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Abstract

Let V_i be short range potential and $\lambda_i(\varepsilon)$ analytic functions. We show that the Hamiltonians $H_\varepsilon = -\Delta + \varepsilon^{-2} \sum_{i=1}^n \lambda_i(\varepsilon) V_i(\frac{1}{\varepsilon}(\cdot - x_i))$ converge in the strong resolvent sense to the point interactions as $\varepsilon \rightarrow 0$, and if V_i have compact support then the eigenvalues and resonances of H_ε which remains bounded as $\varepsilon \rightarrow 0$, are analytic in ε in a complex neighbourhood of zero. We compute in closed form the eigenvalues and resonances of H_ε to the first order in ε .

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1. Introduction.

The point interactions were first studied in [1] where they were introduced as natural objects in non-standard analysis. In [2] and [3] some of their applications to physics were explored. However the short range expansion or the approach to point interaction remained a problem. To explain shortly we consider the Hamiltonian of the form

$$H_\epsilon = -\Delta + \epsilon^{-2} \sum_{i=1}^n \lambda_i(\epsilon) V_i\left(\frac{1}{\epsilon}(\cdot - x_i)\right) \quad (1.1)$$

where V_i are short range potentials and ask if the limit exists as $\epsilon \rightarrow 0$. This problem was attacked in [4] where it was proved that if V_i was of compact support and sufficiently regular then H_ϵ converge in the strong resolvent sense to the Hamiltonian with point interactions as $\epsilon \rightarrow 0$.

However for many physical applications it is of interest to know what happens before one takes the limit, that is to try to expand H_ϵ in powers of ϵ . For the one center problem i.e. (1.1) for $n = 1$ this was solved in [5]. The amazing thing is that (1.1) is actually analytic in ϵ not only for $n = 1$ but for general n . This is what is proved in this paper, namely that the eigenvalues and resonances of H_ϵ that remains bounded as $\epsilon \rightarrow 0$ are analytic if the $\lambda_i(\epsilon)$ are analytic, and the perturbation expansion in ϵ is given and explicitly computed to first order in ϵ . This brings a completely new class of models into the range of the solvable models.

We expect that this discovery will have application not only in potential scattering but also in solid state physics. In solid state physics we have a problem of the type (1.1) with n infinite. The problem of the short range expansion for an infinite number of centers is not attacked in this paper but in a forthcoming paper by the same authors. The short range expansion for a charged particle is studied in [6].

2. Convergence to point interactions

Let x_1, \dots, x_n be n different points in \mathbb{R}^3 and V_1, \dots, V_n n real functions such that $V_j \in R \cap L^1(\mathbb{R}^3)$ for $j=1, \dots, n$ where R is the Rollnik class. (i.e. measurable functions on \mathbb{R}^3 such that $\iint |V(x)V(y)| |x-y|^{-2} dx dy$ is finite. See Simon [7] for general theory concerning Rollnik functions). Let further $\lambda_1, \dots, \lambda_n$ be n real analytic functions defined in a neighbourhood of 0 with $\lambda_1(0) = \dots = \lambda_n(0) = 1$.

Then we can define a family H_ϵ of self-adjoint operators on $L^2(\mathbb{R}^3)$ by means of quadratic forms such that

$$H_\epsilon = -\Delta + \sum_{j=1}^n \epsilon^{-2} \lambda_j(\epsilon) V_j\left(\frac{1}{\epsilon}(\cdot - x_j)\right) \quad (2.1)$$

for small $\epsilon > 0$ where $-\Delta$ is the self-adjoint Laplacian.

In the same way we define the self-adjoint operators

$$H_j = -\Delta + V_j \quad (2.2)$$

Using the notations

$$G_k = (-\Delta - k^2)^{-1} \quad (2.3)$$

with $\text{Im} k > 0$, and

$$v_j = |V_j|^{1/2}, \quad u_j = v_j \text{sgn } V_j \quad (2.4)$$

we have (Simon [7])

$$(H_j - k^2)^{-1} = G_k - G_k v_j (1 + u_j G_k v_j)^{-1} u_j G_k \quad (2.5)$$

when $k^2 \notin \sigma(H_j)$

G_k has an integralkernel which we denote by $G_k(x-y)$ where

$$G_k(x-y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}$$

We will also use the term G_k with $\text{Im} k \leq 0$ for the operator with integralkernel given by (2.6). From Albeverio and Høegh Krohn [4] we take the following definition

Definition 2.1

H_j has a zero energy resonance if and only if -1 is an eigenvalue for the operator $u_j G_k v_j$.

Assume now that H_j has a zero energy resonance. Let $\varphi_j \in L^2(\mathbb{R}^3), \varphi_j \neq 0$, be such that

$$(1 + u_j G_k v_j) \varphi_j = 0 \quad (2.7)$$

From Albeverio, Gesztesy and Høegh Krohn [5] we know that the so called resonance function ψ_j defined by

$$\psi_j = G_k v_j \varphi_j \quad (2.8)$$

is locally in $L^2(\mathbb{R}^3)$ and satisfies

$$H_j \psi_j = 0 \quad (2.9)$$

in the sense of distributions.

But generally ψ_j will not be in $L^2(\mathbb{R}^3)$.

We now distinguish the following cases for the operator $H_j, j=1, \dots, n$ (See Albeverio, Gesztesy and Høegh Krohn [5]).

Case (I)

- 1 is not an eigenvalue of $u_j G_0 v_j$

Case (II)

- 1 is a simple eigenvalue of $u_j G_0 v_j$ and the corresponding

ψ_j is not in $L^2(\mathbb{R}^3)$

Case (III)

- 1 is an eigenvalue of $u_j G_O v_j$ with multiplicity $N_j \geq 1$, and the corresponding ψ_{jr} , $r = 1, \dots, N_j$, are all in $L^2(\mathbb{R}^3)$.

Case (IV)

- 1 is an eigenvalue of $u_j G_O v_j$ with multiplicity $N_j \geq 2$, and at least one of the corresponding ψ_{jr} , $r = 1, \dots, N_j$ is not in $L^2(\mathbb{R}^3)$.

In case (III) and (IV) we will assume that the eigenfunctions

φ_{jr} , $r = 1, \dots, N_j$ are chosen such that

$$(\varphi_{jr}, \tilde{\varphi}_{js}) = 0 \quad \text{for } r \neq s \quad (2.10)$$

and $r, s = 1, \dots, N_j$ where

$$\tilde{\varphi}_{jr} = \varphi_{jr} \operatorname{sgn} V_j \quad (2.11)$$

With some additional assumption on the potentials V_j we have the following useful criterion to decide whether ψ_j is in $L^2(\mathbb{R}^3)$ or not.

