EFFECTIVE COFINALITIES AND ADMISSIBILITY IN E-RECURSION

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§ 0 Introduction.

E-Recursion was introduced by D. Normann [1978] as a natural generalization of normal Kleene recursion in objects of finite type. Unless otherwise stated the E-closed sets we shall consider shall be of the form $E(\alpha)$ for some $\alpha \in \text{OT}$. In § 1 we introduce the RE$^1$ co-RE cofinality and show that the $\Sigma_1$-admissibility of $E(\alpha)$ implies that its greatest cardinal has RE$^1$ co-RE cofinality $\omega$. In addition we show that RE-cofinality $\omega$ does not imply admissibility.

Section § 2 is devoted to a dynamic proof of selection (i.e. $\gamma = \text{cf} E(\alpha)(\alpha)$ then we have uniform selection over RE subsets of any $\delta < \gamma$ on $E(\alpha)$), which can therefore be relativized. This selection theorem thus has among its corollaries the consistency of the extended plus one hypothesis at the type three level with $\neg\text{CH}$.

Applications of the proof of selection given in § 2 are presented in § 3. We show that if $\gamma$ is the cofinality of $\alpha$ in $E(\alpha)$, then the co-RE cofinality of $\gamma$ is $\gamma$. The proof of this gives rise to an effective covering property, namely, any co-RE subset of $\gamma$ can be covered by a REC set of the same order type. The final application makes clear the connection between selection and singularities. We show that for $\alpha < \beta$ such that $\text{cf}(\beta) \leq \alpha$
by a function \( f \) recursive in \( \alpha, \beta \) and some \( \delta < \alpha \), then \( \text{cf}(\beta) \leq \alpha \) by some \( f \) recursive in \( \alpha, \beta \).

The last section (§ 4) treats the interplay between monotone inductive definitions and \( E \)-recursive set functions using methods from Girard's \( \beta \)-logic [1987], without introducing \( \beta \)-logic or its proof theory. If \( \Delta_0 \varphi(x,') \) always has a solution in \( \Gamma_x^\omega \) (the least fixed point of monotone inductive \( \Gamma \) over \( x \)), then the function giving that solution is \( E \)-recursive in \( x \). As a corollary we have an elementary proof of a theorem of Van de Wiele [1981]:

If \( F : V \rightarrow V \) is uniformly \( \Sigma_1 \)-definable and total over all admissible sets, then \( F \) is \( E \)-recursive. Outside of § 4, \( \text{RE} \), \( \text{co-RE} \) etc. are the boldface notions.

§ 1. Effective Cofinalities.

Much attention has been given to various notions of definable cofinality, particularly in connection with priority arguments \( E \)-Recursion. We shall not attempt to give a complete picture and so the interested reader is directed to Griffor [1980], Sacks [1980] or Slaman [1981]. The first question we address here was asked by Sacks, namely, is there a cofinality condition on \( \alpha \) which characterizes when \( E(\alpha) \) is \( \Sigma_1 \)-admissible. The question was motivated by a result of Kirovis that: if \( E(\alpha) \models \text{cf}(\check{\alpha}) = \omega \), then \( E(\alpha) \) is \( \Sigma_1 \)-admissible. Thus an attractive conjecture was that: \( E(\alpha) \) is \( \Sigma_1 \)-admissible if and only if \( E(\alpha) \models \text{cf}(\check{\alpha}) = \omega \). However, Slaman noticed that if \( \gamma \) is the least ordinal where \( E(\gamma) \models \text{cf}(\check{\gamma}) > \omega \), then \( E(\gamma) \) is \( \Sigma_1 \)-admissible. If \( E(\alpha) \) is \( \Sigma_1 \)-admissible Sacks [1980] showed that there is a divergent
computation without a Moschovakis witness in \( E(\alpha) \). This witness induces an \( \omega \)-sequence through \( \bar{\alpha} \) and we will first analyse the level of definability of one such sequence.

**Definition.** Consider \( E(\alpha), \ \alpha \in \text{OR} \), and without loss of generality assume that \( \alpha \) is the greatest cardinal in \( E(\alpha) \). Define the \( \text{RE} \) join co-\( \text{RE} \) cofinality of \( \alpha \) as:

\[
\text{RE} \wedge \text{co-RE-} \text{cf}(\alpha) = \text{least } \tau \leq \alpha \text{ such that there exists an } R \leq \alpha \text{ of order type } \tau \text{ unbounded in } \alpha \text{ and } R \text{ is } \text{RE} \wedge \text{co-RE}, \text{i.e. } R \text{ is the intersection of an } \text{RE} \text{ and a co-RE set.}
\]

