DEFORMATIONS OF REFLEXIVE SHEAVES OF RANK 2 ON $\mathbb{P}^3$  

by 

Jan Oddvar Kleppe  
Inst. of Math., University of Oslo
In this paper we study deformations of reflexive sheaves of rank 2 on \( \mathbb{P}_k \), where \( k \) is an algebraically closed field of any characteristic. Let \( F \) be a reflexive sheaf with a section \( s \in H^0(F) = \mathcal{O}(\mathbb{P},F) \) whose corresponding scheme of zeros is a curve \( \mathcal{C} \) in \( \mathbb{P} \). Moreover let \( M = M(c_1,c_2,c_3) \) be the (coarse) moduli space of stable reflexive sheaves with Chern classes \( c_1, c_2 \) and \( c_3 \). The study of how the deformations of \( \mathcal{C} \subset \mathbb{P} \) correspond to the deformations of the reflexive sheaf \( F \) leads to a nice relationship between the local ring \( \mathcal{O}_{H,\mathcal{C}} \) of the Hilbert scheme \( H = H(d,g) \) of curves of degree \( d \) and arithmetic genus \( g \) at \( \mathcal{C} \subset \mathbb{P} \) and the corresponding local ring \( \mathcal{O}_{M,F} \) of \( M \) at \( F \). In this paper we consider some examples where we use this relationship. In particular we prove that the moduli spaces \( M(0,13,74) \) and \( M(-1,14,88) \) contain generically non-reduced components.

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1. Deformations of a reflexive sheaf with a section.

If \( \text{Def}_F \) is the local deformation functor of \( F \) defined on the category \( \mathcal{L} \) of local artinian \( k \)-algebras with residue field \( k \), then it is well known that \( \text{Ext}_0^1(F,F) \) is the tangent space of \( \text{Def}_F \) and that \( \text{Ext}_0^2(F,F) \) contains the obstructions of deformation. See [H3]. To deform the pair \( (F,s) \) we consider the functor

\[
\text{Def}_{F,s} : \mathcal{L} \rightarrow \text{Sets}
\]
defined by

\[ \text{Def}_{F, s}(R) = \{ \mathcal{O}_F \xrightarrow{s_R} F_R | F_R \in \text{Def}_F(R) \text{ and } s_R \circ \iota_k = s \} / \sim \]

where \( \mathcal{P}_R = \mathcal{P} \times \text{Spec}(R) \) and where \( \iota_k : k - k \) is the identity. Two deformations \((F_R, s_R)\) and \((F'_R, s'_R)\) are equivalent if there exist isomorphisms \( \mathcal{O}_F \xrightarrow{\sim} \mathcal{O}_{F'_R} \) and \( F_R \xrightarrow{\sim} F'_R \) and a commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}_{\mathcal{P}_R} & \xrightarrow{s_R} & F_R \\
\downarrow \cong & & \downarrow \cong \\
\mathcal{O}_{\mathcal{P}_R} & \xrightarrow{s'_R} & F'_R
\end{array}
\]

such that \( s_R \circ \iota_k = s'_R \circ \iota_k \). In fact we also identify the given pair \((F, s)\) with any \((F', s')\) where \( s' \in H^0(\mathcal{P}, F') \) if they fit together into such a commutative diagram.

**Proposition 1.1.** (i) The tangent space of \( \text{Def}_{F, s} \) is

\[
\text{Ext}_{\mathcal{P}}^1(\mathcal{I}_C(c_4), F) \text{ where } \mathcal{I}_C = \ker(\mathcal{O}_F \to \mathcal{O}_C), \text{ and}
\]

\[
\text{Ext}_{\mathcal{P}}^2(\mathcal{I}_C(c_4), F) \text{ contains the obstructions of deformations.}
\]

(ii) The natural

\[ \varphi : \text{Def}_{F, s} \to \text{Def}_F \]

is a smooth morphism of functors on \( \underline{1} \) provided

\[ H^1(F) = 0 \]

By the correspondence [H3, 4.1] there is a curve \( C = (s)_0 \subseteq \mathcal{P} \) and an exact sequence

\[ \xi : 0 \to \mathcal{O}_F \xrightarrow{s} F \to \mathcal{I}_C(c_4) \to 0 \]
associated to \((F, s)\). The condition \(H^1(F) = 0\) is therefore equivalent to
\[ H^1(I_0(c_1)) = 0. \]