Proposition 2.2

Assume $V \in \mathcal{R}$ and $|\cdot|V \in L^1(\mathbb{R}^3)$ and let $\varphi \in L^2(\mathbb{R}^3)$, $\varphi \neq 0$, satisfy

$$(1 + u G_O v) \varphi = 0 \quad (2.12)$$

With $\psi = G_O v \varphi$ we have the following:

$$\psi \in L^2(\mathbb{R}^3) \iff (v, \psi) = 0 \quad (2.13)$$

Proof: See Albeverio, Gesztesy and Høegh Krohn [5]

□

Following Grossmann, Høegh Krohn and Mebkhout [2], [3] we now define the self-adjoint operator $-\Delta_{(X, \alpha)}$ where $X = (x_1, \dots, x_n)$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ by its resolvent $(-\Delta_{(X, \alpha)} - k^2)^{-1}$ with integral kernel

$$(-\Delta_{(X, \alpha)} - k^2)^{-1}(x, y) = \quad (2.14)$$

$$G_k(x-y) + \sum_{\ell, j=1}^n G_k(x-x_\ell) \left[\left(\alpha_\ell - \frac{ik}{4\pi} \right) \delta_{\ell j} - \tilde{G}_k(x_\ell - x_j) \right]_{\ell j}^{-1} G_k(x_j - y)$$

for $\text{Im} k > 0$, $k^2 \notin \sigma(-\Delta_{(X, \alpha)})$, where $\tilde{G}_k(x) = G_k(x)$ if $x \neq 0$

and 0 otherwise.

(We have used $[a_{\ell j}]_{\ell j}^{-1}$ to denote the ℓ, j -th element of the inverse of the matrix $[a_{\ell j}]$).

The self-adjoint operator $-\Delta_{(X, \alpha)}$ represents the formal Hamiltonian with δ -potentials situated at $X = (x_1, \dots, x_n)$ with strength $\alpha = (\alpha_1, \dots, \alpha_n)$.

With these definitions we have the following theorem

Theorem 2.3

Let $V_j: \mathbb{R}^3 \rightarrow \mathbb{R}$ fulfill $(1+|\cdot|)^2 V_j \in R \cap L^1(\mathbb{R}^3)$ for $j = 1, \dots, n$, and assume that for every j the operator H_j is either in case (I) or (II).

Then the operator H_ϵ defined by (2.1) will converge in strong resolvent sense to the operator $-\Delta_{(X, \alpha)}$ defined by (2.14) where

$$\alpha_j = \begin{cases} \infty & \text{in case (I)} \\ \lambda_j^+(0) (\tilde{\varphi}_j, \varphi_j) |(\varphi_j, \varphi_j)|^{-2} & \text{in case (II)} \end{cases} \quad (2.15)$$

Remarks

1. $\alpha_j = \infty$ means that the point x_j shall be removed from the definition of $-\Delta_{(X, \alpha)}$, i.e. we use $-\Delta_{(\tilde{X}, \tilde{\alpha})}$ with \tilde{X} consisting of the points in X which are in case (II). If all points have $\alpha_j = \infty$ we get the free Hamiltonian, i.e. $-\Delta_{(X, \alpha)} = -\Delta$

2. The theorem is proved by other means in Albeverio and Høegh Krohn [4] under the assumption that the potentials have compact support.

Proof:

Define the operator $A = [A_{\ell j}]_{\ell, j=1}^n$ on the Hilbert space

$$\mathcal{H} = \bigoplus_{j=1}^n L^2(\mathbb{R}^3) \quad \text{by}$$

$$A_{\ell j} = \tilde{w}_{\ell} G_k \tilde{v}_j \quad (2.16)$$

for $\ell, j = 1, \dots, n$ where \tilde{v}_j, \tilde{w}_j are given by

$$\tilde{u}_j(x) = u_j\left(\frac{1}{\varepsilon}(x-x_j)\right)$$

$$\tilde{v}_j(x) = v_j\left(\frac{1}{\varepsilon}(x-x_j)\right) \quad (2.17)$$

$$\tilde{w}_j(x) = \varepsilon^{-2} \lambda_j(\varepsilon) \tilde{u}_j(x)$$

As in Simon [7] we have for $\text{Im} k$ sufficiently large that

$$\begin{aligned} (H_{\varepsilon} - k^2)^{-1} &= G_k + \sum_{m=1}^{\infty} (-1)^m \left[G_k \left(\sum_{j=1}^n \tilde{v}_j \tilde{w}_j \right) \right]^m G_k \\ &= G_k + \sum_{m=1}^{\infty} (-1)^m \sum_{\ell, j=1}^n G_k \tilde{v}_{\ell} \left[\sum_{\substack{j_1, \dots, j_{m-2} \\ 1 \leq j_1, \dots, j_{m-2} \leq n}} A_{\ell j_1} A_{j_1 j_2} \dots A_{j_{m-2} j} \right] \tilde{w}_j G_k \end{aligned} \quad (2.18)$$

(For $m = 1$ the last bracket is defined to be $\delta_{\ell j}$, and for

$m = 1$ it is defined to be $A_{\ell j}$.)

We now introduce the operator $B = [B_{\ell j}]_{\ell, j=1}^n : \mathcal{H} \rightarrow \mathcal{H}$ where $B_{\ell j}$

has integral kernel

$$B_{\ell j}(x, y) = \varepsilon \lambda_{\ell}(\varepsilon) u_{\ell}(x) G_k(\varepsilon(x-y) + x_{\ell} - x_j) v_j(y) \quad (2.19)$$

for $\ell, j = 1, \dots, n$. In addition let $C_j, D_j : L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$ have integral kernels

$$C_j(x, y) = G_k(x - \epsilon y - x_j) v_j(y) \quad (2.20)$$

$$D_j(x, y) = \lambda_j(\epsilon) u_j(x) G_k(\epsilon x + x_j - y) \quad (2.21)$$

(We suppress the ϵ and k dependence for the moment to simplify the notations).