**Theorem 1.** Suppose \( E(\alpha) \) is \( \Sigma_1 \)-admissible, then

\[
\text{RE} \wedge \text{co-RE-} \text{cf}(\alpha) = \omega.
\]

**proof.** As above we assume that \( \alpha \) is the greatest cardinal in \( E(\alpha) \) (which is \( L_\kappa \) for some \( \kappa > \alpha \)). If \( e \in w, a \in E(\alpha) \), then associated with the computation tuple \( \langle e, a \rangle \) is the tree of subcomputations \( T_{\langle e, a \rangle} \) (which is recursive in \( \langle e, a \rangle \) if \( \{e\}(a)^\downarrow \), but is in general only \( \text{RE} \) in \( \langle e, a \rangle \)). Assume that \( E(\alpha) \) is \( \Sigma_1 \)-admissible.

By Sacks [1980] there exists an \( e \in w \) and \( a \in E(\alpha) \) such that \( T_{\langle e, a \rangle} \) is not well-founded, but

\[
L_\kappa \models 'T_{\langle e, a \rangle} \text{ is well-founded}'.
\]

**Claim 1.** The leftmost path in \( T_{\langle e, a \rangle} \) is in \( \text{RE} \wedge \text{co-RE} \)

**proof.** We say that \( \sigma \) is on the leftmost path if

\[
\begin{align*}
(i) & \quad \sigma \in T_{\langle e, a \rangle} \quad (\text{RE}) \\
(ii) & \quad \sigma \uparrow \quad (\text{co-RE})
\end{align*}
\]
(iii) If $\tau < \sigma$ in the lexicographical ordering and $n$ is minimal such that $\tau(n) < \sigma(n)$, then $\tau(n+1) \uparrow$ (RE). This proves claim 1.

Now assume that we have an effective coding of all finite sequences from $\alpha$ by $\sigma$ such that

$$\langle \sigma \wedge \tau \rangle > \langle \sigma \rangle,$$

where $\tau \neq \langle \rangle$.

Let $\langle \beta_1, \ldots, \beta_n \rangle \in A$ if $\beta_i$ is the index for the $i$th sequence of the leftmost path through $T\langle e, a \rangle$. Then $A$ is the intersection of an RE set $A_1$ and a co-RE set $A_2$.

Claim 2. $A$ is unbounded in $\alpha$.

Proof. If $A$ is bounded by $\lambda < \alpha$, then use standard properties of the $\Sigma_1$-projectum on admissible ordinals to show that $A_1 \cap \lambda \in E(\alpha), A_2 \cap \lambda \in E(\alpha)$ and so $A \in E(\alpha)$, which is impossible.

This completes the proof of the theorem.

Definition. With $E(\alpha)$ as above let

(i) $REC - cf(\alpha) = \mu \tau \leq \alpha$ such that there exists REC $R \subseteq \alpha$ of order type $\tau$ unbounded in $\alpha$;

(ii) $RE - cf(\alpha) = \mu \tau \leq \alpha$ such that there exists RE $R \subseteq \alpha$ of order type $\tau$ unbounded in $\alpha$.

As one might expect the recursive cofinality is no stronger, on ordinals less than $\kappa$, than the cofinality in the sense of $E(\alpha)$.

Proposition 1.1. If $\gamma < \kappa$, then

$$REC - cf(\gamma) = \sup \{ L_\alpha(\gamma) \}.$$
proof <: let \( f : \text{cf}(\kappa(y)) \to \gamma \), \( f \in L_\kappa \) witness \( \text{cf}(\kappa(y)) \) and without loss of generality we may assume that \( f \) is strictly increasing. Let \( R = \text{im}(f) \), then \( R \) witnesses 

\[
\text{REC} - \text{cf}(\gamma) \leq \text{cf}(\kappa(y)).
\]

\[
\geq: \text{let } R \subseteq \gamma \text{ witness the } \text{REC}-\text{cf}(\gamma) = \tau, \text{ then } R \in L_\kappa \text{ by the bounding principle and the function } f : \tau \to \gamma \text{ given by } \sigma < \tau.
\]

\[
f(\sigma) = \sigma^{\text{th}} \text{ element of } R \]

is in \( L_\kappa \) and witnesses \( \text{cf}(\kappa(y)) \leq \text{REC}-\text{cf}(\gamma) \).

Corollary 1.2. If \( \text{REC}-\text{cf}(\alpha) = \omega \), then \( E(\alpha) \) is \( \Sigma_1 \)-admissible.

proof. Use the proposition and the selection-theorem of Kirousis [1978] stating

\[
E(\alpha) \models \text{cf}(\tilde{\alpha}) = \omega \implies E(\alpha) \text{ is } \Sigma_1 \text{-admissible.}
\]

We shall see now that \( \text{RE}-\text{cf}(\alpha) = \omega \) is not enough to guarantee admissibility.

