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ON THE PROJECTIVE PLANE

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§ 0. Introduction

The starting point of the present work was the following problem, posed by Bănică in [1]: Characterize those rank-2 vector bundles E on P^2 that can be deformed to the trivial bundle $2O_{P^2}$. A necessary condition is clearly that the Chern classes of E be trivial. If we denote by $d(E)$ the largest integer such that $E(-d(E))$ has a global section, it is known that E is always deformable if $d(E) < 2$ [1, 11]. Even in this case, to my knowledge, no reasonably explicit families are known.

The topologically trivial 2-bundles E on P^2 with given type $d(E) = d > 1$ are parametrized in a natural way by an irreducible nonsingular variety $M(d)$ of dimension $(3d^2-1)$, see (2.7) below, or [1] or [11]. Denoting by $M(d;0) \subseteq M(d)$ the set corresponding to bundles that can be deformed to the trivial one, we show in this paper that $M(d;0)$ is closed of codimension $(d-1)(d-2)$ in $M(d)$, thus covering the result of Schafft mentioned above (3.13) and (4.7). We also give one sufficient and one necessary condition for deformability to the trivial bundle (4.6), (5.2). In turns out that the methods we use work for arbitrary Chern classes c_1 and c_2, and hence give similar results on questions like: which bundles are deformable to a stable bundle, or a decomposable bundle, or just to any bundle which is "more stable".

The material is organized as follows: § 1 contains basic preliminaries on 2-bundles on P^2, the invariant $d(E)$ is defined, and dimensions of some cohomology groups are computed. § 2 contains a study of general families of buundles. In § 3 some extra structure is added to a bundle in order to overcome the difficulties arising from the non-existence of moduli for unstable bundles. We describe a certain Quot scheme and determine its irreducible components, allowing us to conclude that if d is large enough, the general bundle of type d is not deformable to smaller type. In § 4 we give an explicit construction of many deformable bundles. § 5 contains some conjectures.

I would like to express thanks to C. Banica for reawakening my interest in bundles on P^2, and to O. A. Laudal for inspiration and patient listening.

<u>Notation</u> and <u>conventions</u>:

k - an algebraically closed field of characteristic $\neq 2$.

P^2 - projective plane over k.

O - structure sheaf on P^2.

$p_T : P^2_T = P^2 \times T \to T$ is the second projection.

Bundle - see (1.1)

If F is a coherent sheaf on a k-scheme X, then $H^0(F) = H^0(X,F)$ and $h^0(F) = \dim_k H^0(F)$. $\chi(F) = \sum_i (-1)^i h^i(F)$ is the Euler characteristic.

P - the polynomial $P(x) = (x-1)(x-2-c_1) - c_2$

χ = the number $4 - 4c_2 - c_1$, see (1.1) for c_1 and c_2.

§ 1. Preliminaries

(1.1) Let c_1 and c_2 be two integers such that $c_1(c_1+1) = 0$. We shall use the term bundle (with no further qualification) to designate a locally free sheaf E of rank 2 on P^2 with Chern classed $c_1(E) = c_1$ and $c_2(E) = c_2$. Recall that a bundle E is stable (in the sense of Mumford-Takemoto) if and only if $h^0(E) = 0$. If E is unstable (not stable), denote by $d = d(E)$ the largest integer such that $h^0(E(-d)) \neq 0$. Then $d(E)$ is called the type of E. If E is stable, we agree to put $d(E) = -1$. If $d \geq -1$ is an integer, we shall denote by $B(d)$ the set of isomorphism classes of bundles of the given type.

(1.2) Proposition: $B(d)$ is nonempty if and only if one of the following two conditions hold:

(i) $d \geq 0$ and $d^2 - dc_1 + c_2 > 0$

(ii) $d = -1$ and $c_2 - c_1 \geq 2$

Proof: For stable bundles, this is well known, see for example [9]. If E is a bundle of type $d \geq 0$, pick a non-zero section s of $E(-d)$. Then its zero-scheme $Y = V(s)$ has codimension two, hence it has finite length $c_2(E(-d)) = d^2 - dc_1 + c_2$. Conversely, suppose (i) holds, and let Y be a closed, locally complete intersection subscheme of finite length $= (d^2 - dc_1 + c_2)$. Then a sufficiently general extension of $I_Y(c_1 - d)$ by $O(d)$ gives a bundle of type d. See for example [5, thm. 1.1] for details.

(1.3) From the above proof we obtain, for each unstable bundle E, a basic exact sequence

$$0 \to O(d) \xrightarrow{s} E \to I_Y(c_1 - d) \to 0$$

If E is not the trivial bundle $2O$, the minimal section s is unique up to a non-zero scalar. In particular, the subscheme $Y = Y(E)$ is uniquely determined by E. The element $\xi(E)=\xi$ of $\mathrm{Ext}^1(I_Y(c_1-d),O(d))$ corresponding to the basic exact sequence is zero if and only if E is decomposable (and $Y = \emptyset$), otherwise it is determined up to a non-zero scalar only. It has the property that the induced global section of the local Ext-sheaf $\underline{\mathrm{Ext}}^1(I_Y(c_1-d),O(d)) \cong \omega_Y$ generates this sheaf.

