SOME CONSEQUENCES OF AD FOR
KLEENE RECURSION IN $\mathcal{E}$

by

E.R. Griffor 1)
Inst. of Math., University of Oslo

1) Partly supported by grant from the National Science Foundation of the U.S.A.
§ 0 Introduction

In this paper we derive some consequences of AD (axiom of Determinacy) for Kleene recursion in $^3\mathbb{E}$ (the type three functional giving the equality predicate for sets of reals). In § 1 we state and sketch the proof of a result of Moschovakis which is key to many of the results in subsequent sections.

A.S. Kechris asked: Does AD imply any "large cardinal properties" for $K_1$ in $L[\mathbb{R}]$? In § 2 we show that $K_1$ is weakly inaccessible under AD, "larger" than measurability need imply ($\mathfrak{S}_1$ is measurable under AD).

A recurrent theme in recursion in higher types and E-recursion is that of selection theorems, i.e. for which sets $Z$ are the classes semirecursive (RE) in $^3\mathbb{E}$ and a real closed under the quantifier $\exists a \in Z$? Under AD we show that these classes are closed under the quantifier $\exists a < \gamma$ for every $\gamma < K_1$. This should be contrasted with the situation under $V = L$ where $K_1 > \mathfrak{S}_1$, but the RE classes are not closed under $\exists a < \gamma$ for any $\gamma$ such that $\mathfrak{S}_1 \leq \gamma < K_1$. We also sketch the proof of a part of the folklore using AD that we can select an element from a set of reals recursive in $^3\mathbb{E}$, a uniformly in $\alpha$ for $\alpha \in 2^\omega$. These results on selection appear in § 3.

It was known that AD implied that the structure of the RE degrees was trivial, i.e. an RE class is either complete RE or REC (recursive). In § 4 we strengthen this result to show that AD
implies that any regular RE class is REC, hence there is no regular complete RE class. This adds some force to the conjecture that there is a model of ZF (assuming ZF is consistent) in which any regular RE set is REC (and hence that the degree structure is trivial). Also in § 4 we employ the tools of § 2 to describe the degrees of subsets of $K_1^3$ under AD. The result here is that any $A \subseteq K_1^3$ is recursive in $3^E$ and a real.

The author is grateful to Dag Normann for many helpful discussions.

§ 1. Prewellorderings and the Moschovakis Lemma.

Following Moschovakis [1980], a prewellordering of $\mathbb{R}$ (with field $\mathbb{R}$) is a relation $\leq$ on $\mathbb{R}$ such that for some ordinal $\delta$, some surjection

$$\varphi : \mathbb{R} \rightarrow \delta$$

and all $x, y \in \mathbb{R}$,

$$x \leq y \iff \varphi(x) \leq \varphi(y),$$

where $\leq$ on the right side is the order relation on $\text{ORD}$ (ordinals).

It is immediate that such $\varphi$ and $\delta$ are unique when they exist; we call $\delta$ the length of $\leq$ and $\varphi$ the canonical surjection (of $\leq$).

A relation is a prewellordering iff it is reflexive, transitive, connected and well-founded (equivalently (DC) has no infinite descending chains.). Let $\Lambda$ be a class of subsets of $\mathbb{R}$ and assume $\Lambda$ contains all singletons and is closed under continuous substitution and $\rightarrow$. An ordinal $\delta$ is realized in $\Lambda$.
if there is a prewellordering \( \leq \) of some subset of \( \mathbb{R} \) in \( A \) with length \( \delta \). These closure properties of \( A \) imply:

(i) \( \delta \) realized in \( A \) and \( \zeta < \delta \Rightarrow \zeta \) is realized in \( A \);

(ii) \( \delta \) realized in \( A \) \( \Rightarrow \) \( \delta + 1 \) is realized in \( A \); and

(iii) \( \delta > 0 \land \delta \) realized in \( A \) \( \Rightarrow \delta \) is the length of some prewellordering of \( \mathbb{R} \) in \( A \).

Let \( \theta(A) = \sup \{ \delta \mid \delta \) is realized in \( A \} \).

**Remark** Let \( A^3 = \{ x \mid x \) is a prewellordering of \( \mathbb{R} \land x \) is recursive in \( ^3 \) and a real \}, then \( \theta(A) = K_A^3 \) and \( A \) clearly satisfies the closure properties above.

