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# EFFECTIVE COFINALITIES AND ADMISSIBILITY IN E-RECURSION

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## § 0 Introduction.

E-Recursion was introduced by D. Normann [1978] as a natural generalization of normal Kleene recursion in objects of finite type. Unless otherwise stated the E-closed sets we shall consider shall be of the form $E(\alpha)$ for some $\alpha \in OR$.

In § 1 we introduce the $RE \wedge co\text{-}RE$ cofinality and show that the $\Sigma_1$-admissibility of $E(\alpha)$ implies that its greatest cardinal has $RE \wedge co\text{-}RE$ cofinality $\omega$. In addition we show that RE-cofinality $\omega$ does not imply admissibility.

Section § 2 is devoted to a dynamic proof of selection (i.e. $\gamma = cf^{E(\alpha)}(\alpha)$ then we have uniform selection over RE subsets of any $\delta < \gamma$ on $E(\alpha)$), which can therefore be relativized. This selection theorem thus has among its corollaries the consistency of the extended plus one hypothesis at the type three level with $\neg CH$.

Applications of the proof of selection given in § 2 are presented in § 3. We show that if $\gamma$ is the cofinality of $\alpha$ in $E(\alpha)$, then the co-RE cofinality of $\gamma$ is $\gamma$. The proof of this gives rise to an effective covering property, namely, any co-RE subset of $\gamma$ can be covered by a REC set of the same order type. The final application makes clear the connection between selection and singularities. We show that for $\alpha < \beta$ such that $cf(\beta) \leq \alpha$

by a function $f$ recursive in $\alpha,\beta$ and some $\delta<\alpha$, then $cf(\beta)\leq\alpha$ by some $f$ recursive in $\alpha,\beta$.

The last section (§ 4) treats the interplay between monotone inductive definitions and E-recursive set functions using methods from Girard's $\beta$-logic [198 ?], without introducing $\beta$-logic or its proof theory. If a $\Delta_0$ $\varphi(x,\cdot)$ always has a solution in $\Gamma_x^\infty$ (the least fixed point of monotone inductive $\Gamma$ over $x$), then the function giving that solution is E-recursive in $x$. As a corollary we have an elementary proof of a theorem of Van de Wiele [1981]:

If $F: V \rightarrow V$ is uniformly $\Sigma_1$-definable and total over all admissible sets, then $F$ is E-recursive. Outside of § 4, RE, co-RE etc. are the boldface notions.

§ 1. Effective Cofinalities.

Much attention has been given to various notions of definable cofinality, particularly in connection with priority arguments E-Recursion. We shall not attempt to give a complete picture and so the interested reader is directed to Griffor [1980], Sacks [1980] or Slaman [1981]. The first question we address here was asked by Sacks, namely, is there a cofinality condition on $\alpha$ which caracterizes when $E(\alpha)$ is $\Sigma_1$-admissible. The question was motivated by a result of Kirousis that: if $E(\alpha)\models cf(\bar{\bar{\alpha}})=\omega$, then $E(\alpha)$ is $\Sigma_1$-admissible. Thus an attractive conjecture was that: $E(\alpha)$ is $\Sigma_1$-admissible if and only if $E(\alpha)\models cf(\bar{\bar{\alpha}})=\omega$. However, Slaman noticed that if $\gamma$ is the least ordinal where $E(\gamma)\models cf(\bar{\bar{\gamma}})>\omega$, then $E(\gamma)$ is $\Sigma_1$-admissible. If $E(\alpha)$ is $\Sigma_1$-admissible Sacks [1980] showed that there is a divergent computation without a Moschovakis witness in $E(\alpha)$. This witness induces an $\omega$-sequence through $\bar{\bar{\alpha}}$ and we will first analyse the level of definability of one such sequence.

Definition. Consider $E(\alpha)$, $\alpha \in OR$, and without loss of generality assume that $\alpha$ is the greatest cardinal in $E(\alpha)$. Define the RE join co-RE cofinality of $\alpha$ as:

$RE \wedge co\text{-}RE\text{-}cf(\alpha)$ = least $\tau \leq \alpha$ such that there exists an $R \leq \alpha$ of order type $\tau$ unbounded in $\alpha$ and $R$ is $RE \wedge co\text{-}RE$, i.e. $R$ is the intersection of an RE and a co-RE set.

