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POSITIVE PROJECTIONS WITH CONTRACTIVE COMPLEMENTS ON C*-ALGEBRAS

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Positive projections with contractive complements on C^*-algebras.

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1. Introduction. In order to obtain a deeper understanding of the structure of positive linear maps on C^*-algebras it seems natural first to study positive unital projections. Such maps, which are central among positive maps, have many of the peculiar properties of general positive maps, and are sufficiently simple to be analyzed in detail. Inspired by a paper of Arazy and Friedman [2], in which they classified the ranges of contractive idempotent maps on the compact operators on a separable Hilbert space, a systematic study of positive projections was initiated in [3] and then continued in [9], [5], [4]. In the last paper Robertson and Youngson studied the problem of whether a positive unital projection with contractive complement on a JB-algebra, see [1], is necessarily of the form $P = \frac{1}{2}(\iota + \theta)$, where ι is the identity map and θ a Jordan automorphism of order 2. They proved some partial results in this direction and expressed a belief that it ought to be true in general. In the present paper we solve this problem affirmatively for projections on C^*-algebras. Our proof will be based on the techniques developed in [9] and consists mainly of solutions to the problem in many special cases. It is our belief that a rather tedious extension of our proof will yield a solution of the same result for JB-algebras.

If A is a C^*-algebra we denote by A_h its self-adjoint part. We denote by M_n the complex $n \times n$ matrices and by $\mathcal{D}_n$ the diagonal matrices in M_n. To simplify notation we shall often write $A = N$ if A is $*$-isomorphic to a C^*-subalgebra N of M_n. In our discussion we shall assume P is a positive unital projection on a C^*-algebra A such that $\iota - P$ is contractive on A_h. To avoid discussion of the norm of $\iota - P$ on A we shall always mean that $\iota - P$ is contractive on A_h when we just say $\iota - P$ is contractive. If B is a Jordan subalgebra of A_h we shall often say P is a projection of A onto B to mean that $B = P(A_h)$. By a Jordan automorphism of A we shall mean a positive unital linear isomorphism θ of A onto itself such that $\theta(x^2) = \theta(x)^2$ for all $x \in A_h$. θ is of order 2 if $\theta^2 = \iota$

2. Positive projections. In this section we state and prove our main result.

Theorem. Let A be a unital C^*-algebra and P a positive unital projection of A into itself. Then $\iota - P$ is contractive if and only if there is a Jordan automorphism θ of order 2 of A such that $P = \frac{1}{2}(\iota + \theta)$.

The proof of this result will be broken up into some lemmas, most of which prove the theorem in special cases. The first five lemmas are concerned with projections onto JW-factors of type I. The first is easily deduced from the literature.

Lemma 1. Let A be a unital C^*-algebra and P a positive unital projection of A onto the scalar operators in A such that $\iota - P$ is contractive. Then $A \subset M_2$, and we have the following possibilities:

(1) $A = \mathbb{C}$ and $P = \iota$.

(2) $A = \mathcal{D}_2$ and $P = \frac{1}{2}(\iota + \theta)$, where $\theta\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$.

(3) $A = M_2$ and $P = \frac{1}{2}(\iota + \theta)$, where θ is the antiautomorphism $\theta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

Proof. By [4, Prop.2.6] $A \subset M_2$ and P is the restriction to A of the normalized trace τ on M_2. Thus (1), (2), (3) are immediate. Q.E.D.

Lemma 2. Let M be a von Neumann algebra, B a subfactor of M of type I and P a normal positive unital projection of M onto B such that $\iota - P$ is contractive. Then we have the following possibilities:

(1) $M = B$ and $P = \iota$.

(2) There exist two central projections e and f in M with sum 1 and an isomorphism α of Me onto Mf such that $B = \{xe + \alpha(xe) : x \in M\}$, and $P = \frac{1}{2}(\iota + \theta)$, where $\theta(xe + yf) = \alpha^{-1}(yf) + \alpha(xe)$.

(3) $M = M_2$, $B = \mathbb{C}$, and P is given by Lemma 1 (3).

<u>Proof</u>. Since B is a subfactor of type I there is a von Neumann algebra N such that $M = B \otimes N$, where B is identified with $B \otimes \mathbb{C}$. By [9, Prop.6.2] $P = \iota \otimes \omega$, where ω is a normal state on N. Since $\iota - P$ is a contraction when restricted to N, $N \subset M_2$ by Lemma 1 and $\omega = \tau | N$. If $N = \mathbb{C}$ we have case (1), and if $N = \mathcal{D}_2$ case (2).