(1.4) Proposition: Let E be a bundle of type $d \geq -1$. Then the following equalities hold:

(i) $\chi(\underline{\mathrm{End}}\ E) = \chi = 4 - c_1 - 4c_2$

(ii) If $d \geq 0$, then

$$h^0(\underline{\mathrm{End}}\ E) = 2d(d-c_1) + 3d - 2c_1 + 2 + \alpha + \alpha\beta$$

$$h^1(\underline{\mathrm{End}}\ E) = 4(d^2 - dc_1 + c_2) - 1 + \alpha - \beta + \alpha\beta$$

$$h^2(\underline{\mathrm{End}}\ E) = 2d(d-c_1) - 3d + c_1 + 1 - \beta$$

where $\alpha = \begin{cases} 0 & \text{if } d^2 - dc_1 + c_2 > 0 \\ 1 & \text{if } d^2 - dc_1 + c_2 = 0 \end{cases}$

and $\beta = \begin{cases} 0 & \text{if } (d,c_1) \neq (0,0) \\ 1 & \text{if } d = c_1 = 0 \end{cases}$

(iii) If $d = -1$, then

$$h^0(\underline{\mathrm{End}}\ E) = 1$$

$$h^1(\underline{\mathrm{End}}\ E) = c_1 + 4c_2 - 3$$

$$h^2(\underline{\mathrm{End}}\ E) = 0$$

Proof: (i) Follows from Riemann-Roch, and (iii) is well known. (ii) is an easy computation, which is left to the reader with the remark that, if $E \neq 2O$, then any endomorphism of E must leave the image of the map $s:O(d) \to E$ invariant.

§ 2. Families of bundles

(2.1) Let T be a k-scheme, and E a locally free sheaf of rank 2 on P^2_T. If t is a k-point of T, we denote by E_t the restriction of E to $P^2 = P^2 \times \{t\} \subseteq P^2_T$, and call it the fiber of E over t. If all the fibers of E are bundles (recall the convention of (1.1)) we shall say that E/T, or simply E, is a family of bundles (parametrized by T). The family is called irreducible if T is an irreducible scheme of finite type over k. If E/T is an irreducible family, the type of E, denoted by $d(E)$, is defined to be the minimum of the $d(E_t)$, $t \in T$. A bundle E is deformable to type d if it is a fiber in some irreducible family of type d.

(2.2) By Serre duality $h^0(E(-d)) = H^2(E^\vee(d-3))$, hence if $d > 0$ we have $d(E) > d$ if and only if $h^2(E^\vee(d-3)) \neq 0$. If E/T is a family of bundles, the coherent sheaf $R^2 p_{T}{*}E^\vee(d-3)$ commutes with base change on T, thus the type of E_t is an upper semicontinuous function on T. From [10, lecture 8] we also obtain a unique maximal locally closed subscheme $T(d) = T(d;E) \subseteq T$ with the property that the restriction to $T(d)$ of $R^2 p_{T}{*}E^\vee(d-3)$ is locally free of rank 1. Let us make the following definition:

(2.3) Let $d > 0$, and let E/T be a family. We say that E has pure type d if $R^2 p_{T}{*}E^\vee(d-3)$ is locally free of rank 1. We agree that E has pure type -1 if all the fibres are stable bundles.

(2.4) Remark: If E/T has pure type $d > 0$, then all the fibers E_t have type d. If T is reduced of finite type over k, the converse also holds.

(2.5) Lemma: Let E/T be a family and $d > 0$ an integer. Then E has pure type d if and only if $p_T{*}E(-d)$ is locally free of rank 1 and commutes with base change on T.

Proof: Follows from the relative duality isomorphism, functorial in N,

$$D^0 : p_T{*}(E(-d) \otimes p_T^*N) \to \underline{\mathrm{Hom}}(R^2 p_T{*}E^\vee(d-3), N)$$

where N is any quasicoherent sheaf on T [7, thm 21].

(2.6) Consider the following (contravariant) functors $\underline{B}(d)$: (schemes/k) $\to$ (sets) defined by $\underline{B}(d)(T) = \{$families E/T of pure type $d\}/\sim$

where $E_1 \sim E_2$ if there exists a linebundle L on T and an isomorphism $E_1 \cong E_2 \otimes p_T^* L$.

(2.7) Proposition: There is an irreducible, nonsingular, quasi-projective and rational variety $M(d)$ acting as a coarse moduli space for $\underline{B}(d)$. If $d = -1$, then $\dim M(d) = 4c_2 + c_1 - 3$, and if $d > 0$ and $d^2 - dc_1 + c_2 = 0$, then $M(d)$ is a point. If $d > 0$ and $d^2 - dc_1 + c_2 > 0$, then $\dim M(d) = 3(d^2 - dc_1 + c_2) - 1$.

