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OPERATOR ALGEBRAS ASSOCIATED WITH FREE PRODUCTS OF GROUPS WITH AMALGAMATION

by

Erik Bédos
University of Oslo

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1. Introduction

In this note, all the groups we consider are supposed to be countable and discrete. For such a group G, we let $C_r^*(G)$ (resp. $U(G)$) denote the C^*-algebra (resp. the von Neumann algebra) generated by the left regular representation of G on $\ell^2(G)$. $C_r^*(G)$ coincides with the full group C^*-algebra $C^*(G)$ if and only if the group G is amenable, and when this happens, it is known that $C_r^*(G)$ possess a (non-trivial) multiplicative linear functional, and thus $C_r^*(G)$ is not simple. In 1974 Powers showed that $C_r^*(\mathbb{F}_2)$ is simple and has a unique tracial state, where $\mathbb{F}_2$ denotes the free group on two generators [11]. Since this time, many mathematicians have refined Powers' method of proof and so enlarged the class of groups possessing the same properties ([1],[2],[3],[4],[9],...).

Among this class one finds

- All free products of the form $G_1 * G_2$, where G_1 and G_2 are two non-trivial groups not both of order 2 ([9]).

This case is subsumed in

- all groups containing a normal non-abelian free subgroup with trivial centralizer ([2]).

At last we mention that the class also includes

- non elementary Fuchsian and Kleinian groups ([4]).

The main purpose of this note is to show that under some suitable assumptions, free products with amalgamation belong to this class too. Such groups appear naturally in topology in connection with fundamental groups and in fact a lot of groups can be written as free products with amalgamation. Under some slightly stronger assumptions, we also show that the group von Neumann algebra of a free product with amalgamation is a II_1-factor which does not possess property Γ of Murray and Neumann.

2. Notation and preliminaries

For general information about C^*-algebra theory we refer to [10], about combinatorial group theory to [5] and [6]. Briefly we recall some of what is needed in this paper and fix some notation.

Given a group G, we denote by $\ell^2(G)$ the Hilbert space of all complex valued functions f on G such that $\sum_{g\in G} |f(g)|^2 < \infty$, and by U the left regular representation of G in $\ell^2(G)$. $C_r^*(G)$ (resp. $U(G)$) is then defined as the closure of the linear span of $\{U(g),\ g\in G\}$ in the operator norm topology (resp. in the weak operator topology). The canonical tracial state τ on $U(G)$ is defined by $\tau(T) = (T\delta,\delta)$, $T \in U(G)$, where δ denotes the characteristic function of the identity of G. We denote also by τ the restriction of τ to $C_r^*(G)$.

A group G is said to be amenable if there exists a state on $\ell^\infty(G)$, the bounded complex valued functions on G, which is invariant under translations by G.

Let now H and K be groups with presentations

$$H = \langle x_1,\ldots;r_1,\ldots\rangle \text{ and } K=\langle y_1,\ldots;s_1,\ldots\rangle$$

Let $A \subseteq H$ and $B \subseteq K$ be subgroups such that there exists an isomorphism $\phi\colon A \to B$. Then <u>the free product of H and K, amalgamating the subgroups A and B by the isomorphism ϕ is the group G given by</u>

$$\underline{G = \langle x_1\ldots,y_1,\ldots;r_1,\ldots,s_1,\ldots,a=\phi(a),\ a\in A\rangle}$$

We will write this more simply as

$$G = \langle H*K;\ A=B,\ \phi\rangle, \text{ or even more } G = H\underset{A}{*}K$$

when no confusion is possible.

The basic idea of the free product with amalgamation is that the subgroup A is identified with its isomorphic image $\phi(A)$. The free product with amalgamation depends on H, K, A, B and ϕ. The groups H and K are called the factors of G, while A and B are called the amalgamated subgroups. Free product with amalgamation clearly reduces to the ordinary free product if the amalgamated subgroup is the identity.

Let now $G = H_1 \underset{A}{*} H_2$ be the free product of two groups H_1 and H_2 with amalgamated subgroup A. If $u \in G$, then either $u \in A$ or else $u = h_1 \dots h_m$ to some $m \geq 1$, where each $h_i \in H_{\nu_i} - A$ for some $\nu_i \in \{1,2\}$ and no $\nu_i = \nu_{i+1}$. In the latter case m and the sequence $(\nu_1, \dots, \nu_m)$ are uniquely determined by u, but not (unless $m = 1$ or $A = 1$) the factors h_i; we nonetheless call the product $h_1 \dots h_m$ a <u>normal form</u> for u, [this is a familiar abuse of lenguage: the normal form is not the product $h_1 \dots h_m$, which is simply the element u, but rather the sequence $(h_1, \dots, h_m)$. Further this departs from the usual usage of "normal form" which involves choosing coset representatives]. For u as above, we define the length $|u|$ of u to be $|u| = 0$ if $u \in A$ and $|u| = m$ otherwise.