**Proof of (i).** Using [L2, §2] or [KL, 1.2] we know that there is a spectral sequence
\[ E^P_2, q = \lim_{\rightarrow} \begin{pmatrix} \Ext^q(F, F) \\ \alpha \end{pmatrix} \Ext^q(\mathcal{O}_P, \mathcal{O}_P) \]
converging to some group \(A(*)\) where \(A^1\) is the tangent space of \(\text{Def}_{F, s}\) and \(A^2\) contains the obstructions of deformation. Since \(E^P_2, q = 0\) for \(p \geq 2\), we have an exact sequence
\[ 0 \rightarrow E^1_2, q-1 \rightarrow A^q \rightarrow E^0_2, q \rightarrow 0. \]
Moreover
\[ \Ext^q(\mathcal{O}_P, \mathcal{O}_P) = 0 \text{ for } q > 0 \text{ and } \Ext^q(\mathcal{O}_P, F) = H^q(F) \text{ for any } q, \]
and this gives
\[ E^0_2, q = \ker \alpha^q \text{ and } E^1_2, q = \coker \alpha^q \text{ for } q > 0. \]

Observe also that
\[ E^1_2, 0 = \lim_{\rightarrow} \begin{pmatrix} \Hom(F, F) \\ \alpha_0 \end{pmatrix} \Hom(\mathcal{O}_P, \mathcal{O}_P) \]
because \(\Hom(\mathcal{O}_P, \mathcal{O}_P) \subseteq \Hom(F, F)\). We therefore have an exact sequence
\[ 0 \rightarrow \coker \alpha^{q-1} \rightarrow A^q \rightarrow \ker \alpha^q \rightarrow 0 \]
for any \(q > 0\). Combining with the long exact sequence
\[ \cdots \rightarrow \text{Hom}(E, F) \xrightarrow{\alpha} H^0(F) \rightarrow \text{Ext}^1(I_G(c_1), F) \xrightarrow{\alpha} \text{Ext}^1(F, F) \rightarrow \cdots \tag{*} \]

deduced from the short exact sequence

\[ 0 \rightarrow O_F \xrightarrow{s} F \rightarrow I_G(c_1) \rightarrow 0, \]

we find isomorphisms

\[ A^q \cong \text{Ext}^q(I_G(c_1), F) \quad \text{for} \quad q > 0. \]

(ii) Let \( S \rightarrow R \) be a morphism in \( l \) whose kernel \( G_l \) is a k-module via \( R \rightarrow k \), let \( s_R : O_F \rightarrow F \) be a deformation of \( s : O_F \rightarrow F \) to \( R \), and let \( F_S \) be a deformation of \( F_R \) to \( S \). To prove the smoothness of \( \varphi \), we must find a morphism \( s_S \),

\[ s_S : O_{F_S} \rightarrow F_S \]

such that \( s_S \circ \varphi = s_R \), i.e. we must prove that \( s_R \in H^0(F_R) \) is contained in the image of \( H^0(F_S) \rightarrow H^0(F_R) \). Since

\[ 0 \rightarrow F \otimes_k GL \rightarrow F_S \rightarrow F_R \rightarrow 0 \]

is exact and since \( H^1(F) = 0 \) by assumption, we see that \( H^0(F_S) \rightarrow H^0(F_R) \) is surjective and we are done.

**Remark 1.2.** In the exact sequence \((*)\) of this proof, \( \varphi^1 \) is the tangent map of \( \varphi : \text{Def}_{E,S} \rightarrow \text{Def}_{E,F} \) and \( \varphi^2 \) maps "obstructions to obstructions". In fact \( \varphi \) is a morphism of principal homogeneous spaces via \( \varphi^1 \). Using this it is in general rather easy to prove the smoothness of \( \varphi \) directly from the surjectivity of \( \varphi^1 \) and the injectivity of \( \varphi^2 \). This gives another proof of \((1.1.ii)\).
The relationship between the deformations of a reflexive sheaf with a section and the deformations of the corresponding curve.

Let $F, s \in H^0(F)$ and $I = I_C = \ker(O_F \to O_C)$ be as in the preceding section, and let $\text{Def}_I : \mathcal{M} \to \text{Sets}$ be the deformation functor of the $O_F$-module $I$. Then there is a natural map

$$\psi : \text{Def}_F, s \to \text{Def}_I$$

defined by

$$\psi(F, s) = M_R \otimes (O_F(-c_1) \otimes_R R)$$

where $M_R = \text{coker} s_R$. If $\text{Hilb}_C : \mathcal{M} \to \text{Sets}$ is the local Hilbert functor at $C \subset F$, we have also a natural map

$$\text{Hilb}_C \to \text{Def}_I$$

of functors on $\mathcal{M}$. Recall that $C$ is locally Cohen Macaulay and equidimensional [H3, 4.1].

