ON THE VARIETIES PARAMETRIZING RATIONAL SPACE CURVES WITH FIXED NORMAL BUNDLE

by

Gianni Sacchiero

Istituto Matematico and Inst. of Mathematics
dell'Università, University of Oslo
Via Machiavelli 35, Norway
44100 Ferrara (Italy)
ON THE VARIETIES PARAMETRIZING RATIONAL SPACE CURVES WITH FIXED NORMAL BUNDLE.

Gianni Sacchiero(*)(**)

(*) Istituto Matematico dell'Università, Via Machiavelli 35, 44100 Ferrara (Italy).

(**) This work was carried out while the author held a CNR research fellowship at the Institute of Mathematics, University of Oslo.
Introduction

This note complements, on the one hand, the results obtained in [G.S] for rational curves, on the other hand it gives an introduction to the study of the normal bundle of curves in \( \mathbb{P}^3 \) (projective space over an algebraically closed field of characteristic 0) of arbitrary genus. In fact, the preliminary results are obtained without hypotheses on the genus of the curves.

Let \( C \subset \mathbb{P}^3 \) be a rational curve with, at worst, ordinary singularities (i.e., if \( C \) is given as the image of a morphism \( \varphi: \mathbb{P}^1 \to \mathbb{P}^3 \), the map \( \varphi^*\Omega^1_{\mathbb{P}^3} \to \Omega^1_{\mathbb{P}^1} \) is surjective), \( \mathcal{N} \) the normal bundle of \( C \) in \( \mathbb{P}^3 \). We prove the following Theorem.

**Theorem.** Let \( U^n \) (resp. \( T^n \)) denote the irreducible space of dimension \( 4n \) parametrizing rational curves in \( \mathbb{P}^3 \) of degree \( n \) with, at worst, ordinary singularities (resp. that are smooth).

There exists a stratification of \( U^n \) (resp. \( T^n \))

\[
\emptyset \neq U^n_{n-3} \subset U^n_{n-4} \subset \ldots \subset U^n_{n-\rho} \subset \ldots \subset U^n \subset U^n
\]

(resp. \( \emptyset \neq T^n_{n-4} \subset T^n_{n-5} \subset \ldots \subset T^n_{n-\rho} \subset \ldots \subset T^n \subset T^n \))

such that

1) \( C \in U^n_{n-\rho} \) (resp. \( C \in T^n_{n-\rho} \)) \( \iff \mathcal{N} \cong \mathcal{O}_{\mathbb{P}^1}(2n-1-\bar{\rho}) \oplus \mathcal{O}_{\mathbb{P}^1}(2n-1+\bar{\rho}) \)

with \( \bar{\rho} \geq \rho \).

2) \( U^n_{n-\rho} \) (resp. \( T^n_{n-\rho} \), \( 1 \leq \rho \leq n-3 \) (resp. \( 1 \leq \rho \leq n-4 \))

is a quasi-projective, integral, locally Cohen-Macaulay.
In this section \( C \subset \mathbb{P}^3 \) denotes a smooth, connected curve of degree \( n \) and genus \( g \). (The smoothness assumption is made to simplify notations, it is enough to assume that \( C \) has at worst ordinary singularities and consider the normalization \( \tilde{C} \rightarrow C \) of \( C \).

Let \( \pi : F \rightarrow C \) be a geometrically ruled surface over \( C \), i.e., \( F \) is of the form \( F = \mathbb{P}(\mathcal{F}) \), where \( \mathcal{F} \) is a locally free sheaf of rank 2 on \( C \) and \( \pi \) is the canonical projection.

Let \( H \) be a unisecant divisor on \( F \), i.e., \( (H,F_x) = 1 \) for all \( x \in C \) and \( F_x = \pi^{-1}(x) \), and assume \( \mathcal{O}_F(H) = \mathcal{O}_F(1) \).

Consider a linear system \( \Sigma \subset |H| \), without base points, such that \( \dim \Sigma = r \geq 3 \) and assume that the corresponding morphism

\[
\delta : F \rightarrow \mathbb{P}^r
\]

is birational onto its image. Then we shall say that \( R = \delta(F) \subset \mathbb{P}^r \) is a ruled surface over \( C \). It results that the fibres of \( F \) become lines in \( \mathbb{P}^r \) and that the degree of \( R \) is \( q = \deg \mathcal{F} \).