Theorem 1.0. Suppose $E(\alpha)$ is $\Sigma_1$-admissible, then

$$RE \wedge co - RE - cf(\alpha) = \omega.$$

proof. As above we assume that $\alpha$ is the greatest cardinal in $E(\alpha)$ (which is $L_\kappa$ for some $\kappa > \alpha$). If $e \in \omega$, $a \in E(\alpha)$, then associated with the computation tuple $\langle e,a\rangle$ is the tree of sub-computations $T_{\langle e,a\rangle}$ (which is recursive in $\langle e,a\rangle$ if $\{e\}(a)\downarrow$, but is in general only RE in $\langle e,a\rangle$). Assume that $E(\alpha)$ is $\Sigma_1$-admissible.

By Sacks [1980] there exists an $e \in \omega$ and $a \in E(\alpha)$ such that $T_{\langle e,a\rangle}$ is not well-founded, but

$$L_\kappa \models \text{'}T_{\langle e,a\rangle} \text{ is well-founded'}.$$

Claim 1. The leftmost path in $T_{\langle e,a\rangle}$ is in $RE \wedge co\text{-}RE$

proof. We say that $\sigma$ is on the leftmost path if

(i) $\sigma \in T_{\langle e,a\rangle}$ (RE)

(ii) $\sigma\uparrow$ (co-RE)

(iii) If $\tau < \sigma$ in the lexicographical ordering and $n$ is minimal such that $\tau(n) < \sigma(n)$, then $\bar{\tau}(n+1)\downarrow$ (RE). This proves claim 1.

Now assume that we have an effective coding of all finite sequences from $\alpha$ by $\alpha$ such that

$$\langle \sigma ^\wedge \tau \rangle > \langle \sigma \rangle, \quad \text{where} \quad \tau \neq \langle\ \rangle.$$

Let $\langle \beta_1, \ldots, \beta_n \rangle \in A$ if $\beta_i$ is the index for the $i^{\underline{th}}$ sequence of the leftmost path through $T_{\langle e,a \rangle}$. Then $A$ is the intersection of an RE set $A_1$ and a co-RE set $A_2$.

Claim 2. $A$ is unbounded in $\alpha$.

proof. If $A$ is bounded by $\lambda < \alpha$, then use standard properties of the $\Sigma_1$-projectum on admissible ordinals to show that $A_1 \cap \lambda \in E(\alpha)$, $A_2 \cap \lambda \in E(\alpha)$ and so $A \in E(\alpha)$, which is impossible.

This completes the proof of the theorem.

Definition. With $E(\alpha)$ as above let

(i) $\text{REC} - \text{cf}(\alpha) = \mu\tau \leq \alpha$ such that there exists REC $R \subseteq \alpha$ of order type $\tau$ unbounded in $\alpha$;

(ii) $\text{RE} - \text{cf}(\alpha) = \mu\tau \leq \alpha$ such that there exists RE $R \subseteq \alpha$ of order type $\tau$ unbounded in $\alpha$.

As one might expect the recursive cofinality is no stronger, on ordinals less than $\varkappa$, than the cofinality in the sense of $E(\alpha)$.

Proposition 1.1. If $\gamma < \varkappa$, then

$$\text{REC} - \text{cf}(\gamma) = \text{cf}^{L_\varkappa}(\gamma).$$

proof $\leq$: let $f : cf^{L_\varkappa}(\gamma) \to \gamma$, $f \in L_\varkappa$ witness $cf^{L_\varkappa}(\gamma)$ and without loss of generality we may assume that $f$ is strictly increasing. Let $R = im(f)$, then $R$ witnesses

$$\mathrm{REC}-cf(\gamma) \leq cf^{L_\varkappa}(\gamma).$$

$\geq$: let $R \subseteq \gamma$ witness the REC-cf$(\gamma) = \tau$, then $R \in L_\varkappa$ by the bounding principle and the function $f : \tau \to \gamma$ given by : $\sigma < \tau$.

$$f(\sigma) = \sigma^{\underline{th}} \text{ element of } R$$

is in $L_\varkappa$ and witnesses $cf^{L_\varkappa}(\gamma) \leq \mathrm{REC}-cf(\gamma)$.

Corollary 1.2. If REC $-$ cf$(\alpha) = \omega$, then $E(\alpha)$ is $\Sigma_1$-admissible.

proof. Use the proposition and the selection-theorem of Kirousis [1978] stating

$$E(\alpha) \models cf(\bar{\bar{\alpha}}) = \omega \Rightarrow E(\alpha) \text{ is } \Sigma_1\text{-admissible.}$$

We shall see now that RE $-$ cf$(\alpha) = \omega$ is not enough to guarantee admissibility.