We write $x \equiv u_1 \dots u_n$ if $x = u_1 \dots u_n$ and $|x| = |u_1| + \dots + |u_n|$. If $x \equiv u_1 u_2$, we say that x begins with u_1 and ends in u_2. One of the most famous group which can be written as a free product with amalgamation is $SL_2(\mathbb{Z})$. It is isomorphic to $\langle a,b; a^4, b^6, a^2 = b^3 \rangle$ $= \mathbb{Z}/_{4\mathbb{Z}} \underset{\mathbb{Z}/_{2\mathbb{Z}}}{*} \mathbb{Z}/_{6\mathbb{Z}}$ (the cyclic groups of order 4 and 6 being generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$, and $\mathbb{Z}/_{2\mathbb{Z}}$ by $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$).

As another example we mention the Higman group

$G = \langle a,b,c,d;\ b^{-1}ab = a^2,\ c^{-1}bc = b^2,\ d^{-1}cd = c^2,\ a^{-1}da = d^2 \rangle$.

It can be written as a free product with amalgamation in the following way:

Let $H_1 = \langle a,b\ c;\ b^{-1}ab = a^2,\ c^{-1}bc = b^2 \rangle$ and

Let $H_2 = \langle c',d,a';\ d^{-1}c'd = c'^2,\ a'^{-1}da' = d^2 \rangle$.

Then $G = \langle H_1 * H_2;\ a = a',\ c = c' \rangle$.

3. Operator algebras associated with free products with amalgamation

Let $G = H \underset{A}{*} K$ be the free product of the two groups H and K with amalgamated subgroup A. It is rather clear that some additional hypotheses must be made on G to ensure that $C_r^*(G)$ will be simple with a unique tracial state. Indeed, if $G = SL_2(\mathbb{Z})$ then G has a non trivial center, so G is not ICC, hence $U(G)$ is not a factor. This easily implies that the canonical tracial state cannot be the only tracial state on $C_r^*(G)$.

We need the following definition:

Let B be a subgroup of a group F. Let $\{x_1,x_2\}$ be a pair of distinct elements of F, neither of which is in B. We say that $\{x_1,x_2\}$ is a blocking pair for B in F if the following condition is satisfied:

(*) If $b \in B$, $b \neq 1$, then $x_i^{\varepsilon} b x_j^{\delta} \notin B$, $1 \leq i,j \leq 2$, $\varepsilon = \pm 1$, $\delta = \pm 1$.

Note that (*) implies that $x_i^{\varepsilon} x_j^{\delta} \notin B$, $1 \leq i,j \leq 2$, $\varepsilon = \pm 1$, $\delta = \pm 1$ unless $x_i^{\varepsilon} x_j^{\delta} = 1$, and that (*) is trivially satisfied if $B = \{1\}$. As remarked in [5], the existence of blocking pairs is not an unreasonable condition in groups in which there is a lot of "freeness". We say that a group G is an amalgam if it can be written $G \cong H \underset{A}{*} K$ with $H \neq A \neq K$.

We shall prove:

Theorem 1: Let $G \cong H \underset{A}{*} K$ be an amalgam such that there exists a blocking pair for A in one of the factors of G. Then $C_r^*(G)$ is simple with a unique tracial state.

As examples of groups for which the theorem is valid, we mention:

a) the free product of two non-trivial groups, not both of order two.

b) $G \cong H \underset{A}{*} K$, where H is a free group, A is a finitely generated subgroup with infinite index in H and K is any group containing as a proper subgroup an isomorphic copy of A (see [5]).

c) the Higman group (the existence of a blocking pair is shown in [12]).

<u>Proof of theorem 1:</u>

It is enough to prove the theorem in the case when there is a blocking pair $\{x_1, x_2\}$ for A in K.