Moreover, \( \delta \) gives a 1-1 correspondence between the sections of \( \pi : F \rightarrow C \) and the directrices of \( R \). Also, if \( D \subset F \) is a section of \( F \), given by the surjection \( \mathcal{F} \rightarrow \mathcal{L}_D \rightarrow 0 \), then the degree of the directrix \( \delta(D) \subset R \) is given by

\[
(1) \quad \deg \delta(D) = \deg \mathcal{L}_D.
\]

It follows that if \( D_1 \) and \( D_2 \) are two directrices of \( R \), then

\[
(2) \quad (D_1,D_2) = \deg D_1 + \deg D_2 - q,
\]
for \( q \in \mathbb{P}^2 \).

We have

\[
(7) \quad \deg C^\vee = \deg (\mathcal{N}(-1)) = 2n + 2g - 2
\]

As an immediate consequence of (2) and (3), we get the following

**PROPOSITION 1-1.**

1) The surjections \( \mathcal{N} \to \mathcal{L} \to 0 \) correspond one-to-one to the directrices of degree \( \deg \mathcal{L} - n \) of \( C^\vee \).

2) \( \mathcal{N} \) decomposes if and only if there exist two directrices \( D_1, D_2 \) such that \( \deg D_1 + \deg D_2 = 2n + 2g - 2 \).
REMARK 2-2. If \( D = C_q \), i.e. if \( D \) is a plane directrix of \( C \), then by duality \( C \) belongs to the cone \( C_q \) with vertex \( q \). Moreover, if \( C_q \) projects \( C \) birationally, a generic plane section of \( C_q \) coincides with the dual plane curve of \( D \) (see [P]).

The construction used in Lemma 2-1 gives a geometrical explanation of the fact observed in [G-S] concerning the biduality that exists between the curves of \( \mathbb{P}^3 \) and the curves of \( \mathbb{P}^3 \) considered as sections of \( \mathcal{N} \).

A consequence of Lemma 2-1 is the existence, for each directrix \( D \) of \( C \) (resp. for each quotient \( \mathcal{N} \to \mathcal{L}_D \to 0 \)), of a "canonical" surface which contains \( C \) simply, and such that its tangent planes along \( C \) give \( D \) (resp. \( \mathcal{L}_D \)).

Clearly this surface is \( D \). Since \( \deg D \leq 2 \deg D + 2g - 2 \), we obtain

COROLLARY 2-3. A directrix of \( C \) of degree \( \leq d \) comes from a surface of degree \( \leq 2d + 2g - 2 \).
This formula also gives the degree of the cone $D^\vee$ in case $D$ is plane.

From Proposition 3-1, taking into account the formula (8), it follows:

**COROLLARY 3-2.** Let $D \subset \mathbb{P}^3$ be a curve. The directrices of degree $\mu$ on $D^\vee$ are parametrized by a smooth, irreducible, quasi-projective variety of dimension $\leq 2\mu - 2d + 1 - K_0(D)$. 
The following hold:

1) $C \in V^n_{\rho} \iff \mathcal{N} \cong \mathcal{O}(2n-1-\bar{\rho}) \oplus \mathcal{O}(2n-1-\overline{\rho})$, with $\overline{\rho} \geq \rho$.

2) $V^n_{\rho} \neq \emptyset \iff 0 \leq \rho \leq n-3$

3) $C \in B^n_{n-3} \iff C$ is on a quadric cone and has a point of multiplicity $n-2$ at the vertex.

4) (see Prop. 3-5 [G.S]) Set $d = n - 1 - \rho$.

Consider the correspondence

$$\Gamma_{n,d} = \{ (\psi, f) \in \mathbb{P}^{4n+3} \times \mathbb{P}^{4d+3} : f \in \mathbb{P}(\text{Ker} \omega_{n+d}(\psi)) \},$$

where $\Gamma_{n,d}$ as a subvariety of $\mathbb{P}^{4n+3} \times \mathbb{P}^{4d+3}$ is defined by the equations

$$(9) \sum_{i=0}^{3} \frac{\partial \psi_i}{\partial t_j} f_i = 0, \quad j = 0,1.$$

