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CHARACTERIZING THE CONTINUOUS FUNCTIONALS

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One of the objectives of mathematics is to construct suitable models for practical or theoretical phenomena and to explore the mathematical richness of such models. This enables other scientists to obtain a better understanding of such phenomena. As an example we will mention the real line and related structures. The line can be used profitably in the study of discrete phenomena like population growth, chemical reactions etc.

Today's version of the real line is a topological completion of the rational numbers. This is so because then mathematicians have been able to work out a powerful analysis of the line. By using the real line to construct models for finitary phenomena we are more able to study those phenomena than we would have been sticking only to true-to-nature but finite structures.

So we may say that the line is a mathematical model for certain finite structures. This motivates us to seek natural models for other types of finite structures, and it is natural to look for models that in some sense are complete.

In this paper our starting point will be finite systems of finite operators. For the sake of simplicity we assume that they all are operators of one variable and that all the values are natural numbers. There is a natural extension of the systems such that they accept several variables and give finite operators as values, but the notational complexity will then obscur the idea of the construction.

Our first systematisation will be to regard such systems as subsystems of a system of infinite operators:

Definition

We define the finitary operators $\mathcal{F} = \bigcup_{k\in\mathbb{N}} \mathcal{F}_k$ of finite type by:

i All numbers n are in $\mathcal{F}_0$.

ii A function $f: \mathbb{N} \to \mathbb{N}$ is in $\mathcal{F}_1$ if f is almost constant.

iii Let $\varphi: \mathcal{F}_k \to \mathbb{N}$, $k \geq 1$. Then $\varphi \in \mathcal{F}_{k+1}$ if there are $\gamma_1, \ldots, \gamma_n \in \mathcal{F}_{k-1}$, i, t such that

* $\varphi(\psi)$ depends only on $\psi(\gamma_1), \ldots, \psi(\gamma_n)$

** If for some $j \leq n$ $\psi(\gamma_j) > i$ then $\varphi(\psi) = t$.

We will now be looking for a suitable completion $\mathcal{G}$ of $\mathcal{F}$. This cannot mean a completion in a precise topological sense since there is no canonical metric or other topological structure on the space $\mathcal{F}$. We rather mean that it in some sense is a natural extension of $\mathcal{F}$ containing the elements it ought to.

Clearly we do not want to add any elements which do not preserve any form of finitarity. This means that we should not for instance include the operator of type 2:

$$E(f) = \begin{cases} 0 & \text{if} \quad f \equiv 0 \\ 1 & \text{if} \quad \exists x \in \mathbb{N}(f(x) \neq 0) \end{cases}$$

since we require information about f that cannot be finitary described in order to find $E(f)$.

Kleene [4] and Kreisel [5] defined independently a hierarchy of functionals called the countable or continuous functionals, $\langle Ct(k) \rangle_{k\in\mathbb{N}}$ and there are several results indicating that the continuous functionals is a natural candidate for $\mathcal{G}$. They both used associates in one form or another; an associate for a functional ψ is a coding of ψ consisting of finite bits of information sufficient to describe ψ completely. The original formal definitions were hard to penetrate and later attempts have been made to give a more "civilized" characterisation of these functionals.

We recommend Hyland [3] for a general and thorough discussion of alternative descriptions of the continuous functionals. Hyland showed that the continuous functionals can be constructed in a natural way as filter-spaces, limit-spaces and topological spaces. In all three cases one start with $\mathbb{N}$ and iterate a standard construction of function spaces. Moreover Bergstra [1] characterized $\langle Ct(k)\rangle_{k\in\mathbb{N}}$ as the maximal type-structure supporting a natural recursion theory and not containing the functional E described above.

Recently Moldestad [6] gave an elegant characterization based on the idea of completing the hereditarily monotone partial operators of finite type with a finite basis and then take the total core. Moldestad's approach is analogue to a more general approach in Ershov [2], but it shows more clearly how these objects can be created from finitary ones.

In this note we will add one more characterization. We will use a few elements from the theory of non-standard analysis to give a natural completion of $\mathcal{T}$. Since the structure we construct turns out to be just the continuous functionals again, we have given support to the thesis that $\langle Ct(k)\rangle_{k\in\mathbb{N}}$ is the natural completion of $\mathcal{T}$ and that it as such deserves the interest of mathematicians.

