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1. In this short note we will discuss regularization of plurisubharmonic functions. More precisely, we will address the following problem:

Question. *Assume* Ω *is a bounded domain in* $\mathbb{C}^n (n \geq 2)$ *with smooth* (C^∞) *boundary and that* $\rho: \Omega \to \mathbb{R} \cup \{-\infty\}$ *is a (discontinuous) plurisubharmonic function. Does there exist a sequence*

$$\{\rho_n\}_{n=1}^{\infty}, \quad \rho_n : \Omega \to \mathbb{R}, \quad \text{of} \quad C^\infty$$

plurisubharmonic functions such that

$$\rho_n \searrow \rho \quad \textit{pointwise?}$$

If ρ is continuous, the answer to the above question is yes (see Richberg [3]). On the other hand, when ρ is allowed to be discontinuous and Ω is not required to have a smooth boundary, the answer is in general no (see [1], [2] for this and related questions).

Our result in this paper is that the answer to the above question is no. We present a counterexample in the next section. The construction leaves open what happens if we make the further requirement that Ω has real analytic boundary. Another question, suggested to the author by Grauert, is obtained by replacing Ω by a compact complex manifold with smooth boundary, and assuming continuity of ρ.

In the next section we need of course both to construct the domain Ω and the function ρ. These constructions are intertwined and therefore we need at first to define approximate solutions Ω_1 and ρ_1 and then use both to define Ω and ρ. The geometric properties we seek of Ω are the following. There exists an annulus $A \subset \overline{\Omega}$ such that $\partial A \subset \Omega$. Furthermore there exist concentric circles C_1, C_2, C_3 in the relative interior of A arranged by increasing radii such that $C_1, C_3 \subset \partial\Omega$ and $C_2 \subset \Omega$. Finally there exists a sequence $\{A_n\}_{n=1}^{\infty}$ of annuli such that $A_n \to A$ and $A_n \subset \Omega \; \forall n$. The properties we seek of ρ are as follows. The function ρ is strictly positive on C_2 and is strictly negative on ∂A. A simple application of the maximum principle now shows that smoothing is impossible.

The example we construct is in $\mathbb{C}^2$. This is with no loss of generality as one obtains then an example in $\mathbb{C}^n$ by crossing with a smooth domain in $\mathbb{C}^{n-2}$, rounding off the edges and pulling back ρ to the new domain.

2. All domains and functions which we will consider in $\mathbb{C}^2(z,w)$ will be invariant under rotations in the z-plane, i.e. will depend only on $|z|$. They will also be invariant under the map $(z,w) \to (1/z,w)$. Because of the latter we will describe only those points (z,w) in these domains or domains of definitions for which $|z| \leq 1$.

If U is a domain in $\mathbb{C}^2(z,w)$, we let U_z denote the part of U over z, i.e. $U_z := \{(\eta,w) \in \mathbb{C}^2 ; \; \eta = z \text{ and } (\eta,w) \in U\}$. Abusing notation we will also take U_z to mean the set $\{w \in \mathbb{C} ; (z,w) \in U\}$.

Similarly, if $\sigma : U \to \mathbb{R} \cup \{-\infty\}$ is a function, then σ_z denotes the restriction of σ to U_z.

Let A be the annulus in $\mathbb{C}^2$ given by $A = \{(z,w);\ w = 0$ and $1/2 \leq |z| \leq 2\}$. This is then the limit of a sequence of annuli $\{A_n\}_{n=1}$ where $A_n = \{(z,w);\ w = 1/n$ and $1/2 \leq |z| \leq 2\}$. We will next describe a bounded domain Ω_1 in $\mathbb{C}^2$ with C^∞ boundary containing all A_n´s (and hence A) in it´s closure. It will suffice to describe $\Omega_{1,z}$ for various z´s. That these can be made to add up to a domain with C^∞ boundary will be clear throughout.

Choose a sequence of positive numbers $\{r_k\}_{k=1}^{\infty}$, $0 < r_1 < r_2 < \ldots < 1$, with $r_3 = 1/2$. We let $\Omega_{1,z} = \emptyset$ if $|z| \leq r_1$ and $\Omega_{1,z}$ be a nonempty disc, concentric about the origin if $r_1 < |z| \leq r_4$. Recall that $\Omega_{1,z} = \Omega_{1,|z|}$ for all z. If $r_2 \leq |z| \leq r_4$ we make the extra assumption that $\Omega_{1,z}$ has radius 2. For $|z| > r_4$ we will break the symmetry in the w - direction at first by letting $\Omega_{1,z}$ gradually approach the shape of an upper-disc. (This is a rough description to be made more precise below.) Increasing $|z|$ further we will rotate this approximate upper half disc 180° clockwise until it becomes approximately a lower half disc. Then we proceed by reversing the process, first by rotating counterclockwise back to an approximate upper half disc and then expanding this back to a disc of radius 2 near $|z| = 1$. As mentioned earlier, if $|z| > 1$, then $\Omega_{1,z} := \Omega_{1,1/z}$.