If the length of \( \leq \) is \( \delta \) and \( \varphi : \text{fld}(\leq) \rightarrow \delta \) is the canonical surjection, then \( \varphi \) gives a 'coding' of \( \delta \) in the space \( \chi \) (a product space on \( \mathbb{R} \)), i.e. we can think of each \( x \in \text{fld}(\leq) \) such that \( \varphi(x) = \eta < \delta \) as a code or name for \( \eta \).

If \( Y \) is such a product space and \( f : \delta \rightarrow Y_2 \) is a function on \( \delta \) to subsets of \( Y \), we can represent it by a subset of \( \chi \times Y \) as follows:

\[
\text{Cod}(f; \leq) = \{ (x, y) : x \leq x \land y \in f(\varphi(x)) \}.
\]

**Definition.** Suppose \( f : \delta \rightarrow Y_2 \) is a function. A **choice sub-function** of \( f \) is any \( g : \delta \rightarrow Y_2 \) such that for all \( \eta < \delta \):

\[
g(\eta) \subseteq f(\eta),
\]

\[
f(\eta) \neq \emptyset \Rightarrow g(\eta) \neq \emptyset, \text{ i.e.}
\]

\( g(\eta) \) chooses a non-empty subset of \( f(\eta) \) (provided \( f(\eta) \neq \emptyset \)).
The following lemma is the main tool of this note and is due to Moschovakis [1970].

The Moschovakis Lemma (ML) assume AD.

Let \( \leq \) be a prewellordering with field a subset of some \( \chi \) and length \( \delta \), let \( f : \delta \to Y_2 \) be a function. Then there exists a choice subfunction \( g \) of \( f \) such that

\[
\text{Cod}(g; \leq) \text{ is a } \Sigma_1^1(\leq) \text{ subset of } \chi \times Y.
\]

pf we give the main idea of the proof - the case we shall be interested in is \( \mathcal{R} = \chi = Y \). For each \( \zeta < \delta \) let \( f_\zeta \) be defined by:

\[
f_\zeta(\eta) = f(\eta), \text{ if } \eta < \zeta \\
\emptyset, \text{ if } \zeta \leq \eta < \delta.
\]

Suppose there is some \( \zeta \leq \delta \) such that \( f_\zeta \) does not have a choice subfunction with Cod in \( \Sigma_1^1(\leq) \), let \( \lambda \) be the last such \( \zeta \). \( \Sigma_1^1(\leq) \) is parameterized, hence the class of \( \Sigma_1^1(\leq) \) subsets of \( \chi \times Y \) is parameterized let

\[
G \subseteq \mathcal{R} \times \chi \times Y \text{ be fixed universal set for it. As usual, let}
\]

\[
G_\alpha = \{(x,y)|(a,x,y) \in G\} \text{ and}
\]

consider the game: I plays \( \alpha \) and II plays \( \beta \), then

II wins \( \Leftrightarrow \exists \eta \left[ g_\eta \text{ is a choice subfunction of } f_\eta \wedge G_\alpha = \text{Cod}(g_\eta; \leq) \vee (\exists \eta < \lambda) \right. \]

\[
(\exists \zeta < \lambda)(\exists g_\eta)(\exists g_\zeta) \left[ g_\eta \text{ is a choice subfn of } f_\eta \wedge g_\zeta \text{ is a choice subfn of } f_\zeta \wedge \eta < \zeta \wedge G_\alpha = \text{Cod}(g_\eta; \leq) \wedge G_\beta = \text{Cod}(g_\zeta; \leq) \right].
\]
If we think of $a$ as a code of a function $g$ when $G_a = \text{Cod}(g;\leq)$ then $\Pi$ wins if either $I$ does not code a choice subfunction of an initial segment of $f$ on $I$ does and $\Pi$ codes a choice subfn of a longer initial segment of $f$.

Moschovakis shows that the existence of a winning strategy for either $I$ or $\Pi$ (AD) yields a choice subfunction of $f_\lambda, g_\lambda$ with $\text{Cod}(g_\lambda;\leq) \in \Sigma^1_1 (\leq)$, contradicting the choice of $\lambda$. For details of the proof see Moschovakis [1970].