Proof: For the case $d = -1$, see [2, 6, 3]. The case $d^2 - dc_1 + c_2 = 0$ is trivial. If $d^2 - dc_1 + c_2 > 0$, we apply the same construction as is carried out in [1] and [11] for topolocally trivial bundles. Note that the correspondence $E \leftrightarrow (Y(E), \xi(E))$ of (1.4) generalizes to families of pure type, by (2.5). Thus, if H denotes the Hilbert scheme parametrizing the $Y(E)$'s, one can obtain $M(d)$ as an open subset of $P_H(D^\vee)$, where D is a certain locally free sheaf on H inducing $\mathrm{Ext}^1(I_Y(c_1 - d), O(d))$ in each point $\{Y\} \in H$. We omit the details.

(2.8) Remark: If E is a bundle, the tangent space of its infinitesimal deformation functor is canonically isomorphic to $\mathrm{Ext}^1(E,E) = H^1(\underline{\mathrm{End}}(E))$. Letting E correspond to the point $m \in M(d)$, $d > 0$, the tangent space of $M(d)$ at m can be identified with the kernel of the map

$$\mathrm{Ext}^1(E,E) \to \mathrm{Ext}^1(O(d),\ I_Y(c_1 - d))$$

induced by the basic sequence of (1.3).

§ 3. Bundles with sections

(3.1) Unfortunately, the functor of all families of bundles is not even prorepresentable. To remedy this we do the fairly standard trick of adding enough structure to the bundles to obtain a representable functor. One way of doing this is to choose a k-basis for the vector space $H^0(E(n))$ for some large but fixed n, leading to the Quot scheme of Grothendieck. To start with, fix a type $\bar{d}$ and an integer n large enough that $E(n)$ is generated by its global sections and $H^i(E(n)) = 0$ for $i \neq 0$ for all $E \in B(d)$ with $d \leq \bar{d}$. Put $N = (n+1)(n+2+c_1) - c_2$, then, by Riemann-Roch, $h^0(E(n)) = N$ for all these E. Let V be a vector space of dimension N. As a matter of notation, put $V(-n) = V \otimes_k O(-n)$ and $V_T(-n) = V \otimes_k O_{P^2_T}(-n)$ for any k-scheme T.

(3.2) Denote by Quot the Grothendieck Quot scheme parametrizing quotients $V(-n) \to F$ with Hilbert polynomial $\chi(F(\nu)) = (\nu+1)(\nu+2+c_1) - c_2$ [4] and let $Q \subseteq$ Quot be the open subscheme defined by the conditions (i) F is locally free of type $\leq \bar{d}$, and (ii) the induced map $V \to H^0(F(n))$ is an isomorphism. Q comes equipped with a universal exact sequence

$$0 \to \mathcal{K} \to V_Q(-n) \xrightarrow{\tilde{q}} \mathcal{F} \to 0.$$

From deformation theory we have the following facts about the local structure of Q:

(3.3) <u>Proposition:</u> <u>Let</u> $q = (0 \to K \to V(-n) \to F \to 0)$ <u>be a closed point of</u> Q. <u>Let</u> $\dim(Q_q)$ <u>be the dimension of</u> Q <u>at</u> q, <u>and denote by</u> A <u>the sheaf</u> <u>Hom</u>(K,F).

(i) <u>The tangent space</u> $T_{Q,q}$ <u>is canonically isomorphic to</u> $H^0(A)$

(ii) $h^0(A) - h^1(A) \leq \dim (Q_q) \leq h^0(A)$

For a proof, see for example [8, (4.2.4)]

(3.4) <u>Corollary</u>: <u>Let</u> $\chi = 4 - 4c_2 - c_1$. <u>Then</u> $N^2 - \chi \leq \dim Q_q \leq N^2 - \chi + h^2(\underline{\mathrm{End}}\ F)$. Q <u>is smooth at</u> q <u>if and only if the rightmost inequality is an equality</u>.

<u>Proof</u>: This is essentially a rephrasing of (3.3), applying <u>Hom</u> (-,F) to the sequence q of (3.3), and noting that $\chi = \chi(\underline{\mathrm{End}}\ F)$.

(3.5) F is a family of vector bundles parametrized by Q. Recalling the definition of (2.3), denote by $Q(d) = Q(d;F)$ the maximal locally closed subscheme such that the restriction F^d of F to $P^2_{Q(d)}$ is of pure type $d \geq 0$. Also denote by $Q(-1)$ the open subscheme parametrizing stable quotients, and by F^{-1} the corresponding restriction. $\bar{Q}(d)$ will denote the closure of $Q(d)$ in Q.

(3.6) Since for each d $(-1 \leq d \leq \bar{d})$, $M(d)$ is a coarse moduli space, we get a uniquely defined morphism

$$\phi = \phi(d) : Q(d) \to M(d)$$

defined by the family F^d. It is clearly surjective by our choice of n.

3.7. <u>Proposition</u>: $\phi(d)$ <u>is smooth of relative dimension</u> $N^2 - h^0(\underline{\mathrm{End}}\ E)$, <u>where</u> E <u>is any bundle in</u> B(d).