Assume $N = M_2$. Then $\tau = \frac{1}{2}(\iota + \theta)$ with θ as in Lemma 1(3). We may write M as 2×2 matrices over B. Since $P = \iota \otimes \tau$ we have

$$P\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{2}\begin{pmatrix} a+d & 0 \\ 0 & a+d \end{pmatrix}, \quad a,b,c,d \in B.$$

In particular if $b \in B$, $P\begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix} = 0$. Since $\iota - P$ is contractive $\|x^2\| = \|P(x^2)\|$ whenever $P(x) = 0$, $x \in M_h$ [4, Thm.2.4]. Thus we have

$$\|b\|^2 = \left\|\begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix}\right\|^2 = \left\|P\left(\begin{pmatrix} 0 & b \\ b^* & 0 \end{pmatrix}^2\right)\right\| = \frac{1}{2}\|bb^* + b^*b\|.$$

If $B \neq \mathbb{C}$ let b be a partial isometry in B such that b^*b is a projection orthogonal to bb^*. Then $\frac{1}{2}\|bb^* + b^*b\| = \frac{1}{2} \neq 1 = \|b\|^2$, a contradiction. Thus $B = \mathbb{C}$, and $M = N = M_2$. Q.E.D.

<u>Lemma 3</u>. Let M be a von Neumann algebra and B a reversible JW-factor of type I contained in M such that the von Neumann

algebra B'' generated by B is a factor and $B \neq (B'')_h$. Suppose P is a normal positive unital projection of M onto B such that $\iota - P$ is contractive. Then $M = B''$, and $P = \frac{1}{2}(\iota + \theta)$ with θ an antiautomorphism of order 2 of M.

Proof. By [6, Thm.8.2] B'' is a factor of type I, hence there is a von Neumann algebra N such that we may write $M = B'' \otimes N$. As in Lemma 2 $N \subset M_2$ and $P = Q \otimes \tau$, where by [9, Prop.6.2] Q is the unique normal positive projection of B'' onto B, described in the proof of [3, Thm.2.2]. We have to show $N = \mathbb{C}$. Assume first $N = \mathcal{D}_2$. Let e and f be the minimal nonzero projections in $\mathcal{D}_2$. Then for $a, b \in B''$,

$$P(a\otimes e + b\otimes f) = \tfrac{1}{2}Q(a+b) \otimes 1.$$

In particular $P(a\otimes e - a\otimes f) = 0$, so by [4, Thm.2.4]

$$\begin{aligned} \|a\|^2 &= \|a\otimes e - a\otimes f\|^2 = \|P((a\otimes e - a\otimes f)^2)\| \\ &= \tfrac{1}{2}\|Q(a^2+a^2)\| = \|Q(a^2)\| \leq \|a\|^2 \end{aligned}$$

for all self-adjoint $a \in B''$. In particular Q is an isometry on the projections in B''. Since $Q = \frac{1}{2}(\iota + \theta)$ with θ an anti-automorphism of order 2 of B'', this means that $\|p + \theta(p)\| = 2$ for all projections p in B''. In particular, if p is a minimal projection so is $\theta(p)$, hence $p = \theta(p)$. Thus $\theta = \iota$, a case we have excluded. Therefore $N \neq \mathcal{D}_2$.

Suppose next $N = M_2$. Write M as 2×2 matrices over B''. As in Lemma 2

$$P\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} Q(a+d) & 0 \\ 0 & Q(a+d) \end{pmatrix},$$

so if b is self-adjoint then $P\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix} = 0$. Again by [4, Thm.2.4]

$$\|b\|^2 = \|\begin{pmatrix} 0 & b \\ b & 0 \end{pmatrix}\|^2 = \|P\begin{pmatrix} b^2 & 0 \\ 0 & b^2 \end{pmatrix}\| = \|Q(b^2)\| \leq \|b\|^2,$$

so Q is the identity map as shown above. Thus $N \neq M_2$, so that $N = \mathbb{C}$. Q.E.D.

Lemma 4. Let M be a von Neumann algebra and $B \subset M$ a reversible JW-factor of type I such that B'' is not a factor. Suppose P is a normal positive unital projection of M onto B such that $\iota - P$ is contractive. Then $M = B''$, and $P = \frac{1}{2}(\iota + \theta)$, where θ is an antiautomorphism of order 2 of M.