**Remark** Note that if $\leq$ is a prewellordering on $\mathbb{R}$ s.t. $\leq \in \Lambda^3 \Xi$ and

$$\text{Cod}(g;\leq) \in \Sigma^1_1 (\leq) \text{ then } \text{Cod}(g;\leq) \in \Lambda^3 \Xi$$

As a corollary of the Moschovakis Lemma we have:

**Corollary 1** Assume AD. Let $\leq$ be a prewellordering on $\mathbb{R}$ with length $\delta$ and let $A \leq \delta$. Then

$$\text{Cod}(A;\leq) \in \Sigma^1_1 (\leq)$$

**pf** fix $a_0, a_1 \in \mathbb{R}$ and set

$$f(\eta) = \begin{cases} [a_0], & \text{if } A(\eta) \\ [a_1], & \text{if } A(\eta). \end{cases}$$

The only choice subfunction of $f$ is $f$ itself and hence by ML,

$$\text{Cod}(f;\leq) \in \Sigma^1_1 (\leq) \text{ and } x \in \text{Cod}(A;\leq) \Rightarrow (x, a_0) \in \text{Cod}(f;\leq)$$
§ 2  $[K_1^E]_{L[R]}$ under AD.

We will now use the tools of § 1 to answer the question of Kechris stated in the introduction positively.

**Theorem 2** Assume AD. In $L[R]$ we have that $K_1^E$ is weakly inaccessible.

**pf** it suffices to show that $K_1^E$ is a regular limit cardinal in $L[R]$. We use the notation $\delta^+(\delta \in OR)$ to denote the least cardinal greater than $\delta$ and $\delta_2 = \{f : \delta \rightarrow \{0, 1\}\}$.

We first show that $K_1^E$ is a cardinal. Suppose not and let $\eta < K_1^E$ and $f : K_1^E \rightarrow \eta$. Also let $\xi \in K_1^E$ with length $\eta$. (we can suppose $\eta$ is a cardinal).

The function $f$ induces a relation on $\eta \times \eta$, (a subset of $\eta \times \eta$), $R_f$, given by $\langle \sigma, \delta \rangle \in R_f \iff f^{-1}(\sigma) \leq f^{-1}(\delta)$.

$R_f$ is a well-founded relation on $\eta^2$ of height $K_1^E$. By an easy generalization of the corollary to ML to n-ary relations

$$\text{Cod}(R_f; \leq) \in \Sigma^1\subset (\leq) \text{ and hence }$$

"\text{Cod}(R_f; \leq) \in \Lambda^3E\" which is absurd since the bounding principle then implies that the height of $R_f$ is less than $K_1^E$.

Suppose now that $K_1^E$ is singular:
let \( \eta < K_1^3 \) be a cardinal s.t. there exists

\[
f : \eta \xrightarrow{\text{increas}} K_1^3.
\]

Let \( \xi \in A^3 \) of length \( \eta \) and for each \( \gamma < K_1^3 \) define

\[
I_\gamma = \{ b \in R \mid b \text{ is an index for } \gamma \}
\]

(i.e. be codes a convergent computation in \( 3^E \) of height \( \gamma \)). Define \( h : \eta \to R^2 \) by for \( \delta < \eta \):

\[
h(\delta) = I_f(\delta). \text{ By ML there exists a choice sub-
\]

function \( g : \eta \to R^2 \) of \( h \) such that

\[
\text{Cod}(g; \leq) \in \Sigma^1_1(\leq). \text{ Note that}
\]

for \( \gamma < K_1^3 \), \( I_\gamma \neq \emptyset \) and hence for \( \delta < \eta \)

\[
g(\delta) \subseteq h(\delta) = I_f(\delta) \quad \text{and}
\]

\[
g(\delta) \neq \emptyset.
\]

Now define \( \tau : \eta \to K_1^3 \) by for \( \delta < \eta \):

\[
\tau(\delta) = \text{supremum}(\|b\|_3 \mid b \in g(\delta)).
\]

Since \( \text{Cod}(g; \leq) \in \Sigma^1_1(\leq) \), \( \text{Cod}(g; \leq) \) is recursive in \( 3^E, \leq \) and clearly \( \forall \delta < \eta \)

\[
\tau(\delta) = f(\delta), \text{ contradicting}
\]

the bounding principle again.