Theorem 1.3. RE $-$ cf$(\alpha) = \omega \not\Rightarrow E(\alpha)$ is $\Sigma_1$-admissible.

proof. Begin with $E(\aleph_1)$ (which is not $\Sigma_1$-admissible) and define the following $\varkappa_r$-sequence:

$$\varkappa_r(0) = \varkappa_r \quad ;$$
$$\varkappa_r(n+1) = \varkappa_r^{\,\varkappa_r(n)}.$$

Now consider $\{x \mid x \in E(\aleph_1)$ and $x \leq_E \varkappa_r(n)$ for some $n \in \omega\} = M$. Let $\bar{\bar{M}}$ be the Mostouski collapse of $M$, then $\bar{\bar{M}}$ is E-closed and satisfirs the Moschovakis Phenomenon (use the MP in $E(\aleph_1)$ and the definition of $\varkappa_r$) and $\bar{\bar{M}}$ is an E-closure of one of its elements.

But $\overline{M}$ has an $\omega$-sequence of $\varkappa_r$'s. Let $\alpha = (\aleph_1)_{\overline{M}}$ and let

$$R = \{x < \alpha \mid x \text{ is the index for an ordinal } \beta \text{ such that } \beta = \varkappa_r^a \text{ for some } a < \alpha\}.$$

$R$ is RE and unbounded in $\alpha$ and clearly of order type $\omega$. Thus $\overline{M}$ is not $\Sigma_1$-admissible, while over $\overline{M}$ RE - cf$(\alpha) = \omega$, where $\alpha = (\aleph_1)_{\overline{M}}$.

§ 2. Dynamic Selection.

We shall give a dynamic proof of the following theorem:

Let $\alpha$ be the greatest cardinal in $E(\alpha)$ and let $\gamma$ be the $E(\alpha)$-cofinality of $\alpha$. Then we have uniform selection for RE subsets of any $\delta < \gamma$.

As it stands, the theorem was proven by Kirousis [1978], but the 'dynamic' proof we shall give can be relativized, whereas Kirousis made use of a Skolem Hull - collapsing argument. A similar proof using a collapsing argument was given by Normann [1979] for the case $\gamma = \alpha$, i.e. $\alpha$ is a regular cardinal in $E(\alpha)$. We now give the dynamic proof.

Let $\delta$ be fixed as in the theorem and let $f$ be a $\delta$-sequence of computations. Let $R$ be the Moschovakis [1967] subcomputation relation which is RE and, finally, let $R_\beta$ denote the $\beta^{\underline{th}}$ approximation to $R$. The relation $R$ is such that for a given computation, the set of immediate subcomputations can uniformly be indexed by a finite set or by $\alpha$ (the case of an

$\alpha$-branching). In the case of composition we let the innermost computation be the leftmost one. If this one is convergent, then we know the other subcomputations.

Following Harrington-MacQueen [1976] we let

$$\min(f) = \inf\{\|f(y)\| : y < \delta\}, \text{ where}$$

$\|\cdot\|$ denotes the function giving the height of a computation, if convergent, and equals $\infty$ otherwise. If $\min(f) < \infty$, i.e. one of the $f(y)$'s is convergent, we shall show that $\min(f)$ is uniformly recursive in $f$ for $f \in E(\alpha)$. The situation $\min(f) < \infty$ corresponds to the non-emptiness of the associated RE subset of $\delta$ and, thus, we have shown selection over $\delta$.

The proof proceeds by transfinite induction on $\min(f)$. An application of the recursion theorem yields the required uniformity.

The relation $\min(f) = 0$ is recursive, so assume that m $\min(f) > 0$ and that we have computed $\min(g)$ for all $g$ such that $\min(g) < \min(f)$.

If $\min(f) > \beta$ (which is recursive in $\beta$) we let

$$g_\beta(y) = \text{leftmost subcomputation } z \text{ of } f(y) \text{ such that } \|z\| \geq \beta \ ;$$

and otherwise we let $g_\beta = f$. Clearly $g_\beta$ is recursive in $f, \beta$ and if $\min(f) > \beta$, then

$$\beta \leq \min(g_\beta) < \min(f).$$

Let $\tau$ be a recursive function defined by:

$$\tau(0) = 1 \ ;$$

$$\tau(\lambda) = \sup\{\tau(\beta) \mid \beta < \lambda\} \text{ if } \lambda \text{ is}$$

a limit ordinal;

$$\tau(\beta+1) = \min(g_{\tau(\beta)+1}).$$

Claim. $\tau(\alpha) \geq \min(f)$.