By a change of variables $(x \mapsto \frac{1}{\epsilon} (x - x_r))$ in (2.18) we obtain the following expression

$$\begin{aligned} & (H_\epsilon - k^2)^{-1} \\ &= G_k + \sum_{m=1}^{\infty} (-1)^m \sum_{\ell, j=1}^n \epsilon C_\ell \left[\sum_{\substack{j_1, \dots, j_{m-2} \\ 1 \leq j_1, \dots, j_{m-2} \leq n}} B_{\ell j_1} B_{j_1 j_2} \dots B_{j_{m-2} j} \right] D_j \\ &= G_k + \sum_{m=1}^{\infty} (-1)^m \sum_{\ell, j=1}^n \epsilon C_\ell [B^{m-1}]_{\ell j} D_j \\ &= G_k - \epsilon \sum_{\ell, j=1}^n C_\ell \left[\sum_{m=0}^{\infty} (-1)^m [B^m]_{\ell j} \right] D_j \\ &= G_k - \epsilon \sum_{\ell, j=1}^n C_\ell [1 + B]_{\ell j}^{-1} D_j \end{aligned} \quad (2.22)$$

Remark the great structural resemblance with the resolvent of

$-\Delta_{(X, \alpha)}$ in equation (2.14).

The validity of (2.22) extends to $\text{Im} k > 0$, $k^2 \notin \sigma(H_\epsilon)$ by analytic continuation of both sides.

What remains to be found is the limit of B, C_j and D_j when ϵ tends to 0 and therefore we introduce the ϵ dependence:

$$B^\epsilon = B, \quad C_j^\epsilon = C_j \quad \text{and} \quad D_j^\epsilon = D_j.$$

From Albeverio, Gesztesy and Høegh Krohn [5] we have that

$$C_j^\epsilon \xrightarrow{n} |G_k(\cdot - x_j) \rangle \langle v_j| \quad \text{as } \epsilon \rightarrow 0 \quad (2.23)$$

where the operator $S = |f\rangle\langle g|$ is defined by $Sh = (g, h)f$.

Similarly

$$D_j^\varepsilon \xrightarrow{n} |u_j\rangle\langle G_k(x_j - \cdot)| \quad \text{as } \varepsilon \rightarrow 0 \quad (2.24)$$

Introducing the operators $E^\varepsilon = [E_{lj}^\varepsilon]$ and $F^\varepsilon = [F_{lj}^\varepsilon]$ with integral kernels

$$E_{lj}^\varepsilon(x, y) = \delta_{lj} \lambda_l(\varepsilon) u_l(x) G_{\varepsilon k}(x-y) v_j(y) \quad (2.25)$$

$$F_{lj}^\varepsilon(x, y) = (1 - \delta_{lj}) \lambda_l(\varepsilon) u_l(x) G_k(\varepsilon(x-y) + x_l - x_j) v_j(y) \quad (2.26)$$

we see that

$$1 + B^\varepsilon = 1 + E^\varepsilon + \varepsilon F^\varepsilon \quad (2.27)$$

To find the limit of $(1+B^\varepsilon)^{-1}$ we see from the following computation that it is necessary to find the limit of $\varepsilon(1+E^\varepsilon)^{-1}$.

$$\begin{aligned} \varepsilon(1+B^\varepsilon)^{-1} &= \varepsilon(1+E^\varepsilon + \varepsilon F^\varepsilon)^{-1} = \\ &= (1 + \varepsilon(1+E^\varepsilon)^{-1} F^\varepsilon)^{-1} \varepsilon(1+E^\varepsilon)^{-1} \end{aligned} \quad (2.28)$$

To this end we expand E_{lj}^ε around $\varepsilon = 0$.

Because $V_l \in R \cap L^1(\mathbb{R}^3)$ we have

$$\lambda_l(\varepsilon) u_l G_{\varepsilon k} v_l = u_l G_0 v_l + \varepsilon L_l + o(\varepsilon) \quad (2.29)$$

for $\varepsilon = 1, \dots, n$ where

$$L_l = \lambda_l'(0) u_l G_0 v_l + \frac{ik}{4\pi} |u_l\rangle\langle v_l| \quad (2.30)$$

and $o(\varepsilon)$ is a bounded operator such that $\frac{1}{\varepsilon} ||o(\varepsilon)|| \rightarrow 0$

as $\varepsilon \rightarrow 0$.

From Albeverio, Gesztesy and Høegh Krohn [5] we have that

$$\varepsilon(1 + \varepsilon u_l G_0 v_l)^{-1} = P_j + o(1) \quad (2.31)$$

where $o(1)$ is a bounded operator such that $||o(1)|| \rightarrow 0$

as $\varepsilon \rightarrow 0$ and

$$P_j = \begin{cases} 0 & \text{in case (I)} \\ \frac{|\varphi_j\rangle\langle\tilde{\varphi}_j|}{(\tilde{\varphi}_j, \varphi_j)} & \text{in case (II)} \end{cases} \quad (2.32)$$

Using this and the expansion (2.29) we obtain

$$\begin{aligned}
 \varepsilon (1+E_{\ell\ell}^\varepsilon)^{-1} &= \varepsilon [1+\varepsilon+u_\ell G_\ell v_\ell + \varepsilon (L_\ell - 1 + o(1))]^{-1} \\
 &= [1+\varepsilon (1+\varepsilon+u_\ell G_\ell v_\ell)^{-1} (L_\ell - 1 + o(1))]^{-1} \varepsilon (1+\varepsilon+u_\ell G_\ell v_\ell)^{-1} \\
 &= [1+(P_\ell + o(1)) (L_\ell - 1 + o(1))]^{-1} (P_\ell + o(1)) \\
 &= [1 + (1+P_\ell L_\ell - P_\ell)^{-1} o(1)]^{-1} (1+P_\ell L_\ell - P_\ell)^{-1} (P_\ell + o(1)) \\
 &= (1+P_\ell L_\ell - P_\ell)^{-1} P_\ell + o(1)
 \end{aligned} \tag{2.33}$$

which implies that

$$\varepsilon (1+E^\varepsilon)^{-1} \xrightarrow{n} K \quad \text{as } \varepsilon \rightarrow 0 \tag{2.34}$$

where

$$K = [\delta_{\ell j} (1+P_\ell L_\ell - P_\ell)^{-1} P_\ell] \tag{2.35}$$

According to Albeverio, Gesztesy and Høegh Krohn [5] we have

$$(1+P_\ell L_\ell - P_\ell)^{-1} P_\ell = \begin{cases} 0 & \text{in case (I)} \\ \left[\frac{ik}{4\pi} | (v_\ell, \varphi_\ell) |^2 - \lambda_\ell'(0) (\tilde{\varphi}_\ell, \varphi_\ell) \right]^{-1} | \varphi_\ell \rangle \langle \tilde{\varphi}_\ell | & \text{in case (II)} \end{cases} \tag{2.36}$$