It remains only to show that \( K_1^3 \) is a limit cardinal. This will follow by an argument first noticed by H. Friedman [1987] for \( \theta(R_2) \). let \( \delta < K_1^3 \), then we show that \( \delta^+ < K_1^3 \): We first show given \( \delta < K_1^3 \) then there exists \( \varphi^* : R \to \delta \) such that \( \varphi^* \)
has
\[ \text{Cod}(\varphi; \leq) \in \Sigma_1^1(\leq) \]
where
\[ \leq \in \Lambda^{3E}_1 \]
of length \( \delta \). By the corollary to ML if \( A \leq \delta \) then
\[ \text{Cod}(A; \leq) \in \Sigma_1^1(\leq), \]
thus

if \( G \) is universal for \( \Sigma_1^1(\leq) \) and hence recursive in \( 3E_1 \leq \)
define for \( \alpha \in \mathbb{R} \):
\[ \varphi^*(\alpha) = \begin{cases} A & \text{if } G_\alpha = \text{Cod}(A; \leq), A \leq \delta \\ \emptyset & \text{otherwise.} \end{cases} \]

Then \( \varphi^* \) is the desired surjection.

Now consider \( \delta < K_1^{3E} \) and assume \( \delta \) is a cardinal w.l.o.g.
Then as above there exists \( \psi: \mathbb{R} \rightarrow 5 \times 5^2 \), but there is a map
\[ \chi: 5 \times 5^2 \rightarrow \delta^+ \]
given by
\[ \chi(A) = \begin{cases} \text{o.t.}(A), & \text{if } A \subseteq 5 \times 5 \text{ is a wellordering,} \\ 0, & \text{otherwise}. \end{cases} \]

So \( \chi \circ \psi: \mathbb{R} \rightarrow \delta^+ \) and \( \delta^+ < K_1^{3E} \). This completes the proof that \( K_1^{3E} \) is weakly inaccessible.

Under AD (i.e. without the AC) the labyrinth, of so called 'large cardinal properties' becomes less manageable. For example, AD \( \rightarrow \kappa_1, \) is measurable, but \( \kappa_1, \) is \( \Pi_1^1 \)-describable and a successor. *) Nevertheless, assuming AD \( K_1^{3E} \) may well be measurable or satisfy \( K \rightarrow (K)^2 \) (weak compactness) etc..

*) Contrast this with the result under AC that every measurable is inaccessible and \( \Pi_1^2 \)-indescribable.
§ 3 Selection under AD.

Section § 2 demonstrates the 'weakness' of the ordinals under AD. An example of previous evidence for this is the ordinal additivity of Lebesgue measure (i.e. \( \lambda \) is \( \gamma \)-additive for every \( \gamma \in \text{OR} \)). Selection over ordinals shall be yet another.

**Definition 3:** Let \( Z \in L^{3 \in \mathbb{R}} \) and say that we have selection \( K_1 \) over \( Z \), if for all \( R(\bar{x},b) \in \text{RE} \), let

\[
Q(\bar{x}) = \exists b \in ZR(\bar{x},b), \text{ then }
\]

\[
Q(\bar{x}) \in \text{RE}.
\]

Some known results are:

1) We do not have selection over \( 2^w \). (Moschovakis [1967]);
2) Selection over \( w \) (Gandy [1962]).

An immediate result of 1) is,

**Proposition 4:** \( (V = L) \) Let \( \gamma \in \text{OR} \) s.t. \( \hat{\gamma} \leq \gamma < K_1 \), then we do not have selection over \( \gamma \).

**Proof:** \( L^{3 \in \mathbb{R}} \) \( \models \hat{\gamma} = \hat{\gamma} \), so use \( f \in L^{3 \in \mathbb{R}} \) s.t. \( f : \gamma \leftrightarrow \), and the \( L \)-wellordering \( \hat{\gamma} \leq \hat{\gamma} \) to show that we have selection over \( 2^w \) contradicting 1).