We now return to the more precise description of $\Omega_{1,z}$ for $|z|>r_4$. Writing $w = u + iv$ in real coordinates u,v, let $v = f(u)$ be a C^∞ function defined for $u \in \mathbb{R}$ with $f(u) = 0$ if $u \leq 0$ or $u \geq 2$, $f \geq 0$ and $f(u) = 0$ on $(0,2)$ if and only if $u = 1/n$ for some positive integer n. We may assume that $|f|$, $|f'|$, $|f''|$ are very small and therefore in particular that the graph of f only intersects the boundary of any disc $\Delta(0;R) = \{|w| < R\}$ in exactly two points. If $r_4 < |z| < r_5$, we let $\Omega_{1,z}$ be a subdomain of $\Delta(0;2)$ containing those $u + iv \in \Delta(0;3/2)$ for which $v \geq f(u)$. When $r_5 \leq |z| \leq r_6$ we choose $\Omega_{1,z}$ independent of z with the properties that $\Omega_{1,z} \subset \Delta(0;7/4) \cap \{v > f(u)\}$ and $\Delta(0;3/2) \cap \{v > f(u)\} \subset \Omega_{1,z}$. Let $\theta(x)$ be a real C^∞ function on $\mathbb{R}$ with $\theta(x) = 0$ if $x \leq r_6$, $\theta(x) = \pi$ if $x \geq r_7$ and $\theta'(x) > 0$ if $r_6 < x < r_7$. Then we can rotate $\Omega_{1,z}$ 180° clockwise for $r_6 \leq |z| \leq r_7$ by defining $\Omega_{1,z} = e^{-i\theta(|z|)}\,\Omega_{1,r_6}$ for such z. Further, we let $\Omega_{1,z} = \Omega_{1,r_7}$ when $r_7 \leq |z| \leq r_8$. Reversing the procedure, we rotate $\Omega_{1,z}$ back 180° when $r_8 \leq |z| \leq r_9$ so that Ω_{1,r_9} again equals Ω_{1,r_6}. Continuing, we let $\Omega_{1,z} = \Omega_{1,r_9}$ whenever $r_9 \leq |z| \leq r_{10}$. Reversing the procedure between r_4 and r_5 we obtain $\Omega_{1,z}$´s, $r_{10} \leq |z| \leq r_{11}$ so that in particular $\Omega_{1,r_{11}}$ is the disc $\Delta(0,2)$. When $r_{11} < |z| \leq 1$, we let $\Omega_{1,z}$ always be this same disc. This completes the construction of Ω_1.

The next step is to define an (almost) plurisubharmonic function ρ_1. Let $\{\varepsilon_n\}_{n=1}^{\infty}$ be a sufficiently rapidly decreasing sequence of positive numbers, $\varepsilon_n \searrow 0$. Then $\sigma_1(w) := \sum_{n=1}^{\infty} \varepsilon_n \log|w - \frac{1}{n}|$

is a subharmonic function on the complex plane and $\sigma_1(0) \in (-\infty, 0)$.

Letting $\sigma(w) = \sigma_1(w) + 1 - \sigma_1(0)$ we obtain a subharmonic function on $\mathbb{C}(w)$ with $\sigma(0) = 1$ and $\sigma(1/n) = -\infty \ \forall n \in \mathbb{Z}^+$. If the constant $K > 0$ is chosen large enough, the plurisubharmonic function $\sigma(w) + K \log(|z|/r_5)$ will be strictly less than -1 at all points $(z,w) \in \Omega_1$ for which $|z| \leq r_4$. The function $\rho_1 : \Omega_1 \to \mathbb{R}$ is defined by the equations $\rho_1(z,w) = \rho_1(1/z,w)$ and $\rho_1(z,w) = \max\{\sigma(w) + K \log(|z|/r_5), -1\}$ when $|z| \leq 1$. Then ρ_1 is the restriction to Ω_1 of the similarly defined function on $\mathbb{C}^2$ and ρ_1 is plurisubharmonic at all points (z,w) with $|z| \neq 1$. This completes the construction of ρ_1.