Proof. By [7, Cor. 3.5] there are exactly two nonzero minimal central projections e and f in B'', $e+f = 1$, and $(B''e)_h = Be$, $(B''f)_h = Bf$. Furthermore, as pointed out on [9, p.35] there is an antiisomorphism $\chi : B''e \to B''f$ such that $B = \{ae + \chi(ae)f : a \in (B''e)_h\}$. Since $B''e$ is a type I factor contained in M_e, $M_e = B''e \otimes N$ for some von Neumann algebra N acting on eH, where H is the underlying Hilbert space. Let p be a minimal projection in $B''e$. Then $q = p + \chi(p)$ is a minimal projection in B. Since $P(q) = q$, P restricted to M_q is a positive unital projection of M_q onto $\mathbb{C}q$ with $\iota - P$ contractive. By lemma 1 $M_q \subset M_2$. In particular M_p is one dimensional, so p is a minimal projection in M. Since this is true for all minimal projections in $B''e$, $B''e = M_e$.

Suppose p_1 and p_2 are two minimal projections in $B''e$ with $p_1 p_2 = 0$. Let $p = p_1 + p_2$ and $q = p + \chi(p)$. Then B_q is isomorphic to D_2 - the JW-algebra consisting of all matrices of the form $\begin{pmatrix} a & 0 \\ 0 & a^t \end{pmatrix}$ in M_4 with $a \in (M_2)_h$ and a^t the transpose of a, see [6]. Furthermore, by the previous paragraph $M_q \subset M_4$. If we combine the restriction $P|M_q$ with a positive projection of M_4 onto M_q we obtain a positive unital projection of M_4 onto D_2. By [9, Lem.5.5]

$$P(p\,a\,\chi(p) + \chi(p)ap) = 0 \quad \text{for all} \quad a \in M_q .$$

Since this holds for all p_1 and p_2, and P is normal, $P(p_1Mf) = 0$, and again by normality $P(eMf) = 0$. But then the conclusions of [9, Prop.6.4] are true, as follows from an inspection of its proof. Thus there are normal positive unital projections $Q : M \to B''$ and $R : B'' \to B$ such that $P = R\circ Q$. Since R is the restriction of P to B'', $\iota - R$ is a contraction. If $x \in M_h$ and $Q(x) = 0$ then $P(x) = 0$, so by [4, Thm.2.4]

$$\|x^2\| \geq \|Q(x^2)\| \geq \|R(Q(x^2))\| = \|P(x^2)\| = \|x^2\|$$

hence $\|x^2\| = \|Q(x^2)\|$, and $\iota - Q$ is a contraction by [4, Thm.2.4].

Since $M_e = B''e$, $M_f = B''f$, and $P(eMf) = 0$, it follows that

$$Q(x) = exe + fxf \quad \text{for} \quad x \in M.$$

Let R_e and R_f be the projections of B'' onto B described in [9, Prop.6.4], i.e.

$$R_e(a) = ae + \chi(ae)f$$

$$R_f(a) = af + \chi^{-1}(af)e.$$

Then $R = \lambda R_e + (1-\lambda)R_f$ with $0 \leq \lambda \leq 1$. Let $x = ae - \chi(ae)f \in B''$. Then

$$R(x) = (2\lambda-1)(ae + \chi(ae)f) ,$$

hence

$$(\iota-R)(x) = 2(1-\lambda)ae - 2\lambda\chi(ae)f,$$

and $\|(\iota-R)x\| = \max(2(1-\lambda),2\lambda)\|x\| \geq \|x\|$. Since $\iota - R$ is a contraction . $\lambda = \frac{1}{2}$, and $R = \frac{1}{2}(R_e+R_f)$.

We assert that $M = B''$. Suppose $M \neq B''$. Since $M_e = B''e$ and similarly for f we have $eMf \neq 0$. Then there are minimal projections p in Be and p' in Bf such that $p \sim p'$ in M. Since $B \neq (B'')_h$, $M_e = B''e \supset M_n$ for some $n \geq 2$, hence we can choose a minimal projection q in M_f such that $p \sim q$ and $q\chi(p) = 0$. Let v be a partial isometry in M such that $vv^* = p$, $v^*v = q$, and let $a = v + v^*$. Then $a \in M_h$, $a^2 = p + q \in B''$ and has norm 1, and $P(a) = 0$ since $a = eaf + fae$. Thus

$$\|P(a^2)\| = \|P(p+q)\| = \tfrac{1}{2}\|R_e(p) + R_f(q)\|$$
$$= \tfrac{1}{2}\|p + \chi(p) + q + \chi^{-1}(q)\| = \tfrac{1}{2},$$

since all four projections involved in the last expression are mutually orthogonal. Since $\|a^2\| = 1$, we have contradicted [4, Thm.2.4], so that $M = B''$, as asserted. But then if θ is the antiautomorphism of order 2 of M defined by

$$\theta(ae+bf) = \chi^{-1}(bf)e + \chi(ae)f, \quad a,b \in M, \text{ then}$$

$P = \frac{1}{2}(R_e+R_f) = \frac{1}{2}(\iota+\theta)$. Q.E.D.