This 'weakness' of the ordinals under AD is demonstrated by the following selection theorem

**Theorem 5:** Assume AD and let \( \gamma < K_1 \). Then we have selection over \( \gamma \).
pf let $\xi \in \mathcal{P} \gamma$ with length $\gamma$ then the map $\varphi^*_\gamma : \mathbb{R} \rightarrow 2^\gamma$ using the universal $\Sigma^1_1$ set $G$ given by:

$$
\varphi^*_\gamma(a) = \begin{cases} 
A, & \text{if } G_a = \text{Cod}(A; \leq), A \subseteq \gamma \\
\emptyset, & \text{otherwise,}
\end{cases}
$$

is recursive in $\mathcal{P} \gamma \leq$. With the power set of $\gamma$ we can now carry out a Harrington-MacQueen [1976]-style argument to show that we have selection over $\gamma$.

A selection result which is part of the folklore (it was noticed by A.S. Kechris and D. Normann) for $\mathcal{P} \gamma$ under AD is that ordinary uniformization holds (this actually only uses determinacy for sets of reals recursive in $\mathcal{P} \gamma$ and a real $\gamma \mathcal{P} \gamma$).

If $Z \subseteq 2^\omega$ and $Z \leq \mathcal{P} \gamma \alpha$ for some real $\alpha$, then uniformly in $\alpha$ we can choose an element of $Z$ (assuming $Z \not= \emptyset$). The proof uses a scale on $Z$ recursive in $\mathcal{P} \gamma \alpha$.

**Theorem 6:** $(\text{Det}(\mathcal{P} \gamma))$ There exists a $S : \mathbb{R} \rightarrow \mathbb{R}$ recursive in $\mathcal{P} \gamma$ such that if $B \subseteq \mathbb{R}$ and $b \leq \mathcal{P} \gamma \alpha$ for some $\alpha \in 2^\omega$, then

$$
B \not= \emptyset \rightarrow S(a) \downarrow \quad \text{and} \quad S(a) \in B.
$$

proof (sketch) For the theory of scales and their construction using determinacy the reader is directed to Moschovakis [1980]. Uniformly by transfinite recursion on the height of a computation construct a scale $\varphi = \{\varphi_n\}_{n \in \omega}$ on

$$
C = \{\langle e, \alpha, n \rangle | \{e\}(\mathcal{P} \gamma, \alpha) \uparrow \wedge \{e\}(\mathcal{P} \gamma, \alpha) = n\},
$$
the set of computation tuples, where a computation tuple $\sigma$ is of the form:

$$\sigma = (e, \bar{a}, n) \quad e, n \in \omega$$

and $\bar{a}$ a finite vector of reals. For pairs $\sigma, \tau \in C$ we proceed by transfinite recursion on

$$\gamma = \max(\|\sigma\|, \|\tau\|),$$

where

$$\|\sigma\| = \begin{cases} \text{height of the computation given by} \\ \sigma, \text{ if } \sigma \in C \\ \infty, \text{ otherwise,} \end{cases}$$

to define prewellorderings $\leq$. A standard argument will then show that a small change in the associated norms $\varphi_i$ gives a scale on $C$.

Computations of height 0 are given in a $\Sigma_0^0$ way and hence have a scale. Let $C_\gamma$ denote computations of height $\gamma$ and $C_{<\gamma}$ those of height less than or equal to $\gamma$. If $\{\varphi_i\}_{i \in \omega}$ are the norms given so far as $C_{<\gamma} \setminus C_\gamma$ we need only extend them to $C_\gamma$.

For $\sigma, \tau \in C_\gamma$ we order them $\sigma \leq_i \tau$ by least index $((\sigma)_0)$ and value $(\sigma)_{lh}(\sigma)$ in that order, given by clopen sets and then proceed by induction on the schemata $S1-S9$ of Kleene. We consider only an application of $S8$:

$$\{e\}(\bar{a}) = \sum (\lambda x[e](\bar{a}, x))$$

which satisfies:

$$\sum (\lambda x[e'](\bar{a}, x)) = \begin{cases} 0, \text{ if } \forall b \in \omega[w(e')(\bar{a}, b)] = 0 \\ 1, \text{ if } \exists b \forall c \exists n \\
[\{e\}(\bar{a}, c) = n \wedge \{e'(\bar{a}, b) \neq 0\}].
\end{cases}$$

corresponding to $\Pi_1(C_{<\gamma})$ and $\Sigma_2(C_{<\gamma})$ respectively. In both
cases the arguments in Moschovakis [1980, pp. 310-17] yield the necessary norms for defining \( \preceq_i \) here.