proof (Claim) Otherwise for each $\beta < \alpha$ let $h_\beta = g_{\tau(\beta)+1}$, then if $\beta_1 < \beta_2$, there is a $y < \delta$ such that

$$h_{\beta_1}(y) < h_{\beta_2}(y). \quad \text{Let} \quad \beta_y = h_\beta(y),$$

then if for some $y$, $\{\beta_y : \beta < \alpha\}$ is unbounded, we have $\|f(y)\| \leq \tau(\alpha)$, so this cannot be the case. Let $\beta_y^* = \sup\{\beta_y | \beta < \alpha\}$. Since

$\delta < \gamma = cf^{E(\alpha)}(\alpha)$, we have that

$\sigma = \sup\{\beta_y^* | y < \delta\} < \alpha$. But for each $\beta < \alpha$

there is one minimal $y$ such that $(\beta+1)_y > \beta_y$. This gives a one-to-one map of $\alpha$ into $\delta \times \sigma$, which is impossible and gives the claim.

Since $\tau(\alpha)$ is recursive, we have computed $\min(f)$ from $f$ giving the theorem.

Corollary 2.0. We have selection over $\gamma = cf^{E(\alpha)}(\alpha)$ if and only if we have selection over $\alpha$.

proof. Selection over $\alpha$ clearly implies selection over $\gamma$. The other direction follows from the theorem and the dynamic proof of selection due to Sacks-Slaman (Theorem 2.8 in Slaman [1981]) which inspired this proof.

Now assume that $E(\alpha)$ is not $\Sigma_1$-admissible and, hence, we do not have selection over $\alpha$. The above corollary tells us we do not have selection over $\gamma$, however the theorem tells us:

Corollary 2.1. Let $\delta < \gamma$, $C \subseteq \delta$ be RE, then $C \in E(\alpha)$.

proof. Since we have selection over $\delta$, it follows that

$$\sup\{\kappa_o^y \mid y < \delta\} < \kappa \quad \text{and}$$

$C$ can be definedthis level in $E(\alpha)$.

Corollary 2.2. (Further Reflection) Let $\delta, C$ be as above, then

(a) $\kappa_o^{C,\delta} < \kappa_r^{\delta}$ ;

(b) if $B \subseteq E(\alpha)$ is RE and $B(C)$ holds, then there exists a $\delta$-recursive $\beta$ such that $B(C_\delta)$ holds.

proof. immediate.

Corollary 2.3. Suppose $2^\omega = \kappa$, $\kappa$ is a regular cardinal and there is a well-ordering of $2^\omega$ of height $\kappa$ recursive in ${}^4\mathbb{E}$ and a real. Then the extended plus one hypothesis is true at the type 3 level.

This last corollary was pointed out to us by T. Slaman. The extended plus-one hypothesis (for reals) states: if $F$ is a normal type $n+2$ object and $n \geq 1$, then there exists a normal type 3 object $G$ such that

$$\tfrac{1}{2}\,sc(G) = \tfrac{1}{2}\,sc(F) \text{ , where}$$

$\frac{1}{2}\,sc(F)$ is the collection of sets of reals recursive in $F$ and some real.

For background and further results on the extended plus-one hypothesis see Sacks [1977] or Slaman [1981].

## § 3. Applications: co-RE Cofinality, Effective Covering and Uniform Computation of Cofinality.

We turn first to an application of the above selection result which will yield a covering property for many co-RE sets 'preserving cofinality' and characterize what will call co-RE cofinality. Let $\alpha$ be an ordinal and consider again $E(\alpha) = L_\kappa$ for some $\kappa > \alpha$. Without loss of generality we assume $\alpha$ is the greatest cardinal in $L_\kappa$ and we let $\gamma = \mathrm{cf}^{L_\kappa}(\alpha)$.

<u>Definition.</u> Let $\beta \leq \kappa$ and define the co-RE cofinality of $\beta$ by:

$$\text{co-RE}-\mathrm{cf}(\beta) = \text{least } \delta \text{ such that there is a co-RE subset } A \text{ of } \beta \text{ of order type } \delta \text{ and unbounded in } \beta.$$

<u>Lemma 3.0.</u> $\text{co-Re}-\mathrm{cf}(\alpha) = \text{co-RE}-\mathrm{cf}(\gamma)$.

<u>proof.</u> Let $f : \gamma \to \alpha$ be increasing and witness that $\mathrm{cf}^{L_\kappa}(\alpha) = \gamma$.

$\leq$: If $A \subseteq \gamma$ is co-RE and of order type $\delta$ then $A_f = \{f(y) | y \in A\}$ is the same order type through $\alpha$. If $A$ is unbounded in $\gamma$, then $A_f$ is unbounded in $\alpha$.