<u>Lemma 5</u>. Let A be a unital C^*-algebra and B a nonreversible spin factor contained in A. Suppose P is a positive unital projection of A onto B. Then $\iota - P$ cannot be contractive.

<u>Proof</u>. We may assume A is the C^*-algebra generated by B, hence A is the CAR algebra on a finite or infinite dimensional Hilbert space, see [8]. Let τ be the unique normalized trace on A, so τ restricted to B is the same on B. Let e be a minimal projection in B. Since B is not reversible, A_e is either M_{2^n} with $n \geq 2$ or the infinite dimensional CAR algebra,

so in particular $A_e \supset M_{2^n}$, $n \geq 2$. Suppose $\iota - P$ is contractive. Since $P(e) = e$, P restricted to A_e is a positive unital projection of A_e onto $\mathbb{C}e$ with $\iota - P$ contractive. Hence by Lemma 1 $A_e \subset M_2$, contradicting the fact that $A \supset M_{2^n}$ for some $n \geq 2$. Q.E.D.

In order to prove the theorem we shall go to the second dual A^{**} of A and restrict the extension of P to maps onto subalgebras of the form $B^{-}e$ with e a central projection of the weak closure B^{-} of B such that $B^{-}e$ is a JW-factor of type I. Since e is not necessarily a central projection in A^{**} we encounter some difficulties which will be treated in the next lemmas. The next lemma is immediate from [4, Prop.2.8].

Lemma 6. Let A be a unital abelian C^*-algebra and P a positive unital projection on A with $\iota - P$ contractive. Then $P = \frac{1}{2}(\iota+\theta)$ with θ an automorphism of order 2 of A. In particular, if e is a minimal projection in A then either $P(e) = e$ or $P(e) = \frac{1}{2}(e+e')$ with e' a minimal projection in A orthogonal to e.

Proof.(*) Let $\theta = 2P - \iota$. We have to show θ is a Jordan homomorphism. Going to the second dual A^{**} of A, since the minimal projections there separate A, it suffices to show $\theta(e)$ is a projection for each minimal projection e in A^{**}. Let e be a minimal projection in A^{**}. Then $eP(e) \neq 0$. Indeed, if $eP(e) = 0$ then $(e - P(e))^2 = e + P(e)^2$ has norm at least 1, so equal to 1 since $\iota - P$ is contractive. Since P is faithful by [4, Lem.2.3], $B = P(A^{**})$ is a JC-subalgebra of A^{**} by [3, Thm. 1.4]. Thus $P(e)^2 \in B$. But for a general projection e,

(*) This proof will only appear in the preprint.

$$P((e-P(e))^2) = P(e) - 2P(eP(e)) + P(P(e)^2)$$
$$= P(e) - 2P(e)^2 + P(e)^2$$
$$= P(e) - P(e)^2.$$

Since by the previous computation $P((e-P(e))^2) = P(e) + P(e)^2$, we have $P(e) = 0$ contrary to assumption. Let

$$P(e) = \lambda e + g \quad \text{with} \quad 0 < \lambda \leq 1, \quad g \in A^{**},$$

and $eg = 0$. We have

$$\lambda(1-\lambda)e + (g-g^2) = P(e) - P(e)^2$$
$$= P(e)(1-P(e))$$
$$= P(e)P(1-e)$$
$$= P(P(e) - P(e)e)$$
$$= P(e - P(e)e)$$
$$= (1-\lambda)P(e)$$
$$= \lambda(1-\lambda)e + (1-\lambda)g.$$

Thus $g^2 = \lambda g$, so $g = \lambda f$ with f a projection, and $P(e) = \lambda(e+f)$. Let $x = e - P(e)$. Then $P(x) = 0$. Since $x^2 = (1-\lambda)^2 e + \lambda^2 f$, $\|x^2\| = \max((1-\lambda)^2, \lambda^2)$ or 0, if $f = 0$. From the above computations $P(x^2) = \lambda(1-\lambda)(e+f)$, hence has norm $\lambda(1-\lambda)$. By [4, Thm.2.4], $\|P(x^2)\| = \|x^2\|$, so that either $f = 0$ and $\lambda = 1$, or $\lambda = \frac{1}{2}$. If $\lambda = 1$ $P(e) = e$. If $\lambda = \frac{1}{2}$, $P(e) = \frac{1}{2}(e+f)$, and $\theta(e) = 2P(e) - e = f$. Q.E.D.