These norms can be shown to give a scale on \( C \) using the recursion theorem \( \overline{\varphi} = \{ \varphi_n \}_{n \in \omega} \).

**Remark** Notice that by the definition of \( \{ \preceq_i \}_{i \in \omega} \) if \( \{ x_i \}_{i \in \omega} \subseteq C \) and
\[
\lim_{i \to \infty} x_i = x \quad \text{such that}
\]
for all \( n \)
\[
\varphi_n(x_i) \quad \text{is constant for all large } i,
\]
then a tail if the sequence lies on the same level of each \( \preceq_i \) by definition of \( \preceq_i \). A straightforward induction on \( \delta \) shows
\[
\lim_{i \to \infty} x_i = x \in C.
\]

To define the uniformizing function \( S \): take an index for \( \overline{\varphi} \) and given \( a \in \mathbb{E} \) such that
\[
B \leq \delta \quad a \text{ and } B \neq \emptyset.
\]
Recursively in a compute \( B^* \subseteq B \) minimizing the height of computation, i.e., if
\[
\{e\}(a,b) = \begin{cases} 0, & \text{if } b \in B \\ 1, & \text{if } b \notin B. \end{cases}
\]
Now let
\[
\alpha = \mu \gamma \in \text{OR} \exists b \left[ \{e\}(a,b) = 0 \wedge \|e,a,b,0\| = \gamma \right] \quad \text{(recall } B \neq \emptyset),
\]
and let \( B^* = \{ b \in B \mid \|e,a,b,0\| = \gamma \} \). The index for \( \overline{\varphi} \) gives a scale on computations of the same height i.e. \( B^* \), call it \( \overline{\varphi}_\alpha = \{ \varphi_{\alpha, n} \}_{n \in \omega} \). If we now compute
\[
\{ b \in B^* \mid \varphi_{\alpha,0}(b) \text{ is minimal} \}, \text{ it}
\]
will be a singleton by the definition of \( \bar{\varphi} \) and gives the value if \( S(\langle e, a \rangle) \).

If \( \Lambda \) is a class of subsets of \( \mathbb{R} \) and we write:

\[
\text{Scale } (\Lambda) \leftrightarrow \text{ for all } Z \in \Lambda \text{ there is a scale on } Z \\
\text{defined by some } w \in \Lambda,
\]

then we have shown,

**Corollary 7:** Scale \( (\mathbb{R}) \)

pf use an index for \( \bar{\varphi} \) on \( C \) and an index for any \( \mathbb{R} \)
class to get an \( \mathbb{R} \) scale on that class.

**Remark** The proof of Theorem 6 is eased by the fact that \( C \) is
given by a positive monotone inductive definition. If one instead
works with Harringtons [1973] representation of the sets of reals
recursive in \( 3^E \) and a real i.e.

\[
\Sigma K_1^E (\mathbb{R}^w), \text{ then the}
\]

fact that levels here are given by first order definability makes
the inherent positivity of the inductive definition less apparent.

The Moschovakis Lemma of § 1 is a powerful tool for analyzing
\( \theta(\mathbb{R}^2) \) under AD. The obvious strengthening of ML, even in the
absence of definability considerations, implies \( \neg AD \).

**Proposition 8:** Let \( \leq \) be a prewellordering of \( \mathbb{R} \) of length
\( \eta \geq \aleph_1 \), and let

\[
f : \eta \to \mathbb{R}^2 \text{ s.t. for uncountably many } \gamma < \eta \\
f(\gamma) \neq \emptyset. \text{ If } \exists g : \eta \to \mathbb{R}^2 \text{ a choice subfunction of } f \text{ such that}
\]

\[
\forall \gamma < \eta
\]
\[ f(\gamma) \neq \emptyset \rightarrow g(\gamma) \] is a singleton, then \( \neg \text{AD} \).

pf the existence of \( g \) would yield an uncountable well-ordered set of reals \( W \). Now define from \( W \) and uncountable set of reals with no perfect subset, contradicting \( \text{AD} \).