By [4, proposition 4] the theorem will be proved if we can show that G satisfies Powers' property:

$$(**)\ \left\{\begin{array}{l}\text{For every finite subset } F \text{ of } G - \{1\} \text{ and for every} \\ \text{natural number } r \geq 1, \text{ there exist elements } b_1, \ldots b_r \text{ of} \\ G \text{ and pairwise disjoints subsets } Z_1, \ldots, Z_r \text{ of } G \text{ such} \\ \text{that} \\ \qquad b_\ell f b_\ell^{-1} y \in Z_\ell, \text{ for all } f \in F,\ y \in G - Z_\ell,\ \ell \in \{1, \ldots, r\}.\end{array}\right.$$

The structure of our proof that G satisfies $(**)$ is inspired by the proof of [9, theorem 1].

Let α be an element of H not in A and set $r = \alpha x_1$, $s = \alpha x_2$.

We need to establish some lemmas.

<u>Lemma 1.</u> For $m \in \mathbb{N} \cup \{0\}$, let $P(m)$ be the following assertion: For all $g \in G - \{1\}$ such that $|g| = m$, $r^{m+1} g\, r^{-(m+1)}$ has a normal form which begins with α and ends with α^{-1} unless m is even and g can be written as $r^{\pm \frac{m}{2}}$. Then $P(m)$ is true for all $m \in \mathbb{N} \cup \{0\}$.

<u>Proof:</u> 1) <u>Suppose $m = 0$.</u>

Let $g \in A - \{1\}$. Then $rgr^{-1} = \alpha x_1 g x_1^{-1} \alpha^{-1}$. Since $\{x_1, x_2\}$ is a blocking pair for A in K, we have that $x_1 g x_1^{-1} \in K - A$ and so clearly $P(0)$ is true for g.

2) <u>Suppose $m = 1$</u>

If $g \in H{-}A$, then $P(1)$ is obviously true for g.
Let $g \in K{-}A$. Then $r^2 g r^{-2} = \alpha x_1 \alpha x_1 g x_1^{-1} \alpha^{-1} x_1^{-1} \alpha^{-1}$. If $x_1 g x_1^{-1} \in K{-}A$, then $P(1)$ is again true for g. So we must check the case when $x_1 g x_1^{-1} \in A - \{1\}$. They are then two possibilities:
- $\alpha(x_1 g x_1^{-1})\alpha^{-1} \in H{-}A$ and so $P(1)$ is clearly true for g.
- $\alpha(x_1 g x_1^{-1})\alpha^{-1} \in A{-}\{1\}$ and it follows that $P(1)$ is true for g by using 1) on $\alpha x_1 g x_1^{-1} \alpha^{-1}$.

3) <u>Suppose $m = 2$</u>

Let $g \in G - \{1\}$, with $|g| = 2$.

a) Suppose g has a normal form $g = kh$, where $k \in K{-}A$, $h \in H{-}A$.
Then $r^3 g r^{-3} = \alpha x_1 \alpha x_1 \alpha x_1 \; k h x_1^{-1} \alpha^{-1} x_1^{-1} \alpha^{-1} x_1^{-1} \alpha^{-1}$

If $x_1 k \in K{-}A$, then $P(2)$ is obviously true for g. So suppose $x_1 k \in A$. They are then three possibilities:
- $\alpha x_1 kh = 1$, i.e. $g = r^{-1}$ and $P(2)$ is true for g.
- $\alpha x_1 kh \in A - \{1\}$. By using 1) on $\alpha x_1 kh$, it follows that $\alpha x_1 (\alpha x_1 kh) x_1^{-1} \alpha^{-1}$ has a normal form which begins with α and ends with α^{-1}. Then clearly the same yields $r^3 g r^{-3}$, i.e. $P(2)$ is true for g.
- $\alpha x_1 kh \in H{-}A$. Then $P(2)$ is obviously true for g.

b) Suppose g has a normal form $g = h'k'$, where $h' \in H{-}A$, $k' \in K{-}A$. We can proceed in the same way as in a) by "looking" at $k' x_1^{-1}$ instead of $x_1 k$, so we omit this.

4) <u>Suppose $P(\ell)$ is true for all $1 \leq \ell \leq m$, where $m \geq 2$.</u>

The lemma will be proved by induction if we show that $P(m+1)$ is then also true.

a) Suppose first $m+1$ is odd, so $m+1 = 2n+1$, $n \in \mathbb{N}$.

If g has a normal form $g = g_1 \cdots g_{2n+1}$ where $g_1, g_{2n+1} \in H-A$, then $P(m+1)$ is obviously true for g.