So far we have only been using the assumption that $v_j \in R \cap L^1(\mathbb{R}^3)$, but from lemma 2.4, proved after this theorem, we have under the assumptions that $(1+|\cdot|)^2 v_j \in R \cap L^1(\mathbb{R}^3)$ that

$$F^\varepsilon \xrightarrow{s} F^O \quad \text{as } \varepsilon \rightarrow 0 \tag{2.37}$$

where

$$F^O = [\tilde{G}_k(x_\ell - x_j) | u_\ell \rangle \langle v_j |] \tag{2.38}$$

From (2.35) and (2.36) we see that the norm of K can be made arbitrarily small when $\text{Im} k$ is large, and (2.44) implies that $\|F^\varepsilon\|$ is uniformly bounded.

$$\begin{aligned}
 &(1+\varepsilon (1+E^\varepsilon)^{-1} F^\varepsilon)^{-1} = \\
 &= (1+KF^O)^{-1} + (1+\varepsilon (1+E^\varepsilon)^{-1} F^\varepsilon)^{-1} (\varepsilon (1+E^\varepsilon)^{-1} F^\varepsilon - KF^O) (1+KF^O)^{-1} \\
 &\xrightarrow{s} (1+KF^O)^{-1} \quad \text{as } \varepsilon \rightarrow 0.
 \end{aligned} \tag{2.39}$$

Using (2.34) and (2.39) we obtain

$$\varepsilon (1+B^\varepsilon)^{-1} \xrightarrow{S} (1+KF^0)^{-1} K \quad (2.40)$$

Taking the limit in (2.22) when ε tends to zero and using equations (2.23), (2.24) and (2.40) we finally obtain after a short computation that

$$(H_\varepsilon - k^2)^{-1} \xrightarrow{S} (-\Delta_{(X,\alpha)} - k^2)^{-1} \quad \text{as } \varepsilon \rightarrow 0 \quad (2.41)$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is given according to (2.15) and remark 1. \square

To establish equation (2.37) we need the following lemma

Lemma 2.4

Let V_1, V_2 be real functions such that $(1+|\cdot|)^2 V_j \in R \cap L^1(\mathbb{R}^3)$ and define $v_j = |V_j|^{1/2}$ and $u_j = v_j \operatorname{sgn} V_j$. Let further λ be a real analytic function in a neighbourhood of 0 with $\lambda(0) = 1$. If $a \in \mathbb{R}^3$, $a \neq 0$, and F^ε is the operator with integral kernel

$$F^\varepsilon(x, y) = \lambda(\varepsilon) u_1(x) G_k(\varepsilon(x-y)+a) v_2(y) \quad (2.42)$$

where G_k is defined by (2.6) and $\operatorname{Im} k > 0$, then

$$F^\varepsilon \xrightarrow{S} F^0 \quad \text{as } \varepsilon \rightarrow 0 \quad (2.43)$$

Proof:

There is no lack of generality to assume that $\lambda(\varepsilon) \equiv 1$.

First we prove that $\|F^\varepsilon\|$ is bounded by estimating the Hilbert-Schmidt norm $\|\cdot\|_2$ of $F^\varepsilon - F^0$.

$$\begin{aligned} \|F^\varepsilon - F^0\|_2^2 &= \frac{1}{16\pi^2} \iint |V_1(x)| |V_2(y)| \left| \frac{e^{ik|\varepsilon(x-y)+a|}}{|\varepsilon(x-y)+a|} - \frac{e^{ik|a|}}{|a|} \right|^2 dx dy \\ &\leq \frac{1}{8\pi^2} \left[\iint |V_1(x)| |V_2(y)| \left| \frac{|a| - |\varepsilon(x-y)+a|}{|\varepsilon(x-y)+a| |a|} e^{ik|\varepsilon(x-y)+a|} \right|^2 dx dy \right. \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{|a|^2} \iint |V_1(x)| |V_2(y)| \left| e^{ik|\varepsilon(x-y)+a|} - e^{ik|a|} \right|^2 dx dy \\
 & \leq \frac{1}{4\pi^2 |a|^2} \left[\iint |V_1(x)| |V_2(y)| \frac{(1+|x|^2)(1+|y|^2)}{|x-y+\frac{1}{\varepsilon}a|^2} dx dy + 2 \|V_1\|_1 \|V_2\|_1 \right] \\
 & \leq \frac{1}{4\pi^2 |a|^2} \left[\|(1+|\cdot|^2)V_1\|_R \|(1+|\cdot|^2)V_2\|_R + 2 \|V_1\|_1 \|V_2\|_1 \right] \quad (2.44)
 \end{aligned}$$

where $\|V\|_R = \left[\iint |V(x)| |V(y)| |x-y|^{-2} dx dy \right]^{1/2}$ is the Rollnik norm.

From this uniform bound on the norm of F^ε we only have to prove

that $\|(F^\varepsilon - F^0)f\| \rightarrow 0$ as $\varepsilon \rightarrow 0$ for $f \in C_0^\infty(\mathbb{R}^3)$

$$\begin{aligned}
 \|(F^\varepsilon - F^0)f\|^2 &= \int |V_1(x)| \left| \int (G_k(\varepsilon(x-y)+a) - G_k(a)) v_2(y) f(y) dy \right|^2 dx \\
 &= \int (1+|x|)^2 |V_1(x)| \left[\frac{1}{1+|x|} \left| \int (G_k(\varepsilon(x-y)+a) - G_k(a)) v_2(y) f(y) dy \right| \right]^2 dx \quad (2.45)
 \end{aligned}$$

For each $x \in \mathbb{R}^3$ we have from Lebesgue's dominated convergence theorem that

$$\int (G_k(\varepsilon(x-y)+a) - G_k(a)) v_2(y) f(y) dy \rightarrow 0 \quad (2.46)$$

as $\varepsilon \rightarrow 0$ because f has compact support.