§ 4 Degree Theory

If \( A \subseteq L_{\mathbb{3}(2^\omega)} \) is \( \text{RE} \), then the degree of \( A \) is:

\[
\{ B \subseteq L_{\mathbb{3}(2^\omega)} \mid \exists a, b \in 2^\omega \quad A \leq \mathbb{3}, B, a \land B \leq \mathbb{3}, A, b \}.
\]

The degrees under the induced ordering form an upper semi-lattice and we say that \( A \in \mathbb{R} \) is complete, if for all \( B \in \mathbb{R} \) there exists \( b \in 2^\omega \) such that \( B \leq \mathbb{3}, A, b \). The degree structure is said to be trivial if every \( A \in \mathbb{R} \) is either complete or \( \text{REC} \). We say that \( A \subseteq L_{\mathbb{3}(2^\omega)} \) is regular (amenable) if \( \forall Z \subseteq L_{\mathbb{3}(2^\omega)} \)

\[
A \cap Z \subseteq L_{\mathbb{3}(2^\omega)}.
\]

It had been remarked (cf. Normann [1979]) that \( \text{AD} \) implied that the degree structure for \( \mathbb{3} \) is trivial. We extend that result here to show any regular \( \text{RE} \) \( A \) is \( \text{REC} \) under \( \text{AD} \).

Theorem 9: (AD) If \( A \subseteq L_{\mathbb{3}(2^\omega, \mathbb{F})} \) is \( \text{RE} \) and regular, then \( A \) is \( \text{REC} \).

Corollary 10: (AD) Any \( \text{RE} \) subset of \( L_{\mathbb{3}(2^\omega, \mathbb{F})} \) is either \( \text{REC} \) or complete.
proof (Cor.) by the theorem $A$ is regular, then $A$ is REC so it suffices to show that if $A$ is not regular, then $A$ is complete. We show this by showing that there is a computation in $A$ with height $K_1$.

Definition 11: A RE a subset of $L_{K_1} (2^w, 3^F)$ is regular, if $\forall \gamma < K_1$

$$A \cap L_{\gamma} (2^w, 3^F) \in L_{K_1} (2^w, 3^F).$$

Remark In what follows we regard reduction procedures on the indexical set (Sacks [1980]) or computations uniform in indices for sets (Normann [1979]): where $X \in L_{K_1} (3^F, 2^w)$ is indexical, if $\exists I_X \subseteq 2^w$ s.t.

(i) $I_X \neq \emptyset \land I_X \subseteq 3^F_X$ and

(ii) $(\forall a \in I_X) [(a_0) (3^F, (a)_1) \downarrow \land

\{(a_0)\} (3^F, (a)_1) = X]$ a set of indices for the set $X$.

To complete the proof of the corollary let $\gamma_0$ be least witness to $A$ not regular, i.e.

$$A \cap L_{\gamma_0} (3^F, 2^w) \notin L_{K_1} (3^F, 2^w)$$

and define $f : L_{\gamma_0} (3^F, 2^w) \rightarrow K_1$ by

$$f(X) = \begin{cases} |\{c\} (3^F, X)|, & \text{if } X \in A \\ \emptyset, & \text{otherwise, where} \end{cases}$$
c is the index witnessing \( A \) \( \text{RE} \). Then \( f \) is \( \text{REC} \) in \( 3^\mathbb{F}, A \) and

\[
\sup_{X \in L_{\gamma_0}} f(X) = K_1^3
\]

by the choice of \( \gamma_0 \), as desired. Cor.

**proof** (theorem) given \( A \) \( \text{RE} \) via \( c \in 2^\omega \) consider the game \( G_A^* \) where

\[
A^* = \{ (a, b) \in 2^\omega \times 2^\omega | a \text{ is an index } \land [b \text{ not index } \lor (b \text{ is index } \land A \cap L_{|a|}(3^\mathbb{F}, 2^\omega) \neq A_{|b|} \cap L_{|a|}(3^\mathbb{F}, 2^\omega) \},
\]

where \( A_a \) for \( a < K_1^3 \) is

\[
\{ X \in L_a(3^\mathbb{F}, 2^\omega) \mid L_a(3^\mathbb{F}, 2^\omega) = \{ c \mid c \supseteq X \} \}, \text{ the portion of} \ A's \ enumeration \ complete \ by \ 'stage' \ a. \ Hence \ Player \ I \ builds \ a \ \text{and Player II builds} \ b. \\

By \ AD \ \( G_A^* \) is determined:

**Case 1:** Player I has a winning strategy \( \sigma \), then

\[
\sigma''[2^\omega] = \{ \sigma \ * \ b | b \in 2^\omega \} \subseteq \text{indices}
\]

and \( \sigma \in 2^\omega \) gives \( \sigma''[2^\omega] \leq 3^\mathbb{F} \sigma \). By the bounding principle

\[
\sup_{c \in \sigma''[2^\omega]} |c| = \delta_0 < K_1^3
\]

for some \( \delta_0 \).