$\geq$: Let $A \subseteq \alpha$ be co-RE, unbounded and of order type $\delta$. Let $y \in A^*$, if there exists $z \in [f(y), f(y+1)\rangle \cap A$. The RE sets are closed under the quantifiers $\forall z \in u$, so the co-RE sets are closed under $\exists z \in u$. Thus $A^*$ is co-RE and clearly unbounded in $\gamma$. In addition $\mathrm{o.t.}(A^*) \leq \mathrm{o.t.}(A)$.

We shall show that $\text{co-RE}-\mathrm{cf}(\gamma) = \gamma$. By the above selection theorem, $\beta < \gamma$ implies that the RE predicates are uniformly closed under $\exists y < \beta$ and, in addition, that

$$L_\varkappa \cap WF(\beta) \in L_\varkappa, \text{ where } WF(\beta)$$

denotes the set of well-founded relations as $\beta \times \beta$ (the latter cannot in general be relativized).

Theorem 3.1. co-RE - cf$(\gamma) = \gamma$.

proof. Let $A \subseteq \gamma$ be co-RE, cofinal in $\gamma$ of order type $\beta$. Let $A_\delta$ be the $\delta^{\text{th}}$ approximation to $A$ from the outside, i.e.

$$A_\delta = \{y \mid L_\delta \not\models y \notin A\}.$$

We will show that there is a recursive $\delta$ such that o.t$(A) =$ o.t.$(A_\delta)$

Let $y < \gamma$, then o.t.$(A \cap y) < \beta$ and by Further Reflection applied to $cA$, there is a $\delta$ recursive in $y$ such that

$$\text{o.t.}(A_\delta \cap y) < \beta .$$

Using this we construct a recursive increasing function $g : \gamma \to \varkappa$ such that

$$\forall y < \gamma \ (\text{o.t.}(Ag(y) \cap y) < \beta).$$

Let $\delta = \sup\{g(y) \mid y < \gamma\}$, then $\delta$ is recursive so let $C = A_\delta$. Thus $C$ is recursive and $A \subseteq C$. If o.t.$(C) > \beta$, then there exists a $y < \gamma$ such that o.t.$(C \cap y) = \beta$. But $C \cap y \subseteq A_{g(y)} \cap y$ since $g(y) < \delta$. Since o.t.$(A_{g(y)} \cap y) < \beta$, we have a contradiction.

Corollary 3.2. (Covering Property) Any co-RE subset $A$ of $\gamma$ can be covered by a REC set of the same order type.

The corollary is proven in the proof of the theorem and we used the ordinal $\beta$ as a parameter. This lack of uniformity makes extension of the result in the corollary to ordinals other than $\gamma$ difficult, however we offer:

Problem. Is there a bounded co-RE set that cannot be covered by a REC set of the same order type?

If $L_\varkappa$ is $\Sigma_1$-admissible, then co-RE - cf($\varkappa$) = $\omega$ (recall that $L_\varkappa = E(\alpha)$), but the converse is not true.

As far as the questions of section § 1 go these results show that

$$\text{co-RE} - \text{cf}(\alpha) = \omega \Rightarrow E(\alpha) \text{ is } \Sigma_1\text{-admissible, however}$$

$$E(\alpha)\ \Sigma_1\text{-admissible} \not\Rightarrow \text{co-RE} - \text{cf}(\alpha) = \omega.$$

Together with the results of § 2 this shows that there is no natural cofinality-assumption that will characterize when $E(\alpha)$ is admissible, the best seems to be the one implicit in the lack of certain Moschovakis Witnesses.

Our next application makes clear the interplay between selection and singularities.

Theorem 3.3. Let $\alpha < \beta$ be ordinals such that $\text{cf}(\beta) \leq \alpha$ by some function $f$ recursive in $\alpha,\beta$ and some $\delta < \alpha$. Then $\text{cf}(\beta) \leq \alpha$ by some function recursive in $\alpha,\beta$.

proof. let $g\colon \alpha \to \beta$ be a list of 'computation tuples' over $\beta$ such that $(\exists\delta < \alpha)[g(\delta)\downarrow]$. The intuition here is that we attempt to carry out a search for the $\delta < \alpha$ in question and we either compute it effectively, and hence the witness to $\text{cf}(\beta) \leq \alpha$, or we don't and in so doing (not doing) obtain a witness to $\text{cf}(\beta) \leq \alpha$.

Let $\min(g) = \min\{\|g(\delta)\| \mid \delta < \alpha\}$.