So suppose g has a normal form $g = g_1 \cdots g_{2n+1}$, where $g_1, g_{2n+1} \in K-A$.

i) if $x_1 g_1 \in A$, then $\alpha x_1 g_1 \in H-A$ and there are two possibilities:

- the first is that $(\alpha x_1 g_1) g_2 \in A$, which implies that $\alpha x_1 g_1 g_2 g_3 \in K-A$, and we can use $P(m-1)$ on $(\alpha x_1 g_1 g_2 g_3) g_4 \cdots g_{2n+1}$ to obtain that $P(m+1)$ is true for g.

- the second is that $(\alpha x_1 g_1) g_2 \in H-A$. Then $|(\alpha x_1 g_1 g_2) g_3 \cdots g_{2n+1}| = 2n = m$. Now obviously $\alpha x_1 g_1 \cdots g_{2n+1} \neq r^{-n}$, but also $\alpha x_1 g_1 \cdots g_{2n+1} \neq r^n$. (Because if $\alpha x_1 g_1 \cdots g_{2n+1} = r^n$, then $g = r^{n-1}$. but $|g| = m+1$ while $|r^{n-1}| = m-2$). Using that $P(m)$ is true for $(\alpha x_1 g_1 g_2) g_3 \cdots g_{2n+1}$, we clearly obtain that $P(m+1)$ is true for g.

ii) if $x_1 g_1 \in K-A$, then we "look" at $g_{2n+1} x_1^{-1}$:

- if $g_{2n+1} x_1^{-1} \in K-A$, then $P(m+1)$ is obviously true for g.
- if $g_{2n+1} x_1^{-1} \in A$, then $g_{2n+1} x_1^{-1} \alpha^{-1} \in H-A$ and we can clearly proceed in the same way as in i).

b) Suppose that $m+1$ is even, so $m+1 = 2p$, $p \geq 2$.

Suppose so that g has a normal form $g = g_1 \cdots g_{2p}$ where $g_1 \in K-A$, $g_{2p} \in H-A$.

If $x_1 g_1 \in K-A$, then $P(m+1)$ is obviously true for g.

If $x_1 g_1 \in A$, then $\alpha x_1 g_1 \in H-A$ and so there are two possibilities:

- the first is that $(\alpha x_1 g_1 g_2) \in H-A$ and $P(m+1)$ is then clearly true for g.

- the second is that $(\alpha x_1 g_1 g_2) \in A$. Then $(\alpha x_1 g_1 g_2) g_3 \in K-A$ so

$(\alpha x_1 g_1 g_2 g_3)g_4 \dots g_{2p}$ has length $2p - 2$ and we can use that $P(2p-2)$ is true on it. Thus we obtain that either $r^{2p-1}(\alpha x_1 g_1 g_2 g_3)g_4 \dots g_{2p} r^{-(2p-1)}$ has a normal form which begins with α and ends with α^{-1}, which implies that the same yields $r^{2p+1} g\, r^{2p+1}$, or $(\alpha x_1 g_1 g_2 g_3)g_4 \dots g_{2p}$ can be written as $r^{-(p-1)}$, (it cannot be written as r^{p-1} since $g_{2p} \in H-A$), which implies that g can be written as r^{-p}. All together this means that $P(m+1)$ is true for g.

At last if g has a normal form $g = g_1 \dots g_{2p}$ where $g_1 \in H-A$, $g_{2p} \in K-A$, then we can proceed in the same way by "looking" at $g_{2p} x_1^{-1}$ instead of $x_1 g_1$.

□ (End of the proof of lemma 1)

The next lemma is an immediate consequence of lemma 1.

Lemma 2.: Let F be a finite subset of $G - \{1\}$, and define $j = 1 + \max_{f \in F} |f|$. Then for all $f \in F$, $r^j f\, r^{-j}$ has a normal form which begins with α and ends with α^{-1}, unless f can be written as a power of r in which case of course $r^j f\, r^{-j} = f$.

For $k = 1,2,\dots,$ let Z_k be the set of all elements w in G such that $s^{-1} r^{-k} w$ has no normal form which begins with an element of $K-A$.

For example, $w = r^k sh$, where $h \in H-A$, is an element of Z_k.

Lemma 3: The Z_k's are pairwise disjoint.

