[2^{w}\right]=\left\{\sigma^{*} b \mid b \in 2^{w}\right\} \subseteq \text { indices }
$$

and $\sigma \in 2^{\omega}$ gives $\sigma^{\prime \prime}\left[2^{\omega}\right] \leq 3_{\mathbb{F}} \sigma_{0}$ ．By the bounding principle

$$
\sup _{c \in \sigma^{\prime \prime}\left[2^{w}\right]}|c|=\delta_{0}<K_{1}^{3} \mathbb{F} \text { for some } \delta_{0} .
$$

If there is no $\gamma$ with $\delta_{0} \leq \gamma<K_{1}^{3}$ IF A．t．

$$
A \cap I_{\delta_{0}}\left({ }^{3} \mathbb{F}, 2^{(w)}\right)=A_{\gamma} \cap I_{\delta_{0}}\left({ }^{3} \mathbb{F}, 2^{w}\right)
$$

then $A$ is complete since $A \cap I_{\delta_{0}}\left({ }^{3} \mathbb{F}, 2^{w}\right) \in I_{K_{1}}\left(2^{w},{ }^{3} \mathbb{F}\right)$（by reg。 and an argument similar to that in the Corollary and if $A$ not regular，then $A$ complete and we＇re done。）

If we let $b$ be an index for any such $\gamma$ ，then II wins playing $b$ against $\sigma$ ，a contradiction with the choice of $\sigma$ ． Thus A is complete RE。

Case 2：Player II has a winning strategy $\sigma$ ：then for all indices a

$$
\begin{gathered}
r^{*} a \text { is an index and } \\
A \cap L_{|a|}\left({ }^{3} \mathbb{F}, 2^{w}\right)=A_{\left|\sigma^{*} a\right|} \cap L_{|a|}\left({ }^{3} \mathbb{F}, 2^{w}\right) .
\end{gathered}
$$

We claim that $A$ is $R E C$ in ${ }^{3} \mathbb{F}, c, \sigma$ where $c$ was the index defining $A$ ：
 in $\sigma$ compute

$$
\sup _{c \in \sigma " I_{X}}(|c|+17)=\gamma<K_{1}^{3} \mathbb{F} \text {. By the choice of } \sigma
$$

$$
X \in A \Leftrightarrow X \in A_{\gamma} \text { and so } A \text { is } R E C \text { in } 3_{\mathbb{F}}, c, \sigma
$$

as desired．

Remark Sacks［1980，Sacks and Griffor］has shown using a well－ ordering of reals，that there exists a regular complete $\underset{\sim}{\mathrm{RE}}$ class。 The foregoing indicates that this assumption is probably necessary。

In closing we employ the tools of § 1 to describe the degrees of arbitrary $A \subseteq K_{1}{ }^{3} \mathbb{E}$ under $A D$ 。
Theorem 12：（AD）Let $A$ be a subset of $K_{1}^{3} \mathbb{E}$ ，then $A$ is recúr－ sive in $3^{\mathbb{E}}$ and a real．
proof We require a lemma guaranteeing the regularity of $A$ ， Lemma 13：If $A \subseteq K_{1}^{3} \mathbb{E}$ ，then $A$ is regular．
pf（lemma）：By Cor 1，§ 1 we have $\forall \gamma<K_{1}^{3} \mathbb{E}$ ，

$$
\operatorname{Cod}(A \cap \gamma ; \leq) \in{\underset{\sim}{1}}_{1}^{1}(\leq), \quad \text { where }
$$

$\leq$ is in $\Lambda^{3} \mathbb{E}$ of length $\gamma_{0}$ Then clearly $\operatorname{Cod}(A \cap \gamma ; \leq)$ is recursive in ${ }^{3} \mathbb{E}, \leq 。$

To complete the proof of the theorem，let $\alpha<K_{1}^{3} \mathbb{E}$ and

$$
\begin{gathered}
A_{\alpha}=\left\{a \in 2^{W} \mid a\right. \text { is an index and } \\
|a| \in \alpha \cap A\} .
\end{gathered}
$$

We consider the following game：As usual I and II play reals a，b respectively．Put

II wins iff a is not an index or

$$
\begin{aligned}
& \text { (a is an index and } b \text { is a code for } \\
& \text { some } A_{\beta} \text { for } \beta \geq|a| \text { ). }
\end{aligned}
$$

This is a so－called Solovay－game and hence $I$ cannot have a winning strategy，if $A$ is regular．Thus $A$ would be recursive in ${ }^{3} \mathbb{E}, \tau$ for $\tau$ any winning strategy for II。

By Lemma 13 every $A \subseteq K_{1}^{3} \mathbb{E}$ is regular and，hence，$A$ is recursive in $3_{\mathbb{E}, a}$ for some real $a$ 。

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