By the selection theorem in section 2: if $E(\beta) \models cf(\beta) \geq \alpha$, we know that $\min(g)$ is computable by some recursive function $M(g)$. In general it is sufficient for $M(g)$ to be defined that $\min(g)$ exists. If $M(g) < \min(g)$ this means that we have

$$E_{M(g)+1}(\alpha) \models cf(\beta) \leq \alpha; \quad \text{where for} \quad \gamma < OR \cap E(\alpha)$$

$E_\gamma(\alpha) = \{x \in E(\alpha) | x$ **computed by a** computation of height $< \gamma\}$. Now let $g(\delta)$ be an index for $f$ recursive in $\delta, \alpha, \beta$ witnessing that $cf(\beta) \leq \alpha$. Since $\min(g)$ exists we have that the selection algorithm $M(g)$ satisfies $M(g)\downarrow$.

If $\min(g) = M(g)$ we have computed the level at which the cofinality map is constructed. If $M(g) < \min(g)$, this is because we know at that ordinal that $cf(\beta) \leq \alpha$. Thus in both cases we can find from $M(g)$ an $f$ collapsing the cofinality of $\beta$ below $\alpha+1$.

If $L_\varkappa = E(\alpha)$ then for all $\gamma$ such that $\alpha < \gamma < \varkappa$ we can find effectively in $\alpha, \gamma$ a map in $L_\varkappa$ witnessing

$$\bar{\bar{\gamma}}^{L_\varkappa} = \bar{\bar{\alpha}}^{L_\varkappa} .$$

The above theorem will enable us to do this in many more cases. Suppose $L_\varkappa$ is E-closed and has a greatest cardinal $(gc(\varkappa))$.

<u>Coraollary 3.4</u>. If $\gamma > gc(\varkappa)$, let $f_\gamma$ be the least (in the sense of $<_L$) collapse of $\gamma$ to $gc(\varkappa)$. <u>If</u> for some $a, \gamma_0 < \varkappa$ we have that

(*) $$(\forall_\gamma > \gamma_0)(\exists z < gc(\varkappa))[f_\gamma \leq_E a, \gamma_0, gc(\varkappa), \gamma, z] ,$$

then the function $\gamma \to f_\gamma$ is uniformly computable in $\gamma_0, a, gc(\kappa)$ and a $gc(\kappa)$-enumeration of $\gamma_0$.

proof. We proceed by induction on $\gamma \geq \gamma_0$.

$\gamma = \gamma_0$ is trivial. If $\gamma > \gamma_0$, let $a_\gamma$ be so large that all $\gamma' < \gamma$ are collapsed to $gc(\kappa)$ by level $\alpha_\gamma$. Let $\alpha \geq \alpha_\gamma$ such that:

$$\text{if } L_{\alpha_\gamma} \models \bar{\bar{\gamma}} > gc(\kappa), \text{ then}$$

$$L_\alpha \models \gamma = (gc(\kappa))^+, \text{ where}$$

$\tau^+$ is the successor cardinal of $\tau$. By the theorem there is an $\alpha$ recursive in $\gamma, a, \gamma_0, gc(\kappa)$ and the collapse of $\gamma_0$ such that

$$L_\alpha \models cf(\gamma) \leq gc(\kappa).$$

But a successor cardinal is regular, so this singularity will demonstrate that $\bar{\bar{\gamma}} = gc(\kappa)$ and the collapsing map can be computed.

Corollary 3.4 can be used to show that under (*) we have

Corollary 3.5. Let $L_\kappa$ be E-closed and let $\alpha = gc(L_\kappa)$. Assume that $L_\kappa \models (*)$. Then the following are equivalent

(i) $L_\kappa$ is RE in an element of $L_\kappa$

(ii) Both $L_\kappa \cap (\alpha)$ and $\kappa$ are RE in an element of $L_\kappa$.

Remark. Using forcing-methods of Sacks [198?] we may show that if * holds, then $L_\kappa$ is not R.E.

## § 4. E-Recursive Functions and Inductive Definability.

In this section we shall give a treatment of monotone inductive definitions using methods from Girard's $\beta$-logic [198?], but without introducing $\beta$-logic and its proof theory. Masseron [1980] has used the proof theory of $\beta$-logic to show that every total $\omega_1^{CK}$-recursive function on $\omega_1^{CK}$ is dominated by a primitive recursive dilator on infinite arguments. As a corollary we give a proof of Van de Wiele's theorem:

If $F : V \to V$ is total uniformly $\Sigma_1$-definable over every admissible set, then $F$ is E-recursive.

The converse for E-recursive functions (lightface) is immediate. Slaman has given an alternate proof, but his proof uses the theory of reflection in E-recursion, whereas we will require only familiarity with the generating schemata of E-recursion.