If there is no \( \gamma \) with \( \delta_0 \leq \gamma < K_1^3 \) A.t.

\[
A \cap L_{\delta_0}(3^\mathbb{F}, 2^\omega) = A \cap L_{\delta_0}(3^\mathbb{F}, 2^\omega)
\]
then $A$ is complete since $A \cap L_0^L(\mathcal{F}^3, 2^\omega) \subseteq L_{\mathcal{F}^3}(2^\omega, \mathcal{F}^3)$ (by reg.
and an argument similar to that in the Corollary and if $A$ not
regular, then $A$ complete and we're done.)

If we let $b$ be an index for any such $\gamma$, then II wins
playing $b$ against $\sigma$, a contradiction with the choice of $\sigma$.
Thus $A$ is complete RE.

Case 2: Player II has a winning strategy $\sigma$: then for all
indices $a$

$$r^*a \text{ is an index and } A \cap L_{a^\sigma}(\mathcal{F}^3, 2^\omega) = A_{\sigma^*a} \cap L_{a^\sigma}(\mathcal{F}^3, 2^\omega).$$

We claim that $A$ is REC in $\mathcal{F}^3, c, \sigma$ where $c$ was the index
defining $A$:

Given $X$ indexical in $L_{\mathcal{F}^3}(\mathcal{F}^3, 2^\omega)$, compute $I_X$ and REC
in $\sigma$ compute

$$\sup(|c| + 17) = \gamma < \mathcal{F}^3_1.$$ By the choice of $\sigma$

$$X \in A \iff X \in A_\gamma \text{ and so } A \text{ is } \text{REC in } \mathcal{F}^3, c, \sigma,$$ as desired.

Remark Sacks [1980, Sacks and Griffor] has shown using a well-
ordering of reals, that there exists a regular complete RE class.
The foregoing indicates that this assumption is probably necessary.

In closing we employ the tools of § 1 to describe the degrees
of arbitrary $A \subseteq \mathcal{F}^3_1$ under AD.

Theorem 12: (AD) Let $A$ be a subset of $\mathcal{F}^3_1$, then $A$ is recursive
in $\mathcal{F}^3_1$ and a real.
proof. We require a lemma guaranteeing the regularity of $A$.

**Lemma 13:** If $A \subseteq K_1^E$, then $A$ is regular.

**pf (lemma):** By Cor 1, § 1 we have $\forall \gamma < K_1^E$, $\text{Cod}(A \cap \gamma; \leq) \in \Sigma_1^1(\leq)$, where $\leq$ is in $\Sigma_1^1$ of length $\gamma$. Then clearly $\text{Cod}(A \cap \gamma; \leq)$ is recursive in $\Sigma_1^1$.

To complete the proof of the theorem, let $\alpha < K_1^E$ and

$$A_\alpha = \{a \in 2^\omega | a \text{ is an index and } |a| \in \alpha \land A\}.$$ 

We consider the following game: As usual I and II play reals $a, b$ respectively. Put

II wins iff $a$ is not an index or

(a is an index and $b$ is a code for some $A_\beta$ for $\beta \geq |a|$).

This is a so-called Solovay-game and hence I cannot have a winning strategy, if $A$ is regular. Thus $A$ would be recursive in $\Sigma_1^1, \tau$ for $\tau$ any winning strategy for II.

By Lemma 13 every $A \subseteq K_1^E$ is regular and, hence, $A$ is recursive in $\Sigma_1^1, \alpha$ for some real $a$. 
References

Friedman, H. and Solovay, R., Large Ordinals and the axiom of determinateness, to appear [198?].

Gandy, R.O., General recursive functionals of finite type and hierarchies of functions, Univ. of Clermont Ferraud [1962].