$$\left| \int (G_k(\varepsilon(x-y)+a) - G_k(a)) v_2(y) f(y) dy \right| \leq$$

$$\frac{1}{4\pi|a|} \left[\left| \int \frac{|a| - |\varepsilon(x-y)+a|}{|\varepsilon(x-y)+a|} e^{ik|\varepsilon(x-y)+a|} v_2(y) f(y) dy \right| + \right.$$

$$\left. \left| \int (e^{ik|\varepsilon(x-y)+a|} - e^{ik|a|}) v_2(y) f(y) dy \right| \right]$$

$$\begin{aligned}
 & \leq \frac{1}{4\pi|a|} \left[2 \int |v_2(y) f(y)| dy + \int \frac{|y| |v_2(y)| |f(y)|}{|x-y+a/\varepsilon|} dy + |x| \int \frac{|v_2(y) f(y)|}{|x-y+a/\varepsilon|} dy \right] \\
 & \leq C(1+|x|) \quad (2.47)
 \end{aligned}$$

where C is a constant independent of ε since $\frac{1}{|\cdot|} \in L^2_{\text{loc}}(\mathbb{R}^3)$
(and $\int \frac{dx}{|x-b|^2}$ is bounded independently of $b \in \mathbb{R}^3$) and
suppf

$|\cdot|_{V_2} f, \quad v_2 f \in L^2(\mathbb{R}^3)$ using Hölder's inequality.

From (2.45), (2.46) and (2.47) we conclude, using dominated convergence, that $F^\varepsilon \xrightarrow{\mathcal{S}} F^0$ as $\varepsilon \rightarrow 0$ thus proving the lemma.

□

We will now strengthen the conditions on the potentials but also improve the conclusion of theorem 2.3, treating all cases (I) to (IV).

Theorem 2.5

Let $V_1, \dots, V_n \in \mathbb{R}$ be real-valued with compact support.

If H_j is in case (III) or (IV) assume in addition that $\lambda_j'(0) \neq 0$.

Then the self-adjoint operator H_ε defined by (2.1) will converge in norm resolvent sense to the self-adjoint operator $-\Delta_{(X, \alpha)}$ defined

by (2.14) where $\alpha = (\alpha_1, \dots, \alpha_n)$ is

$$\alpha_j = \begin{cases} \infty & \text{in case (I) and (III)} \\ \lambda_j'(0) (\tilde{\varphi}_j, \varphi_j) |(\varphi_j, \varphi_j)|^{-2} & \text{in case (II)} \\ \lambda_j'(0) \left[\sum_{r=1}^{N_j} |(\varphi_j, \varphi_{jr})|^2 (\tilde{\varphi}_{jr}, \varphi_{jr})^{-1} \right]^{-1} & \text{in case (IV)} \end{cases} \quad (2.48)$$

Remarks

1. $\alpha_j = \infty$ means that the point x_j shall be removed from the definition of the operator $-\Delta_{(X, \alpha)}$, i.e. we use $-\Delta_{(\tilde{X}, \tilde{\alpha})}$ where

$\tilde{X} \subset X$ consists of the points in case (II) and (IV).

2. Albeverio and Høegh Krohn [4] have proved strong resolvent convergence in case (I) and (II), but in case (III) and (IV) they assume that

the potentials have definite sign.

3. If $\lambda_j'(0) = 0$ in case (III) and (IV) we will not in general have norm resolvent convergence, see Albeverio, Gesztesy and Høegh Krohn [5].

Using the following proof we can also slightly weaken some of the conditions on the potentials in the one-center case, i.e. when $n = 1$, in Albeverio, Gesztesy and Høegh Krohn [5].

Proof:

The proof of this theorem will closely follow the proof of theorem 2.3. From Simon [7] it follows that Rollnik-functions with compact support are in $L^1(\mathbb{R}^3)$ and therefore we can use the proof of theorem 2.3 till equation (2.31)

Instead of (2.32) we now have

$$\varepsilon(1 + \varepsilon + u_j G_o v_j)^{-1} = P_j + o(1) \quad (2.49)$$

where

$$P_j = \begin{cases} 0 & \text{in case (I)} \\ \frac{|\varphi_j\rangle\langle\tilde{\varphi}_j|}{(\tilde{\varphi}_j, \varphi_j)} & \text{in case (II)} \\ \sum_{r=1}^{N_j} \frac{|\varphi_{jr}\rangle\langle\tilde{\varphi}_{jr}|}{(\tilde{\varphi}_{jr}, \varphi_{jr})} & \text{in case (III) and (IV)} \end{cases} \quad (2.50)$$

We still have

$$\varepsilon(1 + E^\varepsilon)^{-1} \rightarrow [\delta_{lj} (1 + P_j L_j - P_j)^{-1} P_j] = K \quad \text{as } \varepsilon \rightarrow 0 \quad (2.51)$$

but now

$$(1 + P_j L_j - P_j)^{-1} P_j = \begin{cases} 0 & \text{in case (I)} \\ \left[\frac{ik}{4\pi} |(v_j, \varphi_j)|^2 - \lambda_j'(0) (\tilde{\varphi}_j, \varphi_j) \right]^{-1} |\varphi_j\rangle\langle\tilde{\varphi}_j| & \text{in case (II)} \\ - \frac{1}{\lambda_j'(0)} \sum_{r=1}^{N_j} \frac{|\varphi_{jr}\rangle\langle\tilde{\varphi}_{jr}|}{(\tilde{\varphi}_{jr}, \varphi_{jr})} & \text{in case (III)} \\ \sum_{r,s=1}^{N_j} \left[\frac{ik}{4\pi} (\varphi_{jr}, v_j) (v_j, \varphi_{js}) - \lambda_j'(0) (\tilde{\varphi}_{jr}, \varphi_{js}) \right]_{rs}^{-1} |\varphi_{jr}\rangle\langle\tilde{\varphi}_{js}| & \text{in case (IV)} \end{cases} \quad (2.52)$$

gative eigenvalue $E(\epsilon)$ with $0 < M_1 \leq |E(\epsilon)| \leq M_2 < \infty$ for small $\epsilon > 0$.

Let $\{\epsilon_n\}$ be a positive sequence converging to zero and let

k_0^2 ($\text{Im} k > 0$) be an accumulation point for $\{E(\epsilon_n)\}$. Then is a multi-valued analytic function $k(\epsilon)$ with $k(0) = k_0$, i.e.

$$k(\epsilon) = k_0 + g(\epsilon^{1/r}) \quad (3.3)$$

where g is analytic, $g(0) = 0$, and $r \in \mathbb{N}$, such that $k^2(\epsilon)$ is a negative eigenvalue for H_ϵ and k_0^2 is a negative eigenvalue for $-\Delta_{(X, \alpha)}$.