<u>Lemma 7</u>. Let M be a von Neumann algebra and P a normal positive unital projection of M onto a JW-subalgebra B with $\iota - P$ contractive. Suppose F is a minimal central projection in M.

Then there exist at most two mutually orthogonal central projections e_1 and e_2 in B such that $e_i F \neq 0$, $i = 1,2$. If $F \notin B$ one of the following two cases must occur:

(1) There exists a minimal central projection $e \in B$ with $F \leq e$, in which case $P(F) = \frac{1}{2}e$.

(2) There are two minimal orthogonal central projections e and f in B with $F \leq e+f$, in which case $P(F) = \frac{1}{2}(e+f)$.

In particular, if there exists a minimal central projection g in B with $g \leq F$ then $F \in B$.

<u>Proof</u>. Suppose there exist three mutually orthogonal central projections e_1, e_2, e_3 in B such that $Fe_i \neq 0$, $i = 1,2,3$. Since M_F is a factor there exist nonzero projections $p_i \leq e_i$ in M_F such that $p_i \sim p_j$, $i,j = 1,2,3$. Let $e_{ii} = p_i$ and e_{ij} be partial isometries in M_F such that $\{e_{ij} : 1 \leq i,j \leq 3\}$ is a complete set of matrix units for a subalgebra of M_F isomorphic to M_3. Let $x = \sum_{i \neq j} e_{ij}$. By [9, Lem.4.1], if $y \in M$ $P(e_i \circ y) = e_i \circ P(y)$. If $i \neq j$ we have $e_i y e_j + e_j y e_i = 4e_i \circ (e_j \circ y)$, so that

$$P(e_i y e_j + e_j y e_i) = 4e_i \circ (e_j \circ P(y)) = 0,$$

since the e_i are central projections in B. Thus $P(e_i y e_j) = 0$ for all $y \in M$ when $i \neq j$. In particular $P(x) = 0$. Now $x^2 = e_{11} + e_{22} + e_{33} + \sum_{i,j=1}^{3} e_{ij}$ has norm 4, while $P(x^2) = 2P(\sum_{i=1}^{3} e_{ii}) \leq 2$, hence $\|P(x^2)\| \leq 2$. This contradicts [4, Thm.2.4], so the existence of e_1, e_2, e_3 is impossible.

By [9, Lem.3.1] P maps the center Z of M onto the center C of B. If A is the abelian von Neumann algebra generated by Z and C, P is a projection of A onto C with contractive complement. By Lemma 6, since $F \notin B$, $P(F) = \frac{1}{2}(F+F')$ with F' another minimal projection in Z. If $F \leq e$ with e minimal in C, $P(F) \leq e$, so is a multiple of e, hence equal to $\frac{1}{2}e$. If $F \leq e+f$ with e and f minimal projections in C, and $F \nleq e, f$, then $P(F) = \lambda e + \mu f$, hence equal to $\frac{1}{2}(e+f)$, since $\|P(F)\| = \frac{1}{2}$.

If there exists a minimal central projection g in B with $g \leq F$ then $P(F) \geq g$ so $\|P(F)\|=1$, while $\|P(F)\| = \frac{1}{2}$ if $F \notin B$. Thus $F \in B$. Q.E.D.

<u>Lemma 8</u>. Let M be a von Neumann algebra with center Z. Suppose P is a normal positive unital projection of M onto a JW-subalgebra B with $\iota - P$ contractive. Suppose e is a minimal central projection in B such that Be is of type I. Let E be the central carrier of e in M. Then $E \in B$, and either E is a minimal projection in Z, or there exist minimal projections F and G in Z belonging to the von Neumann algebra B'' such that $F + G = E = e$.

<u>Proof</u>. If E minimal in Z, $E \in B$ by Lemma 7. Assume E is not minimal. By [9, Lem.4.1] $P(exe) = eP(x)e$ for all $x \in M$, so P restricts to a projection with contractive complement of M_e onto Be. Since E is the central carrier of e and ME is not a factor, M_e is not a factor. Thus by Lemmas 2, 3, and 4 there are minimal central projections f and g in M_e such that $e = f + g$. If F and G are central carriers of f and g respectively then MF and MG are factors since M_f and M_g are, and $E = F + G$.