Theorem 1.3. \( \text{RE}-\text{cf}(\alpha) = \omega \not\implies E(\alpha) \) is \( \Sigma_1 \)-admissible.

proof. Begin with \( E(\kappa(y)) \) (which is not \( \Sigma_1 \)-admissible) and define the following \( \kappa(x) \)-sequence:

\[
\kappa_\tau(0) = \kappa; \quad \kappa_\tau(n+1) = \kappa_\tau(n)^{\kappa_\tau(n)}. \]

Now consider 

\[
\{ x \mid x \in E(\kappa(y)) \text{ and } x \leq \kappa_\tau(n) \text{ for some } n \in \omega \} = M. \]

Let \( \bar{M} \) be the Mostowski collapse of \( M \), then \( \bar{M} \) is \( E \)-closed and satisfies the Moschovakis Phenomenon (use the MP in \( E(\kappa(y)) \) and the definition of \( \kappa_\tau(n) \)) and \( \bar{M} \) is an \( E \)-closure of one of its elements.
But \( M \) has an \( \omega \)-sequence of \( \kappa_r \)'s. Let \( \alpha = \langle \kappa_r \rangle_M \) and let

\[
R = \{ x < \alpha | x \text{ is the index for an ordinal } \beta \text{ such that } \beta = \kappa_r^\alpha \text{ for some } \alpha < \alpha \}.
\]

\( R \) is RE and unbounded in \( \alpha \) and clearly of order type \( \omega \). Thus \( M \) is not \( \Sigma_1 \)-admissible, while over \( M \) \( \text{RE-cf}(\alpha) = \omega \), where \( \alpha = \langle \kappa_r \rangle_M \).

§ 2. Dynamic Selection.

We shall give a dynamic proof of the following theorem:

Let \( \alpha \) be the greatest cardinal in \( E(\alpha) \) and let \( \gamma \) be the \( E(\alpha) \)-cofinality of \( \alpha \). Then we have uniform selection for \( \text{RE} \) subsets of any \( \delta < \gamma \).

As it stands, the theorem was proven by Kiousis [1978], but the 'dynamic' proof we shall give can be relativized, whereas Kiousis made use of a Skolem Hull - collapsing argument. A similar proof using a collapsing argument was given by Normann [1979] for the case \( \gamma = \alpha \), i.e. \( \alpha \) is a regular cardinal in \( E(\alpha) \). We now give the dynamic proof.

Let \( \delta \) be fixed as in the theorem and let \( f \) be a \( \delta \)-sequence of computations. Let \( R \) be the Moschovakis [1967] sub-computation relation which is \( \text{RE} \) and, finally, let \( R_\delta \) denote the \( \delta \)-th approximation to \( R \). The relation \( R \) is such that for a given computation, the set of immediate subcomputations can uniformly be indexed by a finite set or by \( \alpha \) (the case of an
α-branching). In the case of composition we let the innermost computation be the leftmost one. If this one is convergent, then we know the other subcomputations.

Following Harrington-MacQueen [1976] we let

\[ \min(f) = \inf\{\|f(y)\| : y < \delta\} \]

where \(\|\cdot\|\) denotes the function giving the height of a computation, if convergent, and equals \(\infty\) otherwise. If \(\min(f) < \infty\), i.e., one of the \(f(y)\)'s is convergent, we shall show that \(\min(f)\) is uniformly recursive in \(f\) for \(f \in E^\alpha\). The situation \(\min(f) < \infty\) corresponds to the non-emptiness of the associated RE subset of \(\delta\) and, thus, we have shown selection over \(\delta\).

The proof proceeds by transfinite induction on \(\min(f)\). An application of the recursion theorem yields the required uniformity.

The relation \(\min(f) = 0\) is recursive, so assume that \(\min(f) > 0\) and that we have computed \(\min(g)\) for all \(g\) such that \(\min(g) < \min(f)\).

If \(\min(f) > \beta\) (which is recursive in \(\beta\)) we let

\[ g_{\beta}(y) = \text{leftmost subcomputation } z \text{ of } f(y) \text{ such that } \|z\| \geq \beta ; \]

and otherwise we let \(g_{\beta} = f\). Clearly \(g_{\beta}\) is recursive in \(f, \beta\) and if \(\min(f) > \beta\), then

\[ \beta \leq \min(g_{\beta}) < \min(f). \]

Let \(\tau\) be a recursive function defined by:

\[ \tau(0) = 1 ; \]

\[ \tau(\lambda) = \sup\{\tau(\beta) : \beta < \lambda\} \text{ if } \lambda \text{ is} \]
a limit ordinal;

$$\tau(\beta+1) = \min(\xi_{\tau}(\beta)+1).$$

Claim. \(\tau(\alpha) \geq \min(f).\)