Consider the natural projections $\pi_1$ and $\pi_2$

$$\begin{array}{c}
\pi_1 \\
\mathbb{P}^{4n+3} \\
\Gamma_{n,d} \subseteq \mathbb{P}^{4n+3} \times \mathbb{P}^{4d+3} \downarrow \pi_1 \\
\downarrow \pi_2 \\
\mathbb{P}^{4d+3}
\end{array}$$

a) $\pi_1$ is an isomorphism above $V^n_{n-1-d} - V^n_{n-d}$.

b) There exists a unique irreducible component $\Gamma_{n,d}^{(1)}$ of $\Gamma_{n,d}$ such that

$$p_2(\Gamma_{n,d}^{(1)}) = \mathbb{P}^{4d+3}.$$
Set \( A_\rho = A \cap (V^n_\rho - V^n_{\rho+1}) \). Clearly

\[ p(A_\rho) \subset (V^n_{\rho+1} - V^n_{\rho+1} \cap w^{n-1}_{\rho+1}) \]

(see (4, d)).

Let us prove that \( \dim A \leq 4n + 3 - 2\rho \). If \( f \in V^n_{\rho+1} \), denote by \( D_f \) the curve defined by \( f \), then \( k_0(D_f) = 0 \). Taking into account Cor. 3-2, we get \( \dim p^{-1}(f) = 2\rho + 5 \) and hence \( \dim p^{-1}(V^n_{\rho+1}) \leq 4n + 3 - 2\rho \). Now if \( f \in (V^n_{\rho+1} \cap w^{n-1}_{\rho+1} - V^n_{\rho+1}) \), then \( k_0(D_f) \neq 0 \).

Set \( W^n_{\rho+1} = \{ f \in w^{n-1}_{\rho+1} : k_0(D_f) \geq h \} \), \( h \geq 1 \).

There is a stratification of \( W^n_{\rho+1} \)

\[ \cdots \subset W^n_{\rho+1} \subset W^n_{\rho+1} \subset \cdots \subset W^n_{\rho+1} \subset W^n_{\rho+1} \]

Now \( \text{codim}_{w^{n-1}_{\rho+1}} W^n_{\rho+1} \geq 2h \) (as can be easily seen for example by considering rational curves of degree \( n \) in \( \mathbb{P}^3 \) as projections of normal curves in \( \mathbb{P}^n \)), whereas the dimension of the fibre of \( p \) on \( W^n_{\rho+1} - W^n_{\rho+1} \) increases only by \( h \) (Cor. 3-2). This shows that \( \dim A \leq 4n + 3 - 2\rho \).

We thus conclude, using the Observation 4-1. This also proves that \( V^n_\rho \) is locally Cohen-Macaulay, because it is a determinantal variety of maximal codimension. Moreover, it follows that \( \Gamma_{n,n-1-\rho} = \Gamma_{n,n-1-\rho}^{(1)} \) and hence is a complete intersection (defined by the \( 2(n+d) = \text{codim}_{\mathbb{P}^{n+3} \times \mathbb{P}^{4(n-1-\rho)+3}} \Gamma_{n,n-1-\rho} \) equations obtained by (9)).
5.

As we have already seen in the preceding section (3), the curves of $V^n_{n-3}$ have a precise geometrical characterization. Therefore it is natural to ask whether this is so in general. The answer is positive in the case $p = n - 4$.

**Proposition 5-1.** Let $C$ be smooth, of degree $n \geq 6$ ($C \in T^n$). Then:

$$0 \in T_{n-4}^n \iff C \text{ is a directrix of the tangent developable of a twisted cubic.}$$

**Proof.** Assume $N \cong \mathcal{O}(n+3) \oplus \mathcal{O}(3n-5)$, then there exists (Prop. 1-1) a directrix (clearly irreducible and simple) $D \subset C^\vee$ such that $\deg D = 3$. The curve $D$ cannot be plane, otherwise, since $n \geq 6$, $C$ would have a singular point at the vertex of the cone $D^\vee$. We conclude by applying Lemma 2-1, since twisted cubics are self-dual.

The converse follows immediately from similar considerations.

If $p \leq n - 5$, the geometrical situation is not so clear. In fact, for a rational curve $D$ of degree $\geq 4$ there are numerous possibilities for $\deg D^\vee$ and $\deg D^\ast$.

It seems that the best approach to the classification problem would be to consider the curves of $\mathbb{P}^3$ as projections of normal curves in $\mathbb{P}^n$. 
REFERENCES