<u>Proof</u>: Let $q : V(-n) \to E$ be any k-point of $Q(d)$ and put $K = \ker q \subseteq V(-n)$. The smoothness follows easily from the infinitesimal lifting criterion [12], using that $H^1(E(n)) = 0$. For the dimension assertion, apply the functor $\mathrm{Hom}(-,E)$ to the sequence $0 \to K \to V(-n) \to E \to 0$ to get

$$0 \to \mathrm{End}\ E \to \mathrm{Hom}(V(-n),E) \to \mathrm{Hom}(K,E) \xrightarrow{\delta} H^1(\underline{\mathrm{End}}\ E) \to 0$$

Clearly, $T_{Q(d),q} = \delta^{-1} T_{M(d),\{E\}}$ as a subspace of $T_{Q,q} = \mathrm{Hom}(K,E)$. The assertion follows.

(3.8) Remark: There is an obvious action of $\mathrm{Aut}(V) = GL(N)$ on $Q(d)$, transitive on the fibers of $\phi(d)$, and $M(d)$ is the (geometric) quotient of $Q(d)$ by this action. The isotropy subgroup of a point $q : V(-n) \to E$ is easily seen to be isomorphic to $\mathrm{Aut}(E)$.

(3.9) Theorem: Assume that $-1 \leq d \leq \bar{d}$ and that $B(d)$ is nonempty. Let P be the polynomial $P(x) = (x-1)(x-2-c_1) - c_2$ and $\chi = 4 - 4c_2 - c_1$.

(i) $Q(d)$ is irreducible and smooth of dimension

$$\dim Q(d) = \begin{cases} N^2 - \chi & \text{if } d = -1 \text{ or } d = c_1 = c_2 = 0 \\ N^2 - \chi + P(d) - 1 & \text{otherwise} \end{cases}$$

(ii) $\bar{Q}(d)$ is an irreducible component of Q_{red} if and only if $\dim Q(d) \geq N^2 - \chi$.

(iii) $\bar{Q}(d)$ is not an irreducible component of Q_{red} if and only if one of the following two conditions holds:

(a) $0 < c_2 \leq 1 + c_1$ and $1 \leq d \leq 2 + c_1$

(b) $c_2 > 2 + c_1$ and $d > 0$ and $P(d) \leq 0$.

(iv) If $d = -1$ or $d^2 - dc_1 + c_2 = 0$, then Q is nonsingular along $Q(d)$. In the remaining cases, if $\bar{Q}(d)$ is a component of Q_{red}, the embedding codimension of Q at any point of $Q(d)$ equals $(d^2 - dc_1 + c_2)$, hence Q is not reduced along $Q(d)$.

Proof: (i) follows from (3.7), (2.7) and (1.3). (ii) The necessity of the condition has already been noted (3.4). Conversely, if $\bar{Q}(d)$ is not a component, it must be contained in $\bar{Q}(e)$ for some $e < d$. Consequently $\dim Q(d) < \dim Q(e)$. If $e = -1$ or $e = c_1 = c_2 = 0$, we are done, since $\dim Q(e) = N^2 - \chi$ in this case, by (i). In the remaining cases we must have $P(d) < P(e)$. Looking at the definition of P, we must have $e = 0$, and either

$c_1 = 0$ and $1 \leq d \leq 2$, or $c_1 = -1$ and $d = 1$. In either case $P(d) = -c_2 \leq 0$ since $Q(0)$ is nonempty. Hence $\dim Q(d) \leq N^2 - \chi - 1$. (iii) is merely a rephrasing of (ii). (iv) follows from (3.4) and (i) above.

(3.10) We want to interpret thorem (3.9) in terms of the bundles themselves, without reference to the Quot scheme. Recalling the notion of deformability (2.1), let us consider the following subset of $B(d)$, where $-1 \leq e < d$ are integers:

$$B(d;e) = \{D \in B(d) \text{ such that } D \text{ is deformable to type } e\}.$$

(3.11) Lemma: Let $q : V(-n) \to D$ be a point of $Q(d)$. Then q is in $\bar{Q}(e)$ if and only if D is in $B(d;e)$.

Proof: If $\mathcal{D}/T$ is an irreducible family of type e containing D, the sheaf $p_T{*}\mathcal{D}(n)$ is locally free of rank N. Shrinking T if necessary, we may assume it is free. Then the isomorphism $V \to H^0(D(n))$ extends to an isomorphism $V \otimes O_T \to p_T{*}\mathcal{D}(n)$, hence gives a morphism $T \to Q$ hitting the point q. Since the generic point of T lands in $Q(e)$, we are done.