Let A be the abelian von Neumann algebra generated by Z and the center C of B. Since F is minimal in Z and e in C

it is easy to see that $f = Fe$ is minimal in A, and similarly $g = Ge$ is minimal. Thus by Lemma 7 applied to A, f, g and the restriction of P to A, $P(f) = P(g) = \frac{1}{2}e$. If $F = f$ and $G = g$ then $e = f + g = F + G = E$ and we are through.

Assume either $f \neq F$ or $g \neq G$. By symmetry we may assume $f' = F - f \neq 0$. There are three cases.

(i) If $B''e$ is not a factor then by Lemmas 4 and 5 $M_e = B''e$ and P restricted to M_e is of the form $\frac{1}{2}(\iota + \theta)$ with θ an antiautomorphism of order 2 of M_e. Since $P(f) = P(g) = \frac{1}{2}e$, $\theta f = g$.

(ii) If $B''e \neq Be$ and $B''e$ is a factor then by Lemma 3 $M_e = B''e$, contrary to the fact that M_e is not a factor.

(iii) If $B''e = Be$ is a factor then by Lemma 2 either $M_e = Be$, contrary to assumption, or $M_e = M_2$ with $B = \mathbb{C}$, contrary to assumption, or $M_e = Bf + Bg$, and $P(af+bg) = \frac{1}{2}(a+\alpha^{-1}(b))f + \frac{1}{2}(\alpha(a)+b)g$, where α is an isomorphism of Mf onto Mg.

From the above cases we may conclude that $M_e = M_f + M_g$, α is a Jordan isomorphism of M_f onto M_g such that $P(af+bg) = \frac{1}{2}(a+\alpha^{-1}(b))f + \frac{1}{2}(\alpha(a)+b)g$.

We have $f' = F - f \leq 1 - e$, hence $P(f') \leq 1 - e$. Since Be is of type I, so is $B''e$ by [6, Thm.8.2], hence M_f, therefore MF are type I factors. We can thus find a minimal projection $p \leq f$ in M and a partial isometry v in MF such that $v^*v = p$, $vv^* = q \leq f'$. Let $x = v + v^* + p - \alpha(p)$. Then $P(x) = 0$ since $v = (1-e)ve$.
Now $x^2 = q + 2p + v + v^* + \alpha(p)$, which can be identified with the 3×3 matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which has norm $\frac{1}{2}(3+\sqrt{5})$, hence $\|x^2\| = \frac{1}{2}(3+\sqrt{5})$. But we have

$$P(x^2) = P(q) + 2 \cdot \tfrac{1}{2}(p+\alpha(p)) + \tfrac{1}{2}(\alpha(p)+\alpha^{-1}(\alpha(p)))$$
$$= P(q) + \tfrac{3}{2}p + \tfrac{3}{2}\alpha(p),$$

which has norm $\frac{3}{2}$ since $P(q)$ is orthogonal to p and $\alpha(p)$. Thus $\|P(x^2)\| < \|x^2\|$, a contradiction, and we have shown that $F = f$, $G = g$, hence $E = g + f = e$. Q.E.D.

If u is a unitary operator in a von Neumann algebra M we denote by $\mathrm{Ad}u$ the automorphism $x \to uxu^*$.

Lemma 9. Let M be a type I factor and B a JW-subalgebra of type I which is not a JW-factor. Suppose P is a normal positive unital projection of M onto B with $\iota - P$ contractive. Then there exist two nonzero projections e and f in M with sum 1 such that $B = M_e + M_f$, and $P = \frac{1}{2}(\iota + \mathrm{Ad}(e-f))$.

Proof. We may assume $M = B(H)$ - the bounded operators on a complex Hilbert space H. Since 1 is a minimal central projection in M and B is not a JW-factor there exist by Lemma 7 two minimal central projections e and f in B with sum 1. We assert that $M_e = B''e$ and $M_f = B''f$. Suppose for example that $M_e \neq B''e$. Now M_e is a factor of type I and Be a JW-subfactor of type I and P a projection of M_e onto Be. By Lemmas 2, 3 and 4 the only possible case is when $M_e = M_2$ and $B''e = \mathbb{C}$. Suppose $M_e = M_2$ and let p be a minimal projection in M_f. Let $\{e_{ij} : 1 \leq i \leq j \leq 3\}$ be a complete set of matrix units for M_{e+p} such that $e = e_{11} + e_{22}$ and $p = e_{33}$. Then $P : M \to \mathbb{C}(e_{11} + e_{22}) + M_f$ and $P(e_{33}) \in M_f$. Let $x = e_{12} + e_{21} + e_{13} + e_{31}$