We have the following expansion of $k(\epsilon)$

$$k(\epsilon) = k_0 + \epsilon^{1/r} k_1 + o(\epsilon^{1/r}) \quad (3.4)$$

where k_1 is a solution of the implicit equation (3.34) if $r \geq 1$ or (3.35) if $r = 1$.

Proof:

Let $\rho(\epsilon) = \sqrt{E(\epsilon)}$, $\text{Im} \rho(\epsilon) > 0$.

From (2.19) we see that $E(\epsilon)$ is a negative eigenvalue of H_ϵ iff -1 is an eigenvalue of the operator $B^{\epsilon, k}$ with $k = \rho(\epsilon)$ where we have introduced the ϵ and k dependence for the operator defined by (2.19).

We expand the operator $B^{\epsilon, k}$ in powers of ϵ :

$$B^{\epsilon, k} = S + \epsilon T + o(\epsilon) \quad (3.5)$$

where

$$S = [\delta_{\ell j} \ u_j \ G_0 \ v_j] \quad (3.6)$$

and

$$T = [(\lambda_j'(0) u_j G_0 v_j + \frac{ik}{4\pi} |u_j\rangle \langle v_j|) \delta_{\ell j} + \tilde{G}_k(x_\ell - x_j) |u_\ell\rangle \langle v_j|] \quad (3.7)$$

and

$$\frac{1}{\epsilon} \|o(\epsilon)\| \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \quad (3.8)$$

From our assumptions (3.2) concerning zero energy resonance we have that

$$\text{Ker } (1+S) = \{(a_1\varphi_1, \dots, a_n\varphi_n) \mid a_j \in \mathbb{C}\} \quad (3.9)$$

where Ker is the kernel and we recall from section 2 that φ_j is the eigenfunction satisfying

$$(1+u_j G_O v_j)\varphi_j = 0 \quad (3.10)$$

From proposition 2.2 we see that it possible to normalize φ_j such that $(v_j, \varphi_j) = 1$.

Introduce

$$\mathcal{H}_0 = \text{Ker } (1+S) \quad (3.11)$$

$$\mathcal{H}_1 = \text{Ran } (1+S) \quad (3.12)$$

Then

$$P = \left[\delta_{\ell j} \frac{|\varphi_\ell\rangle\langle\tilde{\varphi}_\ell|}{(\tilde{\varphi}_\ell, \varphi_\ell)} \right] \quad (3.13)$$

(recall $\tilde{\varphi}_\ell = \varphi_\ell \text{ sgn } V_\ell$) will be a projection onto \mathcal{H}_0 .

We have

$$\text{Ker } (1+S^*) = \{(a_1\tilde{\varphi}_1, \dots, a_n\tilde{\varphi}_n) \mid a_j \in \mathbb{C}\} \quad (3.14)$$

thus making

$$\text{Ker } P = \text{Ker } (1+S^*)^\perp \quad (3.15)$$

The Fredholm alternative implies

$$\mathcal{H}_1 = \text{Ker } (1+S^*)^\perp = \text{Ker } P. \quad (3.16)$$

i.e. we have that \mathcal{H} is the direct sum of \mathcal{H}_0 and \mathcal{H}_1 . We can also conclude that $(1+S) : \text{Ran } (1+S) \rightarrow \text{Ran } (1+S)$ is a bijection.

We now split the operators S, T and $o(\varepsilon)$ by defining

$$S_{00} = P S P \quad (3.17)$$

$$S_{10} = (1-P) S P \quad (3.18)$$

$$S_{01} = P S (1-P) \quad (3.19)$$

$$S_{11} = (1-P) S (1-P) \quad (3.20)$$

and similarly for T and $o(\varepsilon)$.

Then $S_{00} = -P$, $S_{10} = S_{01} = 0$, thus we can write $B^{\varepsilon,k}$ as

$$B^{\varepsilon,k} = \begin{bmatrix} -1 + \varepsilon T_{00} + o_{00}(\varepsilon) & \varepsilon T_{01} + o_{01}(\varepsilon) \\ \varepsilon T_{10} + o_{10}(\varepsilon) & S_{11} + \varepsilon T_{11} + o_{11}(\varepsilon) \end{bmatrix} \quad (3.21)$$

using the decomposition of \mathcal{H} into \mathcal{H}_0 and \mathcal{H}_1 .

We define the operator $B^{\varepsilon,k}$ by

$$B^{\varepsilon,k} = \begin{bmatrix} -1 + T_{00} + \frac{1}{\varepsilon} o_{00}(\varepsilon) & \varepsilon T_{01} + o_{01}(\varepsilon) \\ T_{10} + \frac{1}{\varepsilon} o_{10}(\varepsilon) & S_{11} + \varepsilon T_{11} + o_{11}(\varepsilon) \end{bmatrix} \quad (3.22)$$

Then we have that

$$(1+B^{\varepsilon,k}) \begin{bmatrix} \psi_0 \\ \varepsilon \psi_1 \end{bmatrix} = \varepsilon (1+B^{\varepsilon,k}) \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix} \quad (3.23)$$

for $\varepsilon > 0$ which shows that $E(\varepsilon)$ is a negative eigenvalue for H_ε iff -1 is an eigenvalue for $B^{\varepsilon,\rho(\varepsilon)}$ where $\rho(\varepsilon) = \sqrt{E(\varepsilon)}$, $\text{Im} \rho(\varepsilon) > 0$.

When $\varepsilon = 0$ we have that

$$(1 + B^{0,k}) \begin{bmatrix} \psi_0 \\ \psi_1 \end{bmatrix} = \begin{bmatrix} T_{00}\psi_0 \\ T_{10}\psi_0 + (1+S_{11})\psi_1 \end{bmatrix} \quad (3.24)$$

and if

$$\psi_0 = (c_1 \varphi_1, \dots, c_n \varphi_n) \quad \text{with } c_i \in \mathbb{C} \quad (3.25)$$

then

$$T_{00} \psi_0 = - \left(\frac{\varphi_\ell}{(\tilde{\varphi}_\ell, \varphi_\ell)} \sum_{j=1}^n [(\alpha_\ell - \frac{ik}{4\pi}) \delta_{\ell j} - \tilde{G}_k(x_\ell - x_j)] c_j \right)_{\ell=1}^n \quad (3.26)$$

Now

$$T_{10} \psi_0 + (1+S_{11}) \psi_1 = 0 \quad (3.27)$$

iff

$$\psi_1 = - [(1+S)|_{\mathcal{H}_1}]^{-1} T_{10} \psi_0 \quad (3.28)$$

Therefore -1 is an eigenvalue for $B^{0,k}$ iff $\text{Ker } T_{00} \neq \{0\}$

and by (3.26) this is the case iff k^2 is a negative eigenvalue for $-\Delta_{(X,\alpha)}$.