(3.12) It follows that the subset $M(d;e) \subseteq M(d)$ corresponding to $B(d;e) \subseteq B(d)$ is closed and that $Q(d) \cap \bar{Q}(e) = \phi(d)^{-1} M(d;e)$. In particular, the number of irreducible components of $M(d;e)$ and their codimensions in $M(d)$ are the same as those for $Q(d) \cap \bar{Q}(e)$ in $Q(d)$. Theorem (3.9) thus translates as follows:

(3.13) <u>Theorem:</u> <u>Let</u> $-1 \leqslant e < d$, <u>and put</u>

$$\gamma = \begin{cases} P(d) & \underline{\text{if}}\ e = -1 \ \underline{\text{or}}\ e = c_1 = c_2 = 0 \\ P(d) - P(e) + 1 & \underline{\text{otherwise}} \end{cases}$$

(i) <u>Each irreducible component of</u> $M(d;e)$ <u>has codimension at least</u> γ <u>in</u> $M(d)$.

(ii) <u>Assume</u> $\gamma \leqslant 0$. <u>Then</u> $e \leqslant 0$. <u>Furthermore, if</u> $e_0 < e$ <u>is the smallest integer such that</u> $M(e_0)$ <u>is nonempty, then</u> $M(d;e_0) = M(d)$.

(iii) <u>Let</u> E <u>be a bundle of type</u> d, <u>and let</u> H <u>denote the prorepresentable hull of the deformation functor of</u> E [12] ("the base of the miniversal deformation of E"). <u>Then, if</u> $d = -1$ <u>or</u> $d^2 - dc_1 + c_2 = 0$, <u>then</u> H <u>is nonsingular. Otherwise,</u> H <u>is non-reduced of embedding codimension</u> $(d^2 - dc_1 + c_2)$, <u>unless</u> $M(d) = M(d;e)$ <u>for some</u> $e < d$.

<u>Proof:</u> (i) By (3.9, i) $\gamma = \dim Q(d) - \dim Q(e) + 1$, clearly a lower bound for the codimension of $Q(d) \cap \bar{Q}(e)$ in $Q(d)$. Then use (3.12). (ii) follows from (3.9, ii) in the same way. For (iii), one uses that the fiber functor of Q at a point $q : V(-n) \to E$ is smooth over H, and (3.9, iv).

§ 4. Existence

(4.1) Our results so far are mainly negative: With a few exceptions (3.13 ii) the general bundle of a given type is not deformable to smaller type. The purpose of this section is to give an explicit family of deformable bundles, large enough to show that all the $M(d;e)$ are non-empty. For our description we shall use monads, i.e. three-term complexes resolving the bundles.

(4.2) Let E be a given bundle of type e, and $\tau \in H^0(E(d - c_1))$ a global section vanishing in a subscheme $Y \subseteq P^2$ of codimension 2, where $d > e$ is some integer. Furthermore, let $F \in H^0(O(2d - c_1))$ be the equation of a curve disjoint from Y. Denote by D the cohomology bundle of the following monad:

$$O(c_1 - d) \xrightarrow{b} O(c_1 - d) \oplus E \oplus O(d) \xrightarrow{a} O(d)$$

where the maps a and b are given by

$$a = (F, \tau\wedge, 0) \quad \text{and} \quad b = (0, \tau, -F)^t.$$

Let $\xi \in \operatorname{Ext}^1(I_Y, O(c_1 - 2d))$ correspond to the short exact sequence induced by the section τ.

(4.3) <u>Proposition</u>: (i) D <u>is deformable to type</u> e.

(ii) $d(D) = d$

(iii) $Y(D) = Y$

(iv) $\xi(D) = F^2\xi$

<u>Proof</u>: (i) Substitute a non-zero scalar λ for the zero in the maps a and b of the monad. Then the cohomology bundle is clearly E. Thus we have a family deforming D to E, letting $\lambda \to 0$.

(ii)-(iv): Consider the display of the monad (4.2):

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & O(c_1 - d) & \longrightarrow & B^{\vee}(c_1) & \longrightarrow & D & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & O(c_1 - d) & \longrightarrow & O(c_1 - d) \oplus E \oplus O(d) & \longrightarrow & B & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & O(d) & = & O(d) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The projection of the middle term onto the first factor $O(c_1 - d)$ factors through B, hence $O(c_1 - d)$ is a direct summand of B. In fact, we have $B = O(c_1 - d) \oplus N(d)$, where $N(d)$ is the cokernel of the map $(\tau, - F)^t\colon O(c_1 - d) \to E \oplus O(d)$. Hence N is locally free of rank 2 (with $c_1(N) = 0$ and $c_2(N) = d^2 - dc_1 + c_2$). Now the proposition is a direct consequence of the following

(4.4) <u>Lemma:</u> <u>There is a commutative diagram with exact rows and cartesian (pushout) left squares:</u>

$$\begin{array}{ccccccccc}
0 & \longrightarrow & O(c_1 - 2d) & \longrightarrow & E(-d) & \longrightarrow & I_Y & \longrightarrow & 0 \\
 & & \downarrow F & & \downarrow & & \| & & \\
0 & \longrightarrow & O & \longrightarrow & N & \longrightarrow & I_Y & \longrightarrow & 0 \\
 & & \downarrow F & & \downarrow & & \| & & \\
0 & \longrightarrow & O(2d - c_1) & \longrightarrow & D(d - c_1) & \longrightarrow & I_Y & \longrightarrow & 0
\end{array}$$

<u>Proof:</u> The first two rows follows directly from the definition of N above. The last two rows follow from the top row of the display in (4.3), noting that $B^{\vee} \cong O(d - c_1) \oplus N(-d)$.