We have two main problems left. The annuli A_n all lie partly in the boundary of Ω_1, so Ω_1 has to be bumped slightly so that they all lie in the interior. However, this bumping should not change the extent to which A lies in the boundary. The other main problem is the failure of plurisubharmonicity of ρ_1 at $|z| = 1$. We will change ρ_1 near $|z| = 1$ so that it will equal $\max\{\sigma(w), -1\}$ in a neighbourhood of this set. In order to deal with both these problems, we will at first construct a subharmonic function $\tau(w)$ which can be used for patching purposes.

Our first approximation to τ will be τ_1.

The domain of τ_1 will be

$D := \{w;\ |w| < 2,\ w \notin (-2,0],\ w \notin \{1/n\}\}$. The properties we will require of τ_1 are that $\tau_1(u+iv) = 0$ when $v \geq f(u)$, $\tau_1(u+iv) \geq 1$ when $v \leq 0$, τ_1 is C^∞ and τ_1 is strongly subharmonic at all points $u + iv$ with $v < f(u)$.

Let K_o denote the compact set $\{w = u + iv;\ |w| \leq 2$ and $v \geq f(u)\}$. Since K_o is polynomially convex, there exists a C^∞ subharmonic function $\lambda_o : \mathbb{C} \to [0,\infty>$ which vanishes precisely on K_o and which is strongly subharmonic on $\mathbb{C} - K_o$. Choose an increasing sequence of compact sets
$F_1 \subset \text{int}\, F_2 \subset F_2 \subset \text{int}\, F_3 \subset \ldots \subset D,\ D = \cup F_\ell$. Letting $K_\ell = K_o \cup F_\ell$ we may even assume that each bounded component of $\mathbb{C} - K_\ell$ clusters at some $1/n$ and in particular therefore that there are only finitely many of these components. With these choices it is possible for each $\ell \geq 1$ to find a non-negative C^∞ function λ_ℓ such that $\lambda_\ell | K_\ell \equiv 0$, $\lambda_\ell \geq 1$ and strongly subharmonic on $\{u+iv \in K_{\ell+2} - \text{int}\, K_{\ell+1}\ ;\ v \leq 0\}$ and λ_ℓ fails to be subharmonic only on a relatively compact subset of $(\text{int}\, K_{\ell+3} - K_{\ell+2}) \cap \{v < 0\}$. But then, if $\{C_\ell\}_{\ell=o}^{\infty}$ is a sufficiently rapidly increasing sequence, $\tau_1 := \sum_{\ell=o}^{\infty} C_\ell \lambda_\ell$ has all the desired properties.

We next want to push the singularities of τ_1 at the points $1/n$ over to the origin. First, let us choose discs $\Delta_n = \Delta(1/n, \rho_n)$ small enough so that $\sigma(w) + K \log 1/r_5 < -1$ on each Δ_n.
We will first perturbe τ_1 inside each Δ_n.
We can make a small perturbation of the situation by making a small translation parallell to the v-axis in the negative direction in a smaller disc about $1/n$ patched with the identity outside a slightly larger disc in Δ_n to obtain a new C^∞ function $\tau_2 \geq 0$ and a new C^∞ function $v = f_1(u)$ with the properties that $f_1 \leq f$, $f_1 < f$ near $1/n$, $f_1 = f$ away from $1/n$ and $\tau_2 = 0$ when $v \geq f_1(u)$, $\tau_2 \geq 1$ when $v \leq 0$ except in very small discs about $1/n$ and

$$\tau_3 = \begin{cases} 0 & \text{when } v \geq f_1(u) \\ \tau_2 + (v-f_1(u))^2 & \text{otherwise} \end{cases}$$ is strongly subharmonic when $v<f_1(u)$.