**Proof (Claim)** Otherwise for each \(\beta < \alpha\) let \(h_\beta = \xi_{\tau}(\beta)+1\), then if \(\beta_1 < \beta_2\), there is a \(y < \delta\) such that

$$h_{\beta_1}(y) < h_{\beta_2}(y).$$

Let \(\beta_y = h_\beta(y)\), then if for some \(y\), \(\{\beta_y : 3 < \alpha\}\) is unbounded, we have

\[\|f(y)\| \leq \tau(\alpha),\]

so this cannot be the case. Let \(\beta_0 = \sup\{\beta_y | \beta < \alpha\}\).

Since \(\delta < \gamma = \text{cf}^{E(\alpha)}(\alpha)\), we have that

$$\sigma = \sup\{\beta_y | y < \delta\} < \alpha.$$ But for each \(\beta < \alpha\)

there is one minimal \(y\) such that \((\beta+1)_y > \beta_y\). This gives a one-to-one map of \(\alpha\) into \(\delta \times \sigma\), which is impossible and gives the claim.

Since \(\tau(\alpha)\) is recursive, we have computed \(\min(f)\) from \(f\) giving the theorem.

**Corollary 2.6.** We have selection over \(\gamma = \text{cf}^{E(\alpha)}(\alpha)\) if and only if we have selection over \(\alpha\).

**Proof.** Selection over \(\alpha\) clearly implies selection over \(\gamma\). The other direction follows from the theorem and the dynamic proof of selection due to Sacks-Slaman (Theorem 2.8 in Slaman [1981]) which inspired this proof.

Now assume that \(E(\alpha)\) is not \(\Sigma_1\)-admissible and, hence, we do not have selection over \(\alpha\). The above corollary tells us we do
not have selection over $\gamma$, however the theorem tells us:

**Corollary 2.1.** Let $\delta < \gamma$, $C \subseteq \delta$ be RE, then $C \in E(\alpha)$.

**Proof.** Since we have selection over $\delta$, it follows that

$$\sup \{ \kappa^y | y < \delta \} < \kappa$$

and $C$ can be defined this level in $E(\alpha)$.

**Corollary 2.2.** (Further Reflection) Let $\delta, C$ be as above, then

(a) $\kappa^C, \delta < \kappa^C$;

(b) if $B \subseteq E(\alpha)$ is RE and $B(C)$ holds, then there exists a $\delta$-recursive $\beta$ such that $B(C_\delta)$ holds.

**Proof.** Immediate.

**Corollary 2.3.** Suppose $2^\omega = \kappa$, $\kappa$ is a regular cardinal and there is a well-ordering of $2^\omega$ of height $\kappa$ recursive in $\omega$, and a real. Then the extended plus-one hypothesis is true at the type 3 level.

This last corollary was pointed out to us by T. Slaman. The extended plus-one hypothesis (for reals) states: if $F$ is a normal type $n+2$ object and $n \geq 1$, then there exists a normal type 3 object $G$ such that

$$\frac{1}{2} sc(G) = \frac{1}{2} sc(F),$$

where $\frac{1}{2} sc(F)$ is the collection of sets of reals recursive in $F$ and some real.

For background and further results on the extended plus-one hypothesis see Sacks [1977] or Slaman [1981].
§ 3. Applications: co-RE Cofinality, Effective Covering and Uniform Computation of Cofinality.

We turn first to an application of the above selection result which will yield a covering property for many co-RE sets 'preserving cofinality' and characterize what will call co-RE cofinality. Let $\alpha$ be an ordinal and consider again $E(\alpha) = L_\kappa$ for some $\kappa > \alpha$. Without loss of generality we assume $\alpha$ is the greatest cardinal in $L_\kappa$ and we let $\gamma = \text{cf}^L_\kappa(\alpha)$.

**Definition.** Let $\beta \leq \kappa$ and define the co-RE cofinality of $\beta$ by:

\[
\text{co-RE-} \text{cf}(\beta) = \text{least } \delta \text{ such that there is a co-RE subset } A \text{ of } \beta \text{ of order type } \delta \text{ and unbounded in } \beta.
\]

**Lemma 3.0.** $\text{co-RE-} \text{cf}(\alpha) = \text{co-RE-} \text{cf}(\gamma)$.

**Proof.** Let $f : \gamma \to \alpha$ be increasing and witness that $\text{cf}^L_\kappa(\alpha) = \gamma$.

\[\leq:\] If $A \subseteq \gamma$ is co-RE and of order type $\delta$ then $A_f = \{f(y) | y \in A\}$ is the same order type through $\alpha$. If $A$ is unbounded in $\gamma$, then $A_f$ is unbounded in $\alpha$.

\[\geq:\] Let $A \subseteq \alpha$ be co-RE, unbounded and of order type $\delta$. Let $y \in A^*$, if there exists $z \in [f(y), f(y+1)) \cap A$. The RE sets are closed under the quantifiers $\forall z \in u$, so the co-RE sets are closed under $\exists z \in u$. Thus $A^*$ is co-RE and clearly unbounded in $\gamma$. In addition $o.t. (A^*) \subseteq o.t. (A)$.