(4.5) Theorem: Let D be a bundle of type $d > 0$, and assume that there exists an exact sequence

$$0 \to O(c_1 - d) \to E \to I_Y(d) \to 0$$

where $Y = Y(D)$ and E is some bundle of type e. Then D is deformable to type e.

Proof: By (4.3) it suffices to show that the map $H^0(O(2d - c_1)) \times \mathrm{Ext}^1(I_Y(d), O(c_1 - d)) \to \mathrm{Ext}^1(I_Y(c_1 - d), O(d))$ given by $(F, \xi) \to F^2\xi$ is dominating. Let ξ_0 be the extension given by the exact sequence above, and fix a form $F_0 \in H^2(O(2d - c_1))$ that defines a curve disjoint from Y. Differentiating the above map at the point (F_0, ξ_0), we obtain the linear map $(F, \xi) \to F_0(F_0\xi + 2F\xi_0)$. Since the map F_0: $\mathrm{Ext}^1(I_Y, O) \to \mathrm{Ext}^1(I_Y(c_1 - d), O(d))$ is an isomorphism by our choice of F_0, it suffices to show that the map $\beta: H^0(O(2d - c_1)) \times \mathrm{Ext}^1(I_Y(d), O(c_1 - d)) \to \mathrm{Ext}^1(I_Y, O)$ given by $\beta(F, \xi) = F_0\xi + 2F\xi_0$ is surjective. To see that this is the case, use the exact sequence above to construct the right-hand column in the following diagram:

$$\begin{tikzcd}
H^0(O(2d - c_1)) \arrow[r, "\alpha"] \arrow[d] & H^0(O(2d - c_1)) \arrow[d] \\
H^0(O(2d - c_1)) \times \mathrm{Ext}^1(I_Y(2d - c_1), O) \arrow[r, "\beta"] \arrow[d] & \mathrm{Ext}^1(I_Y, O) \arrow[d, "p"] \\
\mathrm{Ext}^1(I_Y(2d - c_1), O) \arrow[r, "\gamma"] & \mathrm{Ext}^1(E(-d), O)
\end{tikzcd}$$

The upper right vertical map is $F \to F\xi_0$, the map α is multiplication by 2, the left vertical maps are the obvious ones, and γ is multiplication by F_0 followed by p. The diagram clearly commutes and the columns are exact. Since char $(k) \neq 2$, α is an isomorphism. Thus it suffices to show that γ is surjective. Note that γ must factor through the map $F_0: \mathrm{Ext}^1(E(d - c_1), O) \to \mathrm{Ext}^1(E(-d), O)$. Using that E is locally free, we identify this map with the map $F_0: H^1(E^\vee(c_1 - d)) = H^1(E(-d)) \to H^1(E(d - c_1)) = H^1(E^\vee(d))$, which in turn is nothing but $F_0: H^1(I_Y) \to H^1(I_Y(2d - c_1))$, which is clearly surjective by our choice of F_0.

(4.6) <u>Remark</u>: As a special case $(e = c_1 = c_2 = 0)$ we find that D is deformable to the trivial bundle if $Y(D)$ is a complete intersection of two curves of degree d (or a flat specialization of such complete intersections). For $d = 1$ or 2, we recover once more the result of Banica and Schafft mentioned in the introduction. It is also a special case of (3.13 ii), our proof of which is essentially equivalent to those in [1] and [11]. We shall see later that if D is deformable to the trivial bundle, then at least $h^0(I_{Y(D)}(d)) \geq 2$, thus giving a partial converse to (4.5) in this case. See (5.2) below.

(4.7) <u>Theorem</u>: <u>Let</u> $-1 \leq e < d$ <u>be integers such that</u> $M(e)$ <u>is nonempty. Assume that either</u>

(a) $e = -1$ <u>or</u>

(b) $e \geq 0$ <u>and</u> $\binom{d - e - 1}{2} \geq e^2 - ec_1 + c_2$.

<u>Then</u> $M(d;e)$ <u>contains an irreducible component for which the bound of</u> (3.13 i) <u>is sharp</u>.

<u>Proof</u>: In case (a), if $P(d) = (d - 1)(d - 2 - c_1) - c_2 \leq 0$, the result $M(d; -1) = M(d)$ is covered by (3.13 ii). We will thus assume that $P(d) > 0$ in this case. In both cases, let E be a

general bundle of $M(e)$. I claim that $H^i(E(d - c_1 - 3)) = 0$ for $i \neq 0$. Indeed, in case (a), E has natural cohomology, and $\chi(E(d - c_1 - 3)) = P(d) > 0$ by Riemann-Roch. In case (b), $H^i(E(d - c_1 - 3)) = H^i(I_{Y(E)}(d - e - 3))$ by the basic exact sequence. The assumption in (b) is that $\chi(I_{Y(E)}(d - e - 3)) > 0$. Since $Y(E)$ is in general position, the claim follows also in this case.