and let τ be the normalized trace on M_e. Then by Lemma 2 $P(x) = \tau(e_{12}+e_{21}) + P(e(e_{13}+e_{31})e + f(e_{13}+e_{31})f) = 0$. Now $x^2 = 2e_{11}+e_{22}+e_{23}+e_{32}+e_{33}$, which is twice a 2-dimensional projection, so has norm 2. By Lemma 2 applied to e_{11} and e we have

$$P(x^2) = P(e+e_{11}) + P(e_{33}) + P(e_{23}+e_{32})$$
$$= e + \tfrac{1}{2}e + P(e_{33}),$$

which has norm $3/2$, contradicting the assumption that $\iota - P$ is contractive [4, Thm.2.4]. Therefore $B''e = M_e$ and symmetrically $B''f = M_f$.

We next assert that $Be = B''e$ and $Bf = B''f$. Since $B''e = M_e = B(eH)$ there are three possibilities [6], namely: (1) $Be = B''e$, and we are through, (2) (resp. (3)) Be is the JW-factor such that $(Be)_p$ is isomorphic to the real symmetric matrices (resp. self-adjoint matrices with entries quaternions) whenever p is a finite dimensional projection in Be. Assume $Be \neq B''e$. Then $\dim e \geq 2$. We first show $\dim f = 1$. Indeed, if $\dim f \geq 2$ let $p \leq e$, $q \leq f$ be 2-dimensional projections in B. Then M_{p+q} can be identified with M_4 so that if we write M_4 as 2×2 matrices over M_2, then p is identified with $\begin{pmatrix}1 & \\ & 0\end{pmatrix}$ and q with $\begin{pmatrix}0 & \\ & 1\end{pmatrix}$. Let v be the self-adjoint unitary 2×2 matrix $\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}$, and let

$$x = \begin{pmatrix}0 & v\\ v & v\end{pmatrix} \in M_4 .$$

If B_q is the 2×2 real symmetric matrices then $P\begin{pmatrix}0 & \\ & v\end{pmatrix} = 0$ [9, Lem.6.1]. In the quaternion case $B_q = \mathbb{R}q$, so again by [9, Lem.6.1], $P\begin{pmatrix}0 & \\ & v\end{pmatrix} = 0$. Since $p \leq e$, $q \leq f$, we thus have $P(x) = 0$. Now $x^2 = \begin{pmatrix}1 & 1\\ 1 & 2\end{pmatrix}$, which has norm $\frac{1}{2}(3+\sqrt{5})$, while

$P(x^2) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ has norm 2, contradicting [4, Thm. 2.4]. Thus $\dim f = 1$, as asserted.

Since $\dim f = 1$, M_{p+f} can be identified with M_3. Let $\{e_{ij} : 1 \leq i,j \leq 3\}$ be matrix units for M_{p+f} as above with $e_{33} = f$. Let

$$x = \begin{pmatrix} 0 & i & i \\ -i & 0 & i \\ -i & -i & 0 \end{pmatrix}.$$

Since $B_{e_{11}+e_{22}}$ equals the 2×2 real symmetric matrices or $\mathbb{R}$ we have $P(x) = 0$. But

$$x^2 = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix},$$

so has norm $1 + \sqrt{5}$. We have

$$P(x^2) = P\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + P\begin{pmatrix} 0 & & \\ & 0 & \\ & & 2 \end{pmatrix}.$$

Since $\left\| \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \right\| = 3$, $\|P(x^2)\| \leq 3 < 1 + \sqrt{5}$, again a contradiction. Thus $Be = B''e = M_e$, and $Bf = B''f = M_f$. Therefore $B = M_e + M_f$, and for $x \in M$ we have

$$P(x) = P(exe + fxf) = exe + fxf$$
$$= \tfrac{1}{2}(x + \mathrm{Ad}(e-f)(x)). \qquad \text{Q.E.D.}$$

<u>Lemma 10.</u> Let M be a von Neumann algebra and P a normal, positive unital projection of M onto a JW-subalgebra B with $\iota - P$ contractive. Suppose e is a minimal central projection in B such that Be is of type I. Let E be the central carrier of e in M. Then $E \in B$, and the restriction of P to ME is

of the form $\frac{1}{2}(\iota + \theta)$ with θ a Jordan automorphism of ME of order 2.