**Proposition 2.1.** (i) The natural morphism

$$\text{Hilb}_C \to \text{Def}_I$$

is an isomorphism of functors.

(ii) If $H^1(F(-c)) = 0$, then

$$\psi : \text{Def}_F, s \to \text{Def}_I$$

is a smooth morphism of functors on $\mathcal{M}$.

Observe also that

$$H^1(F(-c)) \simeq H^1(I_C(c_1-c))$$

and moreover by duality that
Proof of (i) If \( N_C = \text{Hom}_{\mathcal{O}_{\mathcal{P}}} (\mathcal{I}, \mathcal{O}_C) \) is the normal bundle of \( C \) in \( \mathcal{P} \), we proved in [Kl, 2.2] that

\[
H^i(N_C) \cong \text{Ext}^{i+1}_{\mathcal{O}_{\mathcal{P}}}(\mathcal{I}, \mathcal{I})
\]

as a consequence of the fact that the projective dimension of the \( \mathcal{O}_{\mathcal{P}} \)-Module \( \mathcal{I} \) is 1, from which the conclusion of (i) is easy to understand. We will, however, give a direct proof.

To construct the inverse of \( \text{Hilb}_C(R) \rightarrow \text{Def}_I(R) \), let \( M_R \) be a deformation of \( I \) to \( R \). Observe that there is an exact sequence

\[
(*) \quad 0 \rightarrow E \rightarrow \bigoplus_{i=1}^{r+1} \mathcal{O}_{\mathcal{P}}(-n_i) \xrightarrow{f} \mathcal{I} \rightarrow 0
\]

where \( E \) is a vector bundle on \( \mathcal{P} \) of rank \( r \). \( \wedge^r E \) is therefore invertible, and we can identify it with \( \mathcal{O}_{\mathcal{P}}(d_1) \) where \( d_1 = -\sum n_i \).

If \( \mathcal{P} = \bigoplus \mathcal{O}_{\mathcal{P}}(-n_i) \), then there is a complex

\[
(**) \quad E \rightarrow \mathcal{P} \cong \left( \wedge^r \mathcal{P} \right)^{\vee}(d_1) \rightarrow \left( \wedge^r E \right)^{\vee}(d_1) = \mathcal{O}_{\mathcal{P}}
\]

and it is well known that the maps \( \mathcal{P} \xrightarrow{f} \mathcal{I} \subseteq \mathcal{O}_{\mathcal{P}} \) and \( \mathcal{P} \rightarrow \mathcal{O}_{\mathcal{P}} \) deduced from (*) and (**) respectively are equal up to a unit of \( k \). We can assume equality. Now since \( M_R \) is a lifting of \( I \) to \( R \), there is a map

\[
f_R : P_R = \bigoplus_{i=1}^{r+1} \mathcal{O}_{\mathcal{P}_R}(-n_i) \rightarrow M_R
\]

such that \( f_R \otimes_R \mathcal{I}_k = f : P \rightarrow \mathcal{I} \). By Nakayama's lemma, \( f_R \) is surjective. Moreover if \( E_R = \ker f_R \), we easily see that \( E_R \otimes_R k = E \).
and $E_R$ is $R$-flat. It follows that $E_R$ is a locally free $O_{\mathbb{P}}$-Module of rank $r$ satisfying

$$r \wedge E_R = O_{\mathbb{P}}(d_\lambda).$$

Furthermore there is a complex

$$E_R \rightarrow P_R \xrightarrow{\cdot} (\wedge P_R)^\vee (d_\lambda) \rightarrow (\wedge E_R)^\vee (d_\lambda) = O_{\mathbb{P}}$$

which proves the existence of an $O_{\mathbb{P}}$-linear map

$$\alpha : M_R \rightarrow O_{\mathbb{P}}$$

which reduces to the natural inclusion $\mathbb{I} \subset O_{\mathbb{P}}$ via $(-) \otimes_R k$. It is easy to see that $\alpha$ is injective, that $\text{coker} \alpha$ is $R$-flat and that $\text{coker} \alpha \otimes_R k = 0$. We therefore have a deformation $C_R \subset \mathbb{P}$ of $C \subset \mathbb{P}$. Finally to see that the inverse of $\text{Hilb}_{C}(R) \rightarrow \text{Def}_{\mathbb{P}}(R)$ is well-defined, let $\beta : M_R \supset M'_R$ and $\alpha' : M'_R \rightarrow O_{\mathbb{P}}$ be $O_{\mathbb{P}}$-linear maps such that $\beta \otimes_R \mathbb{I}_k$ is the identity on $\mathbb{I}_R$ and $\alpha' \otimes_R \mathbb{I}_k$ is the natural inclusion $\mathbb{I} \subset \mathbb{P}$.