We shall show that $\text{co-RE-} \text{cf}(\gamma) = \gamma$. By the above selection
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Theorem, \( \beta < \gamma \) implies that the RE predicates are uniformly closed under \( \exists \gamma < \beta \) and, in addition, that

\[ L_n \cap WF(\beta) \in L_\alpha, \]

where \( WF(\beta) \)
denotes the set of well-founded relations as \( \beta \times \beta \) (the latter cannot in general be relativized).

**Theorem 3.1.** \( \text{co-RE} - \text{cf}(\gamma) = \gamma. \)

**Proof.** Let \( A \subseteq \gamma \) be co-RE, cofinal in \( \gamma \) of order type \( \beta. \)

Let \( A_\delta \) be the \( \delta \)th approximation to \( A \) from the outside, i.e.

\[ A_\delta = \{ y | L_\delta \not\subseteq y \not\in A \}. \]

We will show that there is a recursive \( \delta \) such that \( o.t(A) = o.t(A_\delta). \)

Let \( y < \gamma \), then \( o.t(A \cap y) < \beta \) and by Further Reflection applied to \( cA \), there is a \( \delta \) recursive in \( y \) such that

\[ o.t(A_\delta \cap y) < \beta. \]

Using this we construct a recursive increasing function \( g : \gamma \to \kappa \)
such that

\[ \forall y < \gamma \ (o.t(Ag(y) \cap y) < \beta). \]

Let \( \delta = \sup\{ g(y) | y < \gamma \} \), then \( \delta \) is recursive so let \( C = A_\delta. \)

Thus \( C \) is recursive and \( A \subseteq C. \) If \( o.t.(C) > \delta, \) then there exists a \( y < \gamma \) such that \( o.t.(C \cap y) = \delta. \) But \( C \cap y \subseteq Ag(y) \cap y \) since \( g(y) < \delta. \) Since \( o.t.(A_g(y) \cap y) < \beta, \) we have a contradiction.

**Corollary 3.2.** (Covering Property) Any co-RE subset \( A \) of \( \gamma \)
can be covered by a REC set of the same order type.

The corollary is proven in the proof of the theorem and we
used the ordinal \( \beta \) as a parameter. This lack of uniformity
makes extension of the result in the corollary to ordinals other
than \( \gamma \) difficult, however we offer:

**Problem.** Is there a bounded co-RE set that cannot be covered
by a REC set of the same order type?

If \( L_\alpha \) is \( \Sigma_\alpha \)-admissible, then co-RE - cf(\( \alpha \)) = \( \omega \) (recall
that \( L_\alpha = E(\alpha) \)), but the converse is not true.

As far as the questions of section § 1 go these results show
that

\[
\text{co-RE - cf}(\alpha) = \omega \Rightarrow E(\alpha) \text{ is } \Sigma_\alpha \text{-admissible, however}
\]

\[
\text{E(\alpha) } \Sigma_\alpha \text{-admissible } \not\Rightarrow \text{co-RE - cf}(\alpha) = \omega.
\]

Together with the results of § 2 this shows that there is no
natural cofinality-assumption that will characterize when \( E(\alpha) \)
is admissible, the best seems to be the one implicit in the lack
of certain Moschovakis Witnesses.

Our next application makes clear the interplay between selec-
tion and singularities.

**Theorem 3.3.** Let \( \alpha < \beta \) be ordinals such that \( \text{cf}(\beta) \leq \alpha \) by some
function \( f \) recursive in \( \alpha, \beta \) and some \( \delta < \alpha \). Then \( \text{cf}(\beta) \leq \alpha 
\)
by some function recursive in \( \alpha, \beta \).

**Proof.** Let \( g: \alpha \rightarrow \beta \) be a list of 'computation tuples' over \( \beta
\)
such that \( (\exists \delta < \alpha)[g(\delta) \downarrow] \). The intuition here is that we attempt
to carry out a search for the \( \delta < \alpha \) in question and we either
compute it effectively, and hence the witness to \( \text{cf}(\beta) \leq \alpha \), or we
don't and in so doing (not doing) obtain a witness to \( \text{cf}(\beta) \leq \alpha \).
Let \( \min(g) = \min\{\|g(\delta)\| \mid \delta < \alpha\} \).