<u>Proof</u>. By Lemma 8 $E \in B$, so $P : ME \to BE$. If $E = e$ the lemma follows from Lemmas 2, 3, 4, and 5. Assume $E \neq e$. By Lemma 7 $f = E - e$ is a minimal central projection in B. By Lemma 8 E is a minimal central projection in M, hence ME is a factor, and therefore M_e is a factor. By Lemma 5 Be is reversible, hence by [6, Thm.8.2] $B''e$ is of type I, since Be is. Since $P : M_e \to Be$ is a projection with contractive complement, M_e is of type I by Lemmas 2, 3, and 4. But then ME is a factor of type I, hence so is M_f. Let N denote the commutant of Bf in M_f. If $a \geq 0$ in N and $b \geq 0$ in Bf, then $P(a) \circ b = P(a \circ b) \geq 0$ by [9, Lem.4.1], so $P(a)$ is in the center of Bf by [9, Lem.3.1]. Thus P is a projection of N onto $\mathbb{C}$ with contractive complement. By Lemma 1 $N \subset M_2$, so $B''f$ is of type I. By [7, Thm.6.4] Bf is of type I, hence so is $B = Be + Bf$, and the lemma follows from Lemma 9. Q.E.D.

<u>Proof of Theorem</u>. If $P = \frac{1}{2}(\iota + \theta)$ with θ a Jordan automorphism of order 2 then $\iota - P = \frac{1}{2}(\iota - \theta)$ is a contraction. Conversely assume P is a positive unital projection on the C*-algebra A such that $\iota - P$ is a contraction on A_h. By [4,Lem.2.3] P is faithful, so by [3, Thm.1.4] $B = P(A_h)$ is a JC-subalgebra of A_h. Define $\theta = 2P - \iota$. Then obviously $\theta^2 = \iota$ and θ is a self-adjoint linear isomorphism of A onto itself. It thus suffices to show $\theta(x^2) = \theta(x)^2$ for each $x \in A_h$.

Going to the second dual we may assume $A \subset A^{**}$. Let $M = A^{**}$ and extend P to a normal positive unital projection of M onto B^-

also denoted by P. By Kaplansky's density theorem $\iota - P$ is a contraction on M. Let ρ be a pure state on B. Then ρ has a pure state extension to A, hence to a normal pure state $\bar{\rho}$ on M. Thus the restriction of $\bar{\rho}$ to B^- is normal pure state extending ρ. Let e_ρ be the minimal central projection in B^- such that $\bar{\rho}(e_\rho) = 1$ (for this see also [1]). By [7, Thm.4.3] $B^- e_\rho$ is a JW-factor of type I. Let E_ρ be the central carrier of e_ρ in M. By Lemma 10 $E_\rho \in B^-$ and P maps ME_ρ onto $B^- E_\rho$ and is of the form $\frac{1}{2}(\iota + \theta_\rho)$ with θ_ρ a Jordan automorphism of order 2 of ME_ρ. In particular $2P - \iota$ restricted to ME_ρ is a Jordan homomorphism for all pure states ρ of B. Since P is faithful by [4, Lem.2.3] and the pure states separate B, the projections E_ρ separate A, hence if $\theta = 2P - \iota$ then $\theta(x^2) = \theta(x)^2$ for all $x \in A_h$. Q.E.D.

References

1. E. Alfsen, F.W. Shultz and E. Størmer, A Gelfand-Neumark theorem for Jordan algebras, Advances in Math., 28 (1978), 11-56.

2. J. Arazy and Y. Friedman, Contractive projections in c_1 and c_∞, Mem.Amer.Math.Soc., 200 (1978).

3. E.G. Effros and E. Størmer, Positive projections and Jordan structure in operator algebras, Math.Scand., 45 (1979), 127-138.

4. A.G. Robertson and M.A. Youngson, Positive projections with contractive complements on Jordan algebras, to appear.

5. D.W. Robinson and E. Størmer, Lie and Jordan structure in operator algebras, J. Austral.Math.Soc. (Series A), 29 (1980), 129-142.

6. E. Størmer, Jordan algebras of type I, Acta math., 115 (1966), 165-184.

7. E. Størmer, Irreducible Jordan algebras of self-adjoint operators, Trans.Amer.Math.Soc., 130 (1968), 153-166.

8. E. Størmer, Jordan algebras versus C*-algebras, Quantum Dynamics: Models and Mathematics, ed. L. Streit, pp. 1-13, Springer Verlag, Wien-New York. 1976.

9. E. Størmer, Decomposition of positive projections on C*-algebras, Math.Ann., 247 (1980), 21-41.