Proof: Let $\ell, \ell' \in \{1,2,\dots\}$ and suppose that $\ell < \ell'$. Set $n = \ell' - \ell \in \mathbb{N}$. Let $w \in Z_\ell$. Then

$$s^{-1}r^{-\ell'}w = s^{-1}r^{-n}r^{-\ell}w = s^{-1}r^{-n}s\ s^{-1}r^{-\ell}w$$

$$s^{-1}r^{-(n-1)}x_1^{-1}\ \alpha^{-1}\alpha x_2(s^{-1}r^{-\ell}w)$$

$$= s^{-1}\ r^{-(n-1)}\ x_1^{-1}x_2(s^{-1}r^{-\ell}w)$$

Since $w \in Z$ we have that either $s^{-1}r^{-\ell}w \in A$, or that $s^{-1}r^{-\ell}w$ has a normal form which begins with an element of $H-A$. In both cases, it follows now easily from the fact that $x_1^{-1}x_2 \in K-A$ (since $\{x_1x_2\}$ is a blocking pair for A in K), that $s^{-1}r^{-\ell'}w$ has a normal form which begins with $x_2^{-1} \in K-A$. By definition, this implies that $w \notin A_{\ell'}$, i.e. we have shown that $A_\ell \cap A_{\ell'} = \emptyset$. □

<u>Lemma 4:</u> $s^{-2}r^{-\ell}y$ has a normal form which begins with x_2^{-1} for all $y \in G - Z_\ell$, $\ell = 1,2,\dots$

<u>Proof:</u> Let $\ell \in \mathbb{N}$. For all $y \in G-Z_\ell$, does $s^{-1}r^{-\ell}y$ have a normal form which begins with an element of $K-A$, by definition of Z_ℓ. Thus $s^{-2}r^{-\ell}y = x_2^{-1}\ \alpha^{-1}(s^{-1}r^{-\ell}y)$ has a normal form which begins with x_2^{-1} for all $y \in G-Z_\ell$. □

Now let F be a finite subset of $G - \{1\}$, say $F = \{f_1,\dots,f_n\}$, $n \in \mathbb{N}$. Set so $b_\ell = r^\ell s^2 r^j$, where $j = 1 + \max_{f\in F}|f|$, $\ell = 1,2,\dots$.

If we can show that

(***) $b_\ell f_i b_\ell^{-1} y \in Z_\ell$ for all $y \in G-Z_\ell$, $i\in\{1,\dots,n\}$, $\ell = 1,2,\dots$,

we will have shown that G has Powers' property (lemma 3 shows that the Z_k's are pairwise disjoint) and so the proof of the theorem will be finished.

<u>Proof of (***):</u> Let $y \in G-Z_\ell$, $\ell \in \mathbb{N}$ and $i \in \{1,\dots,n\}$

- Suppose f_i is a positive power of r, $f_i = (\alpha x_1)\dots(\alpha x_1)$.

  Then $b_\ell f_i b_\ell^{-1} y = r^\ell s^2 r^j f_i r^{-j} s^{-2} r^{-\ell} y$
  $= r^\ell s^2 f_i s^{-2} r^{-\ell} y$
  $= r^\ell s(\alpha x_2)(\alpha x_1)\dots(\alpha x_1)(s^{-2}r^{-\ell}y)$

By lemma 4 $s^{-2}r^{-\ell}y$ has a normal form which begins with x_2^{-1}. Since $x_1x_2^{-1} \in K-A$, it follows easily that $s^{-1}r^{-\ell}(b_\ell f_i b_\ell^{-1}y)$ $= (\alpha x_2)(\alpha x_1)\dots(\alpha x_1)(s^{-2}r^{-\ell}y)$ has a normal form which begins with $\alpha \in H-A$, i.e. $b_\ell f_i b_\ell^{-1} y \in Z_\ell$.

- Suppose f_i is a negative power of $r, f_i = (x_1^{-1}\alpha^{-1})\dots(x_1^{-1}\alpha^{-1})$ Then $b_\ell f_i b_\ell^{-1}y = r^\ell s(\alpha x_2)(x_1^{-1}\alpha^{-1})\dots(x_1^{-1}\alpha^{-1})(s^{-2}r^{-\ell}y)$, and again, using lemma 4 and the fact that $x_1x_2^{-1} \in K-A$, we obtain that $b_\ell f_i b_\ell^{-1} y \in Z_\ell$.

- At last suppose f_i is not a power of r.

  Then $b_\ell f_i b_\ell^{-1}y = r^\ell s^2 (r^j f_i r^{-j})(s^{-2}r^{-\ell}y)$.