The singularities of τ_1 at the points $1/n$ have thus been moved down to the points $\rho_n = 1/n + if_1(i/n)$. Let $\Delta_n' = \Delta(1/n, \rho_n')$, $0 < \rho_n' << \rho_n$ be discs on which $\tau_3 \equiv 0$. We may assume that $p_n \notin \bar{\Delta}_n'$. Let γ be a curve from p_1 to 0 passing in the lower half plane through all the p_n' s and avoiding all the $\bar{\Delta}_n'$ ´s. We can assume say that γ is linear between p_n and p_{n+1}. Let V be a narrow tubular neighbourhood of $\gamma - \{0\}$ also lying in the lower half-plane and avoiding all the $\bar{\Delta}_n'$ ´s. The restriction $\tau_3|V$ is C^∞, subharmonic and ≥ 1 except for singularities at each p_n. Let $\tau_4 \geq 1$ be a C^∞ function on V which agrees with $\tau_3|V$ on $V \cap V'$, V' some open set containing $\partial V - \{0\}$. A construction similar to the one for τ_1 yields a C^∞ subharmonic function $\tau_5 \geq 0$ on $\mathbb{C} - (0)$ which vanishes outside V and is such that $\tau_4 + \tau_5$ is subharmonic on V. Finally, let $\tau : \{(w) < 2,\ w \notin [-2,0]\} \to \mathbb{R}^+$ be the C^∞ subharmonic function given by $\tau = \tau_3$ outside V and $\tau = \tau_4 + \tau_5$ on V. Then $\tau = 0$ on each Δ_n' and $\tau(w) = 0$ when $v \geq f_1(u)$ except possibly on a concentric disc Δ_n', $\Delta_n' \subset\subset \Delta_n'' \subset\subset \Delta_n$. Also, $\tau(w) \geq 1$ when $v \leq 0$, $w \notin \cup \Delta_n''$. This completes the construction of the patching function τ.

The construction of Ω can now be completed. A point $(z,1/n) \in A_n$ lies in the boundary of Ω_1 only when $|z|$ or $1/|z|$ is in $[r_5,r_6] \cup [r_7,r_8] \cup [r_9,r_{10}]$. This set is contained in the open set $\{(z,w);\ |z| \text{ or } 1/|z| \in (r_4,r_{11}) \text{ and } w \in \Delta_n'\} =: U_n$. We let

Ω be a domain with C^∞ boundary which agrees with Ω_1 outside $\cup U_n$ and which contains all A_n's in it's interior.

Next we define the plurisubharmonic function $\rho : \Omega \to \mathbb{R}$. Let $\sigma' = \max\{\sigma, -1\}$ and choose a constant $L >> 1$ such that $\rho_1 \leq L - 1$ on $\bar{\Omega}$. If $|z| \leq r_6$, let $\rho_z := \rho_{1,z}$. For $r_5 \leq |z| \leq r_6$, this definition agrees with $\rho_z = \max\{\rho_{1,z}, \sigma'+L\tau\}$ since τ is then 0 and $\rho_1 = \sigma' + K\log(|z|/r_5)$. If $r_6<|z|\leq r_8$, let $\rho_z := \max\{\rho_{1,z}, \sigma' + L\tau\}$. For $r_7 \leq |z| \leq r_8$, this definition agrees with $\rho_z = \sigma' + L\tau$. To see this, observe that if $w \in \Delta_n''$, then $\rho_{1,z} = -1$ and $\sigma' = -1$ while $\tau \geq 0$. If on the other hand $w \notin \cup \Delta_n''$, then $v < 0$ and $\sigma' + L\tau \geq -1 + L \geq \rho_1$. If $r_8 < |z| \leq r_{10}$, let $\rho_z := \sigma' + L\tau$. For $r_9 \leq |z| \leq r_{10}$ this definition agrees with $\rho_z = \sigma'$ since $\tau = 0$. Also, if $r_{10} \leq |z| \leq 1$, let $\rho_z := \sigma'$, and if $|z| > 1$, let $\rho_z := \rho_{1/z}$. Then ρ is plurisubharmonic on Ω, $\rho(e^{i\theta},0) = 1$ $\forall\theta \in \mathbb{R}$ and $\rho(e^{i\theta}/2,0) = \rho(2e^{i\theta},0) = -1$ $\forall\theta \in \mathbb{R}$. If there exists a sequence of C^∞ plurisubharmonic functions $\rho_m : \Omega \to \mathbb{R}$, $\rho_m \searrow \rho$, then there exists an m for which $\rho_m(e^{i\theta}/2,0)$, $\rho_m(2e^{i\theta},0) < 0$ $\forall\theta\in \mathbb{R}$. Hence, for all large enough n, $\rho_m(e^{i\theta}/2, 1/n)$, $\rho_m(2e^{i\theta}, 1/n) < 0$ $\forall\theta \in\mathbb{R}$. By the maximum principle applied to the annuli $A_n \subset \Omega$, it follows that $\rho_m(e^{i\theta},1/n) < 0$ $\forall\theta \in\mathbb{R}$ and all large enough n. Hence, by continuity of ρ_m, $\rho_m(e^{i\theta},0) \leq 0$ $\forall\theta \in\mathbb{R}$. This contradicts the assumption that $\rho_m \geq \rho$ and therefore completes the counter-example.

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