(We do not assume $\alpha' \beta = \alpha$). We claim that $\text{Im} \alpha' = \text{Im} \alpha$. In fact since

$$\text{Ext}^{i}_{O_{\mathbb{P}}}(O_C, O_{\mathbb{P}}) = 0$$

for $i = 0, 1$, we have

$$k = \text{Hom}_{O_{\mathbb{P}}}(O_C, O_{\mathbb{P}}) \supset \text{Hom}_{O_{\mathbb{P}}}(\mathbb{I}_R, O_{\mathbb{P}}).$$

We deduce that the map

$$R = \text{Hom}_{O_{\mathbb{P}}} (O_{\mathbb{P}}, O_{\mathbb{P}}) \rightarrow \text{Hom}_{O_{\mathbb{P}}} (M_R, O_{\mathbb{P}})$$

induced by $\alpha$, is surjective. Hence

$$\alpha' \beta = \alpha \beta.$$
for some $r \in R$, and since $\alpha \otimes 1_k = \alpha \otimes 1_k$ is the natural inclusion $I \subset O_D$, $r$ is a unit and we are done.

(ii) Let $S \to R$, $\mathcal{O}_R$ and $s_R : O_\mathcal{F}_R \to \mathcal{F}_R$ be as in the proof of (1.1 ii). Moreover let $M_R = \text{coker} s_R$, and let $M_S$ be a deformation of $M_R$ to $S$. To prove smoothness we must find a deformation

$$s_S : O_\mathcal{F}_S \to \mathcal{F}_S$$

with cokernel $M_S$ such that $s_S \otimes 1_R = s_R$. By theory of extensions it is sufficient to prove that the map

$$\Ext^1_{O_S}(M_S, C_{O_S}) \to \Ext^1_{O_R}(M_R, O_{\mathcal{F}_R})$$

induced by $(-) \otimes_S R$ is surjective. Modulo isomorphisms we find this map in the long exact sequence

$$\cdots \to \Ext^1(M_S, O_{\mathcal{F}_S} \otimes \mathcal{O}_L) \to \Ext^1(M_S, O_{\mathcal{F}_S}) \to \Ext^1(M_S, O_{\mathcal{F}_R}) \to \Ext^2(M_S, O_{\mathcal{F}_S} \otimes \mathcal{O}_L).$$

Since $\Ext^2_{O_{\mathcal{F}_S}}(M_S, O_{\mathcal{F}_S} \otimes \mathcal{O}_L) \simeq \Ext^2_{O_{\mathcal{F}_S}}(I_{\mathcal{O}}(c_1), O_{\mathcal{F}}) \otimes \mathcal{O}_L = 0$ by assumption, we are done.

**Remark 2.2.** The short exact sequence

$$\xi : 0 \to O_{\mathcal{F}} \to \mathcal{F} \to I_{\mathcal{O}}(c_1) \to 0$$

induces a long exact sequence

$$\cdots \to \Ext^1_{O_{\mathcal{F}}}(I_{\mathcal{O}}(c_1), O_{\mathcal{F}}) \to \Ext^1_{O_{\mathcal{F}}}(I_{\mathcal{O}}(c_1), \mathcal{F}) \xrightarrow{\psi^1} \Ext^1_{O_{\mathcal{F}}}(I_{\mathcal{O}}, I_{\mathcal{O}}) \to$$

$$\Ext^2_{O_{\mathcal{F}}}(I_{\mathcal{O}}(c_1), O_{\mathcal{F}}) \to \Ext^2_{O_{\mathcal{F}}}(I_{\mathcal{O}}(c_1), \mathcal{F}) \xrightarrow{\psi^2} \Ext^2_{O_{\mathcal{F}}}(I_{\mathcal{O}}, I_{\mathcal{O}})$$
where $\psi$ is the tangent map of $\psi$ or more generally, $\psi$ is a map of principal homogeneous spaces via $\psi^1$ and $\psi^2$ maps "obstructions to obstructions". As remarked in (1.2), the smoothness of $\psi$ follows therefore from the surjectivity of $\psi^1$ and the injectivity of $\psi^2$.

Remark 2.3. Let $\xi$ be the extension

$$0 \rightarrow O_{\mathcal{P}} \xrightarrow{\xi} F \