Like the completeness theorem for $\beta$-logic this proof is based on the Henkin-type construction of term models, otherwise the proof is elementary. For each set $x$ let $\Gamma_x$ be a uniformly $\Delta_0(x)$ positive inductive definition on $x$. Let $\leq_x$ denote the stage comparison relation on $x$. The following lemma is valid for monotone inductive definitions in general.

<u>Lemma 4.0.</u> Let $Y \subseteq x$, $\leq$ be a relation on $y$ such that

(i) $\Gamma(Y) = Y$ ; and

(ii) for each $y \in Y$

$\{y' | y' \leq y\} = \Gamma(\{y' | y' < y\})$ , then

$\Gamma_x^\infty \leq Y$ and $\leq_x$ is the well-founded initial segment of $\leq$ ($\Gamma_x^\infty$ is the least fixed-point of $\Gamma_x$).

For each $x$, let $\tau_x$ be the closure ordinal of $\Gamma_x$ and let $\varphi$ be a $\Delta_o$-formula such that

$$\forall x\, \exists \gamma < \tau_x\, \varphi(x, \Gamma_x^{\gamma+1}) .$$

<u>Theorem 4.1.</u> There is an E-recursive function $G$ such that

$$\forall \alpha\, \forall x(\mathrm{rank}(x) \leq \alpha \Rightarrow \exists_\gamma \leq \min(G(\alpha), \tau_x) \varphi(x, \Gamma_x^{\gamma+1})) ;$$

<u>Definition.</u> Let $T = T_{\Gamma,\varphi}$ be the following first order theory:

| | |
|---|---|
| unary predicates | $\underline{x}$, $\underline{Y}$, $\underline{ON}$ |
| binary predicates | $\underline{P}$ (for $\leq_x$) and $\underline{\in}$ |
| unary function | R (for rank) |
| constants | $\underline{c_o}$, $\underline{c_1}$, ... |

Take standard axioms like regularity, extensionality, etc. together with:

(i) $\underline{Y} = \Gamma(\underline{Y})$ ;

(ii) $\varphi(x, \{y | \underline{P}(y, \underline{c_o})\}) \to \forall z \in \underline{Y}(\varphi(x, \{y | \underline{P}(y,z)\}) \to \underline{P}(\underline{c_o}, z))$;

(iii) $\underline{P}(\underline{c}_{i+1}, \underline{c_i}) \wedge \neg \underline{P}(\underline{c_i}, \underline{c}_{i+1})$ ; and

(iv) $\forall z \in \underline{Y}(\{y | \underline{P}(y,z)\} = \Gamma(\{y | \underline{P}(y,z) \wedge \neg \underline{P}(z,y)\}))$ .

<u>Definition</u> (a) Let $T_n$ denote the part of $T$ that does not contain any $\underline{c_i}$ for $i \geq n$ ;

(b) Let $T^*$, $T_n^*$ denote the respective Henkin-extensions ;

(c) Let $\underline{e_o}$, $\underline{e_1}$ ,... be a recursive enumeration of the terms of $T^*$ such that $\forall i(\underline{e_i} \in T_i^*)$ .

Now if $f : \mathbb{N} \to ON$, let $T^f$ be $T^*$ extended with the following axioms:

$$\{R(\underline{e_i}) \leq R(\underline{e_j}) \mid f(i) \leq f(j)\}.$$

Lemma 4.2. Let $f : \mathbb{N} \to ON$ and $T^f$ be as above, then $T^f$ is inconsistent.

proof. Assume $T^f$ is consistent for a contradiction and let $T^f$ denote a consistent completion of $\overline{T}^f$. The term model for $\overline{T}^f$ will then be a model of $T$ and since the rank-relation is well-founded, the model will be isomorphic to a set $z$ where $\underline{x}$ is interpreted as a subset of $z$. Let $\gamma < \tau_x$ be such that $\varphi(x, \Gamma_x^{\gamma+1})$. By lemma the interpretation $c_0$ of $\underline{c_0}$ must be in $\Gamma_x^{\infty}$ and have rank $\leq \gamma+1$. But then interpretations of $\underline{c_i}$ will form an $\leq$-infinite descending sequence, which is absurd.

If $\sigma$ is a finite sequence of ordinals we define $T^\sigma$ as an extension of $T^*_{lh(\sigma)}$ as before. Thus we have

$$\forall f : \mathbb{N} \to OR\, \exists r \in \mathbb{N}[T^{\overline{f(n)}} \text{ is inconsistent}].$$

Definition. Let $\sigma$ be a sequence of ordinals of length $n$, then we say $\sigma$ is good if we cannot prove a contradiction from $T^\sigma$ using a proof of length $\leq n$ and at most the $n$ first axioms of $T^\sigma$ (in some uniform enumeration of $T^f$'s).