By lemma 2, $r^j f_i r^{-j}$ has a normal form which begins with $\alpha \in H-A$ and ends with $\alpha^{-1} \in H-A$. Using lemma 4 this clearly implies that

$s^{-1}r^{-\ell}(b_\ell f_i b_\ell^{-1}y) = (\alpha x_2)(r^j f_i r^{-j})(s^{-2}r^{-\ell}y)$ has a normal form which begins with $\alpha \in H-A$, i.e. $b_\ell f_i b_\ell^{-1} y \in Z_\ell$.

□ (End of the proof of theorem 1)..

<u>Remarks.</u> 1) The existence of a blocking pair for A in one of the factors of G is essential in the proof of theorem 1.

2) Any group G satisfying the assumptions of theorem 1 is ICC. This can be shown directly, but it is also a consequence of the theorem.

3) The theorem is also valid if G is the free product of subgroups H_ν, for ν in an index set I, with a subgroup A is a proper subgroup of H_ν and such that there exists a blocking pair in one of the factors of G.

In another direction we have:

<u>Theorem 2:</u> Let $G \cong H \underset{A}{*} K$ be an amalgam such that:

a) there exists a blocking pair $\{x_1, x_2\}$ for A in K,

b) there exists an element $\alpha \in H-A$ such that $\alpha^{-1}h\alpha \notin A$ for all $h \in A - \{1\}$.

Then $U(G)$ is a II_1-factor which does not possess property Γ of Murray and von Neumann (see [7]).

Proof: G is ICC so $U(G)$ is a II_1-factor.

Define $F = \{w \in G \mid w$ has a normal form which begins with an element of $H-A\}$.

We will show that the following two statements are true:

i) $F \cup \alpha F \alpha^{-1} = G - \{1\}$,

ii) F, $x_1 F x_1^{-1}$, $x_2 F x_2^{-1}$ are pairwise disjoint.

By [7], this will prove the theorem.

Let $g \in G - \{1\}$, $g \notin F$

If $|g| = 0$, i.e. $g \in A - \{1\}$, then b) implies that $\alpha^{-1}g\alpha \in H-A$, so $\alpha^{-1}g\alpha \in F$, hence $g \in \alpha F \alpha^{-1}$

If $|g| \geq 1$, then g has a normal form which begins with an element of $K-A$ and it is easy to see that $\alpha^{-1}g\alpha \in F$ so again $g \in \alpha F \alpha^{-1}$

Thus i) is true.

It is easy to verify that F and $x_1 F x_1^{-1}$ (resp. $x_2 F x_2^{-1}$) are disjoint. We show at last that $x_1 F x_1^{-1}$ and $x_2 F x_2^{-1}$ are disjoint.

Let $\beta = x_2^{-1}x_1$. Since $\{x_1, x_2\}$ is a blocking pair for A in K, $\beta \in K-A$. It follows clearly that $\beta f \beta^{-1} \notin F$ for all $f \in F$. This implies that

$x_1 f x_1^{-1} \notin x_2 F x_2^{-1}$ for all $f \in F$, i.e. $x_1 F x_1^{-1} \cap x_2 F x_2^{-1} = \emptyset$

□

Theorem 2 is valid for example when G is the Higman group (because [12] shows the existence of a blocking pair in both factors of G)

or of course when G is the free product of two non-trivial groups not both of order two ([7]).

We conclude this note with a negative remark.

Let C be a group. C is said to be SQ-universal if every countable group can be embedded in a quotient group of C. (see [5]).
The main result of [12] is that any group satisfying the hypotheses of theorem 1 is SQ-universal. It is also shown in [8] that if G is a finitely generated Fuchsian group which is not elementary, then G is SQ-universal. Thus it is natural to ask: Is $C_r^*(G)$ simple with a unique tracial state whenever G is ICC and SQ-universal? To see that this question has a negative answer, we need the following result of [9]:
If G has a normal amenable subgroup $\neq \{1\}$, then $C_r^*(G)$ is not simple and the canonical tracial state is not the only tracial state on $C_r^*(G)$. Now, let G be the direct product of an amenable ICC group H_1 and of a ICC SQ-universal group H_2 (for example take H_1 to be the group of all finite permutations of $\mathbb{N}$ and H_2 to be $\mathbb{F}_2$. Then $G = H_1 x H_2$ is ICC, has a normal amenable subgroup $\neq \{1\}$ and is easily seen to be SQ-universal.
This provides also an example of a ICC SQ-universal group such that its group von Neumann algebra possess property Γ (in contrast to theorem 2).

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