By the selection theorem in section 2: if \( E(\beta) \models \text{cf}(\beta) > \alpha \), we know that \( \min(g) \) is computable by some recursive function \( M(g) \). In general it is sufficient for \( M(g) \) to be defined that \( \min(g) \) exists. If \( M(g) < \min(g) \) this means that we have

\[ E_{M(g)+1}(\alpha) = \text{cf}(\beta) \leq \alpha; \text{ where for } \gamma < \text{OR} \cap E(\alpha) \]

\[ E_\gamma(\alpha) = \{x \in E(\alpha) \mid x \text{ computed by a computation of height } < \gamma\}. \]

Now let \( g(\delta) \) be an index for \( f \) recursive in \( \delta, \alpha, \beta \) witnessing that \( \text{cf}(\beta) \leq \alpha \). Since \( \min(g) \) exists we have that the selection algorithm \( M(g) \) satisfies \( M(g) \).

If \( \min(g) = M(g) \) we have computed the level at which the cofinality map is constructed. If \( M(g) < \min(g) \), this is because we know at that ordinal that \( \text{cf}(\beta) \leq \alpha \). Thus in both cases we can find from \( M(g) \) an \( f \) collapsing the cofinality of \( \beta \) below \( \alpha + 1 \).

If \( L_\kappa = E(\alpha) \) then for all \( \gamma \) such that \( \alpha < \gamma < \kappa \) we can find effectively in \( \alpha, \gamma \) a map in \( L_\kappa \) witnessing

\[ \frac{L_\kappa}{\gamma} = \frac{L_\kappa}{\alpha} \].

The above theorem will enable us to do this in many more cases.

Suppose \( L_\kappa \) is \( E \)-closed and has a greatest cardinal \( (\text{gc}(\kappa)) \).

**Corollary 3.4**. If \( \gamma > \text{gc}(\kappa) \), let \( f_\gamma \) be the least (in the sense of \( <_{L_\kappa} \)) collapse of \( \gamma \) to \( \text{gc}(\kappa) \). If for some \( a, \gamma_0 < \kappa \) we have that

\[(*) \quad (\forall \gamma > \gamma_0)(\exists z < \text{gc}(\kappa))[f_\gamma \leq a, \gamma_0, \text{gc}(\kappa), \gamma, z], \]

Then...
then the function $y \rightarrow f_y$ is uniformly computable in $\gamma_0, a, \text{gc}(\kappa)$ and a $\text{gc}(\kappa)$-enumeration of $\gamma_0$.

**proof.** We proceed by induction on $\gamma \leq \gamma_0$.

$\gamma = \gamma_0$ is trivial. If $\gamma > \gamma_0$, let $a_y$ be so large that all $\gamma' < \gamma$ are collapsed to $\text{gc}(\kappa)$ by level $\alpha_y$. Let $a \geq a_y$ such that:

if $L_a \models \gamma > \text{gc}(\kappa)$, then

$L_a \models \gamma = (\text{gc}(\kappa))^+$, where

$\tau^+$ is the successor cardinal of $\tau$. By the theorem there is an $\alpha$ recursive in $\gamma, a, \gamma_0, \text{gc}(\kappa)$ and the collapse of $\gamma_0$ such that

$L_a \models \text{cf}(\gamma) \leq \text{gc}(\kappa)$.

But a successor cardinal is regular, so this singularity will demonstrate that $\bar{\gamma} = \text{gc}(\kappa)$ and the collapsing map can be computed.

**Corollary 3.4** can be used to show that under (i) we have

**Corollary 3.5.** Let $L_\kappa$ be $E$-closed and let $\alpha = \text{gc}(L_\kappa)$. Assume that $L_\kappa \models \text{(*)}$. Then the following are equivalent

(i) $L_\kappa$ is RE in an element of $L_\kappa$

(ii) Both $L_\kappa \cap (\alpha)$ and $\kappa$ are RE in an element of $L_\kappa$.

**Remark.** Using forcing-methods of Sacks [1987] we may show that if $\star$ holds, then $L_\kappa$ is not RE.
§ 4. E-Recursive Functions and Inductive Definability.

In this section we shall give a treatment of monotone inductive definitions using methods from Girard's $\beta$-logic [1987], but without introducing $\beta$-logic and its proof theory. Masseron [1980] has used the proof theory of $\beta$-logic to show that every total $\omega^1_{CK}$-recursive function on $\omega^1_{CK}$ is dominated by a primitive recursive dilator on infinite arguments. As a corollary we give a proof of Van de Wiele's theorem:

If $F : V \rightarrow V$ is total uniformly $\Gamma^0_1$-definable over every admissible set, then $F$ is E-recursive.