Now let $A(d;e) \subseteq M(e)$ be the open subset of those E for which $H^i(E(d - c_1 - 3)) = 0$ for $i \neq 0$. Denote by Z the set of pairs (E,τ) where $E \in A(d;e)$ and $\tau \in H^0(E(d - c_1))$ is a section vanishing in codimension two, and let W be the subset of the Hilbert scheme of P^2 consisting of the $V(\tau)$ for $(E,\tau) \in Z$. From $H^i(E(d - c_1 - 3)) = 0$ for $i \neq 0$ one deduces easily that $\dim \operatorname{Ext}^1(I_Y(d), O(c_1 - d)) = 1$ for any $Y \in W$. Hence, if (E,τ) and (E',τ') are in Z with $V(\tau) = V(\tau')$, then $E \cong E'$ and τ and τ' are equivalent under the natural action of $\operatorname{Aut}(E)$ on $H^0(E(d - c_1))$. Hence the dimension of W is $\dim W = \dim M(e) + \chi(E(d - c_1)) - h^0(\underline{\operatorname{End}}\, E)$, hence its codimension in the Hilbert scheme is $(2(d^2 - dc_1 + c_2) - \chi(E(d - c_1)) - (\dim M(e) - h^0(\underline{\operatorname{End}}\, E)) = (P(d) - \chi) - (\dim Q(e) - N^2)$. Using (3.9), this is precisely the bound γ of (3.13). The proof is completed using (4.5).

(4.8) In the remaining case, $e > 0$ and $\binom{d - e - 1}{2} < e^2 - ec_1 + c_2$, a little more care is needed to compute the dimension of the family satisfying the hypothesis of (4.5). The differences from the above proof are the following: For $A(d;e) \subseteq M(e)$ take the (locally closed) set of those E for which $Y(E)$ is contained in some irreducible curve of degree $(d - e)$, but otherwise in general position. Then proceed as above, noting that $\dim \operatorname{Ext}^1(I_Y(d), O(c_1 - d)) = 1 + h^1(I_{Y(E)}(d - e - 3))$. The codimension of $A(d;e)$ in $M(e)$ is positive if $\chi(I_{Y(E)}(d - e)) < 0$, but the dimension of $H^0(E(d - c_1))$ increases by the same amount, so that the formula $\dim Z = \dim M(e) + \chi(E(d - c_1))$ still holds. We end up having proved:

(4.9) Theorem: Let $0 < e < d$ be integers such that

$$\delta = e^2 - ec_1 + c_2 - \binom{d - e - 1}{2} > 0$$

Then $M(d;e)$ contains an irreducible component of codimension at most $P(d) - P(e) + 1 + \delta$ in $M(d)$.

(4.10) Example: If $d = e + 1$, a bundle $D \in M(d)$ such that $Y(D)$ is on a line is in $M(d;d - 1)$. The family of such D has codimension $(d^2 - dc_1 + c_2 - 2)$, which agrees with the formula of (4.9).

§ 5. Open questions

(5.1) Some problems arising naturally from our results so far are the following:

(I) Is the family constructed by the method of (4.2) dense in $M(d;e)$?

(II) If D is of type $d > 0$ and $H^1(I_{Y(D)}(2d - c_1 - 3)) = 0$, does it follow that D is not deformable to smaller type?

(III) In the same situation, will $H^1(I_{Y(D)}(2d - c_1 - 3)) \neq 0$ imply that D is deformable?

(IV) If Y is a finite subscheme of P^2 with $H^1(I_Y(n)) \neq 0$, does it follow that Y is the limit of a family $Y_t \subseteq P^2$ such that (Y_t, n) satisfy the Cayley-Bacharach condition for $t \neq 0$? (The CB condition is that any curve of degree n containing all but one point of Y contains Y.)

Clearly, (I) => (II) and (IV) => (III) (using 4.5). As evidence for (I), at least in the case the D can be deformed to a decomposable bundle, let ut prove the following necessary condition for deformability.

(5.2) Proposition: Let $D \in M(d;e)$ and suppose that $e^2 - ec_1 + c_2 = 0$. Then there are curves C_1 and C_2 of degrees $(d-e)$, $(d+e-c_1)$ respectively, C_2 not containing C_1, such that $Y(D)_{red} \subseteq C_1 \cap C_2$.