For $\alpha \in OR$ we let

$$S_\alpha = \{\sigma \mid \sigma \text{ is good and } \forall i < lh(\sigma)(\sigma(i) < \alpha)\}$$

and set $G(\alpha)$ = height of $S_\alpha$. Then $G$ is E-recursive since we can uniformly compute the height of any well-founded relation in E-recursion.

Lemma 4.3. Let $\mathrm{rank}(x) \leq \alpha$, then we can find $\gamma \leq G(\alpha)$ such that $\varphi(x, \Gamma_x^{\gamma+1})$ holds.

proof. Fix $x$ and let $\gamma$ be minimal such that $\varphi(x, \Gamma_x^{\gamma+1})$ and choose $y \in \Gamma_x^{\gamma+1} - \Gamma_x^{\gamma}$. Let $p$ denote the ordinal norm function on $\Gamma_x^{\infty}$ induced by $\Gamma_x$. Then we have $p(y) = \gamma$. Assume that $y_0, \ldots, y_{n-1}$ is a sequence from $\Gamma_x^{\infty}$ such that $y_0 = y$ and $p(y_i) < p(y_{i-1})$ for $1 \leq i < n$.

We shall construct a model for $T_n$ using $TC(x)$ as the domain, $x$ for $\underline{x}$, $\Gamma_x^{\infty}$ for $\underline{Y}$, $\leq_x$ for $\underline{P}$ and $y_0, \ldots, y_{n-1}$ for $\underline{c_0}, \ldots, \underline{c_{n-1}}$. This model can be extended to a model for $T_n^*$ since $T_n^*$ is a conservative extension of $T_n$ and we do not change the domain. For $i < n$ let $\sigma(i) = \mathrm{rank}(e_i)$ ($e_i$ is the interpretation of $\underline{e_i}$). Note that if we extend $\vec{g}$ in a consistent way, then we may extend $\sigma$ (i.e. we cannot choose $\sigma$ such that it is inconsistent with the construction based on extensions of $\vec{y}$).

If $\alpha = \mathrm{rank}(x)$, then $\mathrm{rank}(e_i) < \alpha$ by our choice of domain as $TC(x)$) and so $\sigma \in S_\alpha$. By induction on $p(y_{n-1})$ we can show that $p(y_{n-1}) \leq \|\sigma\|_{S_\alpha}$. The induction is trivial by the above remark on the consistency-considerations and, hence, the lemma follows. The theorem follows from the lemma.

Remark. The theory $T$ in the proof asserts that $x$ is a relation on a transitive set $y$; $\langle Y, P \rangle$ is the prewellordering induced by $\Gamma$ over $x$ and there is no $z \in \Gamma_x^{\infty}$ satisfying $\varphi$. If $T'$ is a primitive recursive theory in the language of set theory, then the same proof gives:

Corollary 4.4. Let $\Gamma$, $\varphi$ and $\tau_x$ be as above. If

$\forall x(x \models T' \Rightarrow \exists\gamma < \tau_x \varphi(x, \Gamma_x^{\gamma+1}))$ then there is an E-recursive function G such that

$$\forall x(x \models T' \Rightarrow \exists\gamma < \min\{\tau_x, G(\mathrm{rank}(x))\}\varphi(x,\Gamma_x^{\gamma+1}))$$

Examples of such theories are:

(i) x is transitive, infinite and closed under finite subsets;

(ii) x is rudimentarily closed.

Now if x is transitive, infinite and closed under finite subsets, then we have a notation system for the next admissible (HYP(x)) and that notation system is defined by a monotone inductive definition. If $\exists y \in \mathrm{HYP}(x)\varphi(x,y)$, then there is a $\Delta_0$ formula $\varphi'$ such that $\varphi'(x,\Gamma_x^{\gamma})$ for the least $\gamma$ such that $\exists y \in L_\gamma[x]\varphi(x,y)$ where $\Gamma$ defines that notation system.

Using this we have proven the following theorem of J. Van de Wiele:

Corollary 4.5. (Van de Wiele) Let $F: V \to V$ be uniformly $\Sigma_1$-definable and total over all admissible sets, then F is E-recursive.

proof follows immediately from the theorem and the above remarks on the inductive generation of HYP(x).

Note that we actually show that F is computable in a weaker system than E-recursion, since we use elementary functions together with the operator which computes the height of a well-founded relation.

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