The converse for $E$-recursive functions (lightface) is immediate. Slaman has given an alternate proof, but his proof uses the theory of reflection in $E$-recursion, whereas we will require only familiarity with the generating schemata of $E$-recursion.

Like the completeness theorem for $\beta$-logic this proof is based on the Henkin-type construction of term models, otherwise the proof is elementary. For each set $x$ let $\Gamma^x_0$ be a uniformly $\Delta^0_0(x)$ positive inductive definition on $x$. Let $\leq^x_x$ denote the stage comparison relation on $x$. The following lemma is valid for monotone inductive definitions in general.

**Lemma 4.0.** Let $Y \subseteq x$, $\leq$ be a relation on $y$ such that

(i) $\Gamma(Y) = Y$; and

(ii) for each $y \in Y$

\[ \{y' \mid y' \leq y\} = \Gamma(\{y' \mid y' < y\}) \], then

$\Gamma^\infty_x \leq Y$ and $\leq^x_x$ is the well-founded initial segment of $\leq$

$(\Gamma^\infty_x$ is the least fixed-point of $\Gamma^x_0)$. 
For each $x$, let $\tau_x$ be the closure ordinal of $\Gamma_x$ and let

$\varphi$ be a $\Delta_0$-formula such that

$$\forall x \exists y < \tau_x \varphi(x, \Gamma_x^{y+1}).$$

**Theorem 4.1.** There is an $E$-recursive function $G$ such that

$$\forall \alpha \forall x (\text{rank}(x) \leq \alpha \Rightarrow \exists y < \min(G(\alpha), \tau_x) \varphi(x, \Gamma_x^{y+1})).$$

**Definition.** Let $T = T_{\Gamma, \varphi}$ be the following first order theory:

- unary predicates $x, Y, ON$
- binary predicates $P$ (for $\leq_x$) and $\in$
- unary function $R$ (for rank)
- constants $\omega, \omega_1, \ldots$

Take standard axioms like regularity, extensionality, etc., together with:

(i) $Y = \Gamma(Y)$;

(ii) $\varphi(x, [y \in P(y, c_0)]) \rightarrow \forall z \in Y(\varphi(x, [y \in P(y, z)]) \rightarrow P(c_0, z));$

(iii) $P(c_{i+1}, c_{i+1}) \land \neg P(c_{i+1}, c_{i+1})$; and

(iv) $\forall z \in Y([y \in P(y, z)]) = \Gamma([y \in P(y, z) \land \neg P(z, y)]).

**Definition**

(a) Let $T_n$ denote the part of $T$ that does not contain any $c_i$ for $i \geq n$;

(b) Let $T^*, T_n^*$ denote the respective Henkin-extensions;

(c) Let $e_0, e_1, \ldots$ be a recursive enumeration of the terms of $T^*$ such that $\forall i (e_i \in T_i^*)$. 
Now if \( f : \mathbb{N} \rightarrow \mathbb{ON} \), let \( T^f \) be \( T^* \) extended with the following axioms:

\[
\{ R(e_i) \leq R(e_j) \mid f(i) \leq f(j) \}.
\]

**Lemma 4.2.** Let \( f : \mathbb{N} \rightarrow \mathbb{ON} \) and \( T^f \) be as above, then \( T^f \) is inconsistent.

**Proof.** Assume \( T^f \) is consistent for a contradiction and let \( T_f \) denote a consistent completion of \( T^f \). The term model for \( T_f \) will then be a model of \( T \) and since the rank-relation is well-founded, the model will be isomorphic to a set \( z \) where \( x \) is interpreted as a subset of \( z \). Let \( \gamma < \tau_x \) be such that \( \varphi(x, \Gamma_x^{\gamma+1}) \). By lemma the interpretation \( c_0 \) of \( \varphi \) must be in \( \Gamma_x^\infty \) and have rank \( \leq \gamma + 1 \). But then interpretations of \( c_i \) will form an \( \leq \)-infinite descending sequence, which is absurd.

If \( \sigma \) is a finite sequence of ordinals we define \( T^\sigma \) as an extension of \( T^*_{lh(\sigma)} \) as before. Thus we have

\[
\forall f : \mathbb{N} \rightarrow \mathbb{ON} \exists r \in \mathbb{N}[T^f(r) \text{ is inconsistent}].
\]

**Definition.** Let \( \sigma \) be a sequence of ordinals of length \( n \), then we say \( \sigma \) is **good** if we cannot prove a contradiction from \( T^\sigma \) using a proof of length \( \leq n \) and at most the \( n \) first axioms of \( T^\sigma \) (in some uniform enumeration of \( T^f \)'s).