We do not need to know much about non-standard theory. It is based on the interplay between two structures, $V_{\omega+\omega}$ and the non-standard extension $^*V_{\omega+\omega}$, with an elementary imbedding $* : x \rightsquigarrow {}^*x$. Elementarity of $*$ is often called the <u>transfer principle</u> and means that all truths and falsities are transferred by $*$. If we want to use non-standard theory at a more elaborate level, we must be far more precise, but for our purpose this will be sufficient. Let us just mention that a set is hyperfinite when $^*V_{\omega+\omega}$ believes that it is finite, i.e. in 1-1-correspondance with an element of $^*\mathbb{N}$ by a function in $^*V_{\omega+\omega}$. $*$ is the identity only on truely finite objects.

One of the methods used in non-standard theory is to regard a hyperfinite version of some finite but large structure. The strength of the theory lies in the interplay between the discrete hyper-finite structure and its standard part. The standard part will often be a complete standard version of the finite structures one originally had. This is also the basis of our construction, we just take the standard parts of the hyper-finitary functionals in $^*\mathcal{T}$.

Our motivation is not to give an alternative definition of the continuous functionals from which the theory naturally developes, only to show how natural the structure is as the completion of finite systems of finite operators. Thus we do not hesitate to use known facts about $\langle Ct(k)\rangle_{k\in\mathbb{N}}$ and we give no applications of the main theorem.

Definition

Let $^*\mathcal{F}$ be the non-standard set of hyper-finitary functionals. By induction on $k \in \mathbb{N}$ we define A_k and the equivalence-relation $\simeq_k$ by

i $A_0 = \mathbb{N}$, $n \simeq_0 m$ if and only if $n = m$.

ii Let $f \in {}^*\mathcal{F}_{k+1}$. Then $f \in A_{k+1}$ if

$$\forall g, h \in A_k (g \simeq_k h \Rightarrow f(g) = f(h) \in \mathbb{N})$$

Let $f_1, f_2 \in A_{k+1}$. Then

$$f_1 \simeq_{k+1} f_2 \quad \text{if} \quad \forall g \in A_k (f_1(g) = f_2(g)).$$

Let $\mathcal{G}_k = A_k/\simeq_k$. $\langle \mathcal{G}_k \rangle_{k\in\mathbb{N}}$ can in a canonical way be seen as a hierarchy of functionals.

Remark

At each stage we let A_{k+1} be the set of functions f such that $f/\simeq_k$ is well defined and a standard function from $A_k/\simeq_k$ into $\mathbb{N}$. $f \simeq_{k+1} g$ if $f/\simeq_k = g/\simeq_k$.

Our main result will be

Theorem

$\langle \mathcal{G}_k \rangle_{k\in\mathbb{N}}$ is isomorphic to $\langle Ct(k) \rangle_{k\in\mathbb{N}}$.

In order to prove this theorem we need to define the continuous functionals, and we will stick to Kleene's original definition.

By an effective enumeration all finite sequences of natural numbers can be enumerated, let $\{\sigma_n\}_{n\in\mathbb{N}}$ be the enumeration, n is called the sequence-number of σ_n and we will identify a sequence with its sequence-number.

If $f\colon \mathbb{N} \to \mathbb{N}$ then $\bar{f}(n) = (f(0),\ldots,f(n-1))$.

Definition

$Ct(0) = \mathbb{N}$, $Ct(1) = \mathbb{N}^{\mathbb{N}}$ and a function is its own associate.

Assume that $Ct(k)$ is defined together with the associates for the functionals in $Ct(k)$.

Let $\psi\colon Ct(k) \to \mathbb{N}$. Then $\psi \in Ct(k+1)$ if ψ has an associate, where an associate for ψ is any function α satisfying the following:

Whenever $\varphi \in Ct(k)$ and β is an associate for φ then

$$\exists n \forall m((m < n \Rightarrow \alpha(\bar{\beta}(m)) = 0) \wedge (m \geq n \Rightarrow \alpha(\bar{\beta}(m)) = \psi(\varphi) + 1)).$$

The finitary functionals can be regarded as a natural subclass of the continuous functionals.