If we define the analytic function

$$f(\varepsilon, k) = \det_2(1 + B^{\varepsilon, k}) \quad (3.29)$$

where \det_2 is the modified Fredholm-determinant (see e.g. Simon [8])

then $f(\varepsilon, \rho(\varepsilon)) = 0$ for small $\varepsilon > 0$.

Let k_0 be an accumulation point for $\{\rho(\varepsilon_n)\}$ where $\{\varepsilon_n\}$ is a positive sequence converging to zero. Then $f(0, k_0) = 0$ which shows that k_0^2 is a negative eigenvalue for $-\Delta_{(X,\alpha)}$.

The analytic function $f(\cdot, 0)$ is not identically zero, and from implicit function theory (see e.g. Rauch [9]) we know that there is a multivalued analytic function $k(\varepsilon)$ with $k(0) = k_0$, i.e.

$$k(\varepsilon) = k_0 + g(\varepsilon^{1/r}) \quad (3.30)$$

with g analytic, $g(0) = 0$ and $r \in \mathbb{N}$ such that

$$f(\varepsilon, k(\varepsilon)) = 0 \quad (3.31)$$

for small $\varepsilon > 0$. $k(\varepsilon)^2$ is then a negative eigenvalue for H_ε .

Returning now to the operator $B^{\varepsilon, k}$ and putting $\kappa = k(\varepsilon^r)$, we have an analytic Hilbert-Schmidt operator $B^{\varepsilon^r, \kappa(\varepsilon)}$ with -1 as an eigenvalue for ε small (for $\varepsilon = 0$ $B^{0, k}$ will always have -1 as an eigenvalue independently of k as will be seen from the definition of $B^{\varepsilon, k}$ (2.19) and the assumption (3.2) on the potentials.) By first reducing the problem to a finite dimensional space by standard methods (See e.g. Reed and Simon [10] ch. XII sec. 1 and 2) and using a theorem of Baumgärtel [11] we can find an eigenvector ϕ^ε with $\varepsilon \mapsto \phi^\varepsilon$ analytic such that

$$(1 + B^{\varepsilon^r, \kappa(\varepsilon)}) \phi^\varepsilon = 0 \quad (3.32)$$

Let $\phi^\varepsilon = (\varphi_1^\varepsilon, \dots, \varphi_n^\varepsilon)$ and put $\varphi_j' = \frac{\partial \varphi_j^\varepsilon}{\partial \varepsilon} \Big|_{\varepsilon=0}$.

From (3.32) we see that if $\varepsilon = 0$ we have

$$(1 + u_j G_O v_j) \varphi_j^O = 0 \quad (3.33)$$

By taking the derivative $r + 1$ times in $\varepsilon = 0$ in (3.32) and taking inner product with $\tilde{\varphi}_j^O$ we obtain the following equations ($k_1 = \kappa'(0)$)

$r > 1$

$$\frac{ik_O}{4\pi} (\varphi_\ell^O, v_\ell) (v_\ell, \varphi_\ell') - \lambda_\ell' (0) (\tilde{\varphi}_\ell^O, \varphi_\ell') \quad (3.34)$$

$$+ \frac{ik_1}{4\pi} \sum_{j=1}^n e^{ik_O |x_\ell - x_j|} (\varphi_\ell^O, v_\ell) (v_j, \varphi_j^O) + \sum_{j=1}^n \tilde{G}_{k_O} (x_\ell - x_j) (\varphi_\ell^O, v_\ell) (v_j, \varphi_j') = 0$$

$r = 1$

$$\begin{aligned} & -\lambda_\ell'' (0) (\tilde{\varphi}_\ell^O, \varphi_\ell^O) - \frac{k_O^2}{4\pi} \iint \tilde{\varphi}_\ell^O (x) v_\ell (x) |x-y| v_\ell (y) \varphi_\ell^O (y) dx dy \\ & + 2\lambda_\ell' (0) \alpha_\ell | (v_\ell, \varphi_\ell^O) |^2 - 2\lambda_\ell' (0) (\tilde{\varphi}_\ell^O, \varphi_\ell') + \frac{ik_O}{2\pi} (\varphi_\ell^O, v_\ell) (v_\ell, \varphi_\ell') \\ & + 2 \sum_{\substack{j=1 \\ j \neq \ell}}^n \iint \tilde{\varphi}_\ell^O (x) v_\ell (x) \nabla G_{k_O} (x_\ell - x_j) \cdot (x-y) v_j (y) \varphi_j^O (y) dx dy \\ & + \frac{ik_1}{2\pi} \sum_{j=1}^n e^{ik_O |x_\ell - x_j|} (\varphi_\ell^O, v_\ell) (v_j, \varphi_j^O) \\ & + 2 \sum_{j=1}^n \tilde{G}_{k_O} (x_\ell - x_j) (\varphi_\ell^O, v_\ell) (v_j, \varphi_j') = 0 \end{aligned} \quad (3.35)$$

(In the $r = 1$ equation we have used the equation one obtains by taking the derivative r times in $\varepsilon = 0$ in (3.32) to simplify the expression).

□

We now want to reverse theorem 3.1 by starting with an eigenvalue for $-\Delta_{(X,\alpha)}$. Using the norm resolvent convergence we can formulate the following theorem

Theorem 3.2

Assume that k_0^2 ($\text{Im} k_0 > 0$) is a negative eigenvalue for $-\Delta_{(X,\alpha)}$ with multiplicity m .

Then there exist m (not necessarily different) multivalued analytic functions $k_j(\epsilon)$ in a neighbourhood of 0 with $k_j(0) = k_0$, i.e.

$$k_j(\epsilon) = k_0 + g_j(\epsilon^{1/r_j}) \quad (3.36)$$

with g_j analytic, $g_j(0) = 0$, and $r_j \in \mathbb{N}$ such that $\{k_j^2(\epsilon)\}$ are all the eigenvalues for H_ϵ in a neighbourhood of k_0^2 for all sufficiently small ϵ .