Proof: One shows easily that there is a nonsingular affine curve T with distingnished point $0 \in T$ parametrizing a family E with $E_0 \cong D$ and $E_t \cong O(e) \oplus O(c_1 - e)$ for $t \in T - \{0\}$. Let λ be a uniformizing parameter at 0. For each integer m the base change map $H^0(E(m)) \otimes_T k(0) \to H^0(E_0(m))$ is injective. Let $\sigma^i \in H^0(E(m_i))$ $i=1,2$ be chosen so that: (i) $m_1 = -e$, $m_2 = e - c_1$, (ii) the map $\sigma^1 \oplus \sigma^2$: $O_{P_T^2}(e) \oplus O_{P_T^2}(c_1 - e) \to E$ is an isomorphism over $T - \{0\}$, and (iii) σ^i is not divisible by λ. Let σ_0^i be the image of σ^i in $H^0(E_0(m_i))$. If $s \in H^0(E_0(-d))$ is the minimal section, there are forms G^i of degree $d + m_i$ such that $\sigma_0^i = G^i s$. Let $C_i = V(G^i)$. Since $V(\sigma^i)$ has no component of codimension 3, and furthermore has support in $V(\sigma_0^i) = Y(D) \cup C_i$, we conclude that $Y(D)_{red} \subseteq C_i$. The assertion that C_1 is not contained in C_2 follows from the injectivety of the base-change map for m_2.

(5.3) Remark: We have not ruled out the possibility of C_1 and C_2 having a common component. Under the additional hypothesis that the curve of minimal degree containing $Y(D)$ be irreducible, it does follow that $Y(D)$ is a complete intersection. In particular, if $d = e + 1$ (and $e^2 - ec_1 + c_2 = 0$) then we can answer (I) affirmatively. In the general case $(e^2 - ec_1 + c_2 > 0)$ this method seems only to show that at least $(d^2 - dc_1 + c_2) - (e^2 - ec_1 + c_2)$ points of $Y(D)$ are contained in a curve of degree $(d-e)$. A slightly different approach would be to consider the case $m = (d-c_1)$ and consider the exact sequence

$$H^0(E(d-c_1)) \to H^0(E_0(d-c_1)) \to H^1(E(d-c_1)).$$

All sections of $H^0(E_0(d-c_1))$ are of the form Fs, where $F \in H^0(O(2d-c_1))$. The strategy would be to find an F such that

(i) Fs is the image of a section $\sigma \in H^0(O(d-c_1))$, and (ii) $V(F) \cap Y(E_0)$ is empty. It would follow that $V(\sigma) = X$ would have codimension 2, and that $Y(E_0)$ would be the (flat) limit of X_t. A first-order study of the coboundary map seems to indicate that in "most cases" such an F does in fact exist, but there are technical difficulties involved. In the case of the families of (4.2), the F that occurs there will do the job.

(5.4) The question (II) is a weaker version of (I). A possible approach is to study the base of the versal deformation by actually computing Massey products in the algebra $\mathrm{Ext}^{\bullet}(D,D)$. The first of these, the cup product, shows an intimate relation to the condition of (II). (III) is the converse of (II), and might be approached in a similar way. Question (IV) is a stronger version of (III). The answer is yes if there is an irreducible curve of minimal degree containing Y. A possible approach would be to compute dimensions of the locally closed subsets of the Hilbert scheme with a given Hilbert function or minimal resolution.

(5.5) Another question of interest would be to describe intersections $M(d;e) \cap M(d;e^1)$, or more generally to determine, for a given $D \in M(d)$, the set of integers $e < d$ for which $D \in M(d;e)$. An extreme example is given by (4.10): If $Y(D)$ is on a line, then D is in $M(d;e)$ for all $e < d$ for which $M(e)$ is nonempty. On the other hand, if $c_1 = c_2 = 0$ and $d = 2$, then D is in $M(2;0)$, but not in $M(2;1)$ unless $Y(D)$ is on a line.

REFERENCES

1. Bǎnicǎ, C.: 2-bündel auf P^2 and P^3, topologisch trivial. To appear in Crelles Journal für die reine und ang. Math.

2. Barth, W: Moduli of vector bundles on the projective plane. Invent. math. 42, 63-91 (1977).

3. Ellingsrud, G. and Strømme, S. A.: On the moduli space for stable rank-2 vector bundles on P^2. Preprint, Oslo (1979).

4. Grothendieck, A.: Techniques de construction et théorèmes d'existence en géométrie algébrique IV: Les schémas de Hilbert. Séminaire Bourbaki 221 (1960/61).

5. Hartshorne, R.: Stable Vector Bundles of Rank 2 on P^3. Math. Ann. 238, 229-280 (1978).

6. Hulek, K.: Stable Rank-2 Vector Bundles on P^2 with c_1 Odd. Math. Ann. 242, 241-266 (1979).

7. Kleiman, S.: Relative duality for quasi-coherent sheaves. Compositio Math. 41, 39-60 (1980)

8. Laudal, O. A.: Formal Moduli of Algebraic Structures. Springer Lecture Notes 754 (1979).

9. Maruyama, M.: On a family of algebraic vector bundles. Number Theory, Algebraic Geometry and Commutative Algebra, in Honor of Y. Akizuki, Tokyo, 95-146 (1973).

10. Mumford, D.: Lectures on curves on an algebraic surface. Annals of Math. Studies 59, Princeton Univ. Press (1966).

11. Schäfft, U.: Nichtsepariertheit instabiler Rang-2-Vektorbündel auf P^2. Dissertation, Göttingen (1981).

12. Schlessinger, M.: Functors of Artin Rings. Trans. Am. Math. Soc. 130, 208-222 (1968).