Let $B_\sigma^k = \{\psi \in Ct(k);\ \psi$ has an associate α extending $\sigma\}$. Kleene [4] showed that if B_σ^k is nonempty then B_σ^k contains a finitary element. If $\varphi \in Ct(k)$, then the principal associate for φ is the function α defined by:

$$\alpha(\sigma) = \begin{cases} t+1 & \text{if } \varphi \text{ is constant } t \text{ on } B_\sigma^{k-1} \\ 0 & \text{otherwise.} \end{cases}$$

Again by Kleene [4] φ is constant t on B_σ^{k-1} if and only if φ is constant t on $B_\sigma^{k-1} \cap \mathcal{F}$. This means that we can define non-standard associates for $f \in {}^*\mathcal{F}$ without reference to a non-standard version of $\langle Ct(k)\rangle_{k\in\mathbb{N}}$.

The theorem will now follow from

Lemma

Let $k \in \mathbb{N}$, $\omega \in {}^*\mathbb{N} \setminus \mathbb{N}$. Then

i Let α be an associate for $\psi \in Ct(k)$ and let f_α be a hyper-finitary element of $B^k_{\alpha(\omega)}$. Then $f_\alpha \in A_k$. Moreover if β is another associate for ψ then $f_\alpha \simeq_k f_\beta$.

ii Let $f \in A_k$. Let α_f be the principal non-standard associate for f. Then $\alpha_f \restriction \mathbb{N}$ is an associate. Moreover, if $f \simeq_k g$ then α_f and α_g are associates for the same functional.

Proof

We use induction on k.

For $k = 0$ and $k = 1$ this is trivial (For $k = 0$ we have not even defined the concepts of the lemma). So assume that $k > 1$ and that the lemma holds for all $l < k$.

i Let α, ψ be given, $g \in A_{k-1}$. Then $\alpha_g \restriction \mathbb{N}$ is an associate for a functional φ, and there is some n, t such that $\alpha(\bar{\alpha}_g(n)) = t+1$. Then $f_\alpha(g) = \psi(\varphi)$ since f is constant t on $B^{k-1}_{\alpha_g(n)}$ and $g \in B^{k-1}_{\alpha_g(n)}$. If $g \simeq_{k-1} h$ then α_g and α_h are associates for the same functional φ so in both cases $t = \psi(\varphi)$. The value t is also only dependent of ψ and φ and not of α.

ii Let $f \in A_k$ be given. Let β be an associate. Let α_f be the non-standard associate for f. If $\forall n \in \mathbb{N}\, \alpha_f(\bar{\beta}(n)) = 0$ then $\forall n \in \mathbb{N} \exists h_1, h_2 \in B^{k-1}_{\bar{\beta}(n)}(f(h_1) \neq f(h_2))$.

Since nothing drastic can take place on the borderline between $\mathbb{N}$ and ${}^*\mathbb{N} \setminus \mathbb{N}$ (The non-standard world does not recognise $\mathbb{N}$

as a set) there is an $\omega_1 \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that

$$\exists h_1, h_2 \in B^{k-1}_{\beta(\omega_1)}(f(h_1) \neq f(h_2)).$$

By the induction hypothesis $h_1 \in A_{k-1}$ and $h_2 \in A_{k-1}$ and $h_1 \simeq_{k-1} h_2$ so $f(h_1) = f(h_2)$ since $f \in A_k$.

This ends the proof of the lemma, and the theorem.

Remarks

a Let $f \in A_k$. α_f as constructed above is not necessarily a principal associate. We can, in fact get all associates for type k functionals from elememnts in A_k in this way.

b Instead of using the hyper-finite functionals we could have used a hyper-finite collection of them.

If $\omega \in {}^*N \setminus \mathbb{N}$ let $\mathcal{J} = \langle T_i \rangle_{i \leq \omega}$ be defined by

$T_o = \{i; i \leq \omega\}$

$T_{j+1} = \{\sigma; \sigma : T_j \to T_o\}$.

If we use $\mathcal{J}$ instead of $\mathcal{T}$ in defining A_k and $\simeq_k$ then we still get the continuous functionals.

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