We have the following expansion

$$k_j(\epsilon) = k_0 + \epsilon^{1/r_j} k_{1,j} + o(\epsilon^{1/r_j}) \quad (3.37)$$

where $k_{1,j} = k_1$ is a solution of (3.34) if $r_j > 1$ or (3.35) if $r_j = 1$.

Proof:

From the norm resolvent convergence proved in theorem 2.5. we can conclude using the convergence of the spectral families that there are m functions $E_j(\epsilon)$ where $E_j(\epsilon)$ is an eigenvalue for H_ϵ , converging to k_0^2 .

As in the proof of theorem 3.1 we obtain the multivalued analytic functions $k_j(\epsilon)$ and the expansion stated in the theorem.

4. Resonances

In this section we will use the same assumptions on the potentials as in section 3, i.e.

$$(i) \quad V_j \in R \quad \text{and} \quad \text{supp } V_j \quad \text{is compact} \quad (4.1)$$

$$(ii) \quad H_j = -\Delta + V_j \quad \text{is in case (II)} \quad (4.2)$$

From (2.22) we have for $\text{Im} k > 0$, $k^2 \notin \sigma(H_\epsilon)$

$$(H_\epsilon - k^2)^{-1} = G_k - \epsilon \sum_{\ell, j=1}^n C_\ell [1+B]_{\ell j}^{-1} D_j \quad (4.3)$$

But recalling the definitions (2.19-21) of the operators B, C_ℓ, D_j we see (because of our assumption (4.1)) that the right hand side of (4.3) is a meromorphic function of k also for $\text{Im} k \leq 0$. In analogy with the properties of negative eigenvalues, we define resonances as follows. (We now introduce the ϵ and k dependence for B , i.e. $B^{\epsilon, k} = B$)

Definition 4.1

We say that $k(\epsilon)$, $\text{Im} k(\epsilon) < 0$, is a resonance for H_ϵ if and only if -1 is an eigenvalue for $B^{\epsilon, k}$.

For the operator $-\Delta_{(X, \alpha)}$ negative eigenvalues and resonances are in complete analogy.

Definition 4.2

We say that k , $\text{Im} k < 0$, is a resonance for $-\Delta_{(X, \alpha)}$ if and only if

$$\det [(\alpha_\ell - \frac{ik}{4\pi}) \delta_{\ell j} - \tilde{G}_k(x_\ell - x_j)]_{\ell, j=1}^n = 0$$

This definition makes it possible to study how the resonances vary with $\alpha = (\alpha_1, \dots, \alpha_n)$ for simple geometric arrangements of

$X = (x_1, \dots, x_n)$. See Albeverio and Høegh Krohn [12] for details.

With these definitions we can formulate the following theorem.

Theorem 4.3

Assume that H_ε has a resonance $\kappa(\varepsilon)$ with

$$0 < M_1 \leq |\operatorname{Im} \kappa(\varepsilon)| \leq |\kappa(\varepsilon)| \leq M_2 < \infty \quad \text{for } \varepsilon \text{ small}$$

Let $\{\varepsilon_n\}$ be a positive sequence converging to zero and let k_0 be an accumulation point for $\{\kappa(\varepsilon_n)\}$. Then there exists a multivalued analytic function $k(\varepsilon)$ in a neighbourhood of zero with $k(0) = k_0$, i.e.

$$k(\varepsilon) = k_0 + g(\varepsilon^{1/r}) \tag{4.4}$$

with g analytic, $g(0) = 0$, and $r \in \mathbb{N}$, where $k(\varepsilon)$ is a resonance for H_ε and k_0 is a resonance for $-\Delta_{(X, \alpha)}$. We have the following

ε expansion

$$k(\varepsilon) = k_0 + \varepsilon^{1/r} k_1 + o(\varepsilon^{1/r}) \tag{4.5}$$

where k_1 is a solution of (3.34) if $r > 1$ or (3.35) if $r = 1$.

Proof: The proof is identical to that of theorem 3.1 except for one fact. For eigenvalues we have to appeal to (4.3) to say that -1 is an eigenvalue for $B^{\varepsilon, k}$, for resonances this follows from definition 4.1. The assumption $|\operatorname{Im} \kappa(\varepsilon)| \geq M_1 > 0$ enables us to say that $\operatorname{Im} k_0 < 0$.

□

If we want to have an analogue to theorem 3.2 for resonances, we cannot use the same sort of proof because we do not have the spectral projections for resonances. We can now instead formulate the following theorem which is also valid for eigenvalues.

Theorem 4.4

Assume that k_0 ($\text{Im} k_0 < 0$) is a resonance for $-\Delta_{(X,\alpha)}$. Then there exists a multivalued function $k(\epsilon)$ in a neighbourhood of 0 with $k(0) = k_0$, i.e.

$$k(\epsilon) = k_0 + g(\epsilon^{1/r}) \quad (4.6)$$

where g is analytic, $g(0) = 0$, and $r \in \mathbb{N}$, such that $k(\epsilon)$ is a resonance for H_ϵ for small $\epsilon > 0$. We have the following expansion

$$k(\epsilon) = k_0 + k_1 \epsilon^{1/r} + o(\epsilon^{1/r}) \quad (4.7)$$

where k_1 is a solution of (3.34) if $r > 1$ or (3.35) if $r = 1$.

Proof:

The proof will depend heavily upon the proof of theorem 3.1 and we will use the same terminology.

Let

$$f(\epsilon, k) = \det_2(1 + \mathcal{B}^{\epsilon, k}) \quad (4.8)$$

where $\mathcal{B}^{\epsilon, k}$ is defined by (3.22).

From the properties of $\mathcal{B}^{\epsilon, k}$ we have that -1 is an eigenvalue for $\mathcal{B}^{0, k}$ iff k is a resonance for $-\Delta_{(X,\alpha)}$ which implies that

$$f(0, k_0) = 0 \quad (4.9)$$

$$f(0, k) \neq 0 \quad (4.10)$$

From implicit function theory (See e.g. Rauch [10]) we have that there exists a multivalued analytic function $k(\epsilon)$ with $k(0) = k_0$ and

$$f(\epsilon, k(\epsilon)) = 0 \quad (4.11)$$

for small ϵ . We are now in the situation covered by theorem 3.1 and we obtain the same expansions.

□

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