For \( \alpha \in \mathbb{OR} \) we let

\[
S_\alpha = [\sigma \mid \sigma \text{ is good and } \forall i < lh(\sigma)(\sigma(i) < \alpha)]
\]

and set \( G(\alpha) = \text{height of } S_\alpha \). Then \( G \) is \( E \)-recursive since we can uniformly compute the height of any well-founded relation in \( E \)-recursion.
Lemma 4.3. Let \( \text{rank}(x) \leq \alpha \), then we can find \( \gamma \leq \Theta(\alpha) \) such that \( \varphi(x, \Gamma_{x}^{\gamma+1}) \) holds.

**proof.** Fix \( x \) and let \( \gamma \) be minimal such that \( \varphi(x, \Gamma_{x}^{\gamma+1}) \) and choose \( y \in \Gamma_{x}^{\gamma+1} - \Gamma_{x}^{\gamma} \). Let \( p \) denote the ordinal norm function on \( \Gamma_{x}^{\infty} \) induced by \( \Gamma_{x}^{\gamma} \). Then we have \( p(y) = \gamma \). Assume that \( y_{0}, \ldots, y_{n-1} \) is a sequence from \( \Gamma_{x}^{\infty} \) such that \( y_{0} = y \) and \( p(y_{i}) < p(y_{i-1}) \) for \( 1 \leq i < n \).

We shall construct a model for \( T_{n} \) using \( TC(x) \) as the domain, \( x \) for \( x \), \( \Gamma_{x}^{\infty} \) for \( \gamma \), \( \leq_{x} \) for \( \leq \) and \( y_{0}, \ldots, y_{n-1} \) for \( c_{0}, \ldots, c_{n-1} \). This model can be extended to a model for \( T_{n}^{*} \) since \( T_{n}^{*} \) is a conservative extension of \( T_{n} \) and we do not change the domain. For \( i < n \) let \( \sigma(i) = \text{rank}(e_{i}) \) (\( e_{i} \) is the interpretation of \( e_{i} \)). Note that if we extend \( \sigma \) in a consistent way, then we may extend \( \sigma \) (i.e. we cannot choose \( \sigma \) such that it is inconsistent with the construction based on extensions of \( y \)).

If \( \alpha = \text{rank}(x) \), then \( \text{rank}(e_{i}) < \alpha \) by our choice of domain as \( TC(x) \) and so \( \sigma \in S_{\alpha} \). By induction on \( p(y_{n-1}) \) we can show that \( p(y_{n-1}) \leq \| \sigma \|_{S_{\alpha}} \). The induction is trivial by the above remark on the consistency-considerations and, hence, the lemma follows. The theorem follows from the lemma.

**Remark.** The theory \( T \) in the proof asserts that \( x \) is a relation on a transitive set \( \gamma \); \( \langle Y, P \rangle \) is the prewellordering induced by \( \Gamma \) over \( x \) and there is no \( z \in \Gamma_{x}^{\infty} \) satisfying \( \varphi \).

If \( T' \) is a primitive recursive theory in the language of set theory, then the same proof gives:
Corollary 4.4. Let $\Gamma$, $\varphi$ and $\tau_\chi$ be as above. If

$$\forall x(x \models T' \Rightarrow \exists y < \tau_\chi \varphi(x, \Gamma_x^{y+1}))$$

then there is an $E$-recursive function $G$ such that

$$\forall x(x \models T' \Rightarrow \exists y < \min(\tau_\chi, G(\text{rank}(x))) \varphi(x, \Gamma_x^{y+1}))$$

Examples of such theories are:

(i) $x$ is transitive, infinite and closed under finite subsets;
(ii) $x$ is rudimentarily closed.

Now if $x$ is transitive, infinite and closed under finite subsets, then we have a notation system for the next admissible ($\text{HYP}(x)$) and that notation system is defined by a monotone inductive definition. If $\exists y \in \text{HYP}(x) \varphi(x, y)$, then there is a $\Delta_0$ formula $\varphi'$ such that $\varphi'(x, \Gamma_x^y)$ for the least $y$ such that $\exists y \in L_\gamma[x] \varphi(x, y)$ where $\Gamma$ defines that notation system.

Using this we have proven the following theorem of J. Van de Wiele:

Corollary 4.5. (Van de Wiele) Let $F: V \rightarrow V$ be uniformly $\Sigma_1$-definable and total over all admissible sets, then $F$ is $E$-recursive.

proof follows immediately from the theorem and the above remarks on the inductive generation of $\text{HYP}(x)$.

Note that we actually show that $F$ is computable in a weaker system than $E$-recursion, since we use elementary functions together with the operator which computes the height of a well-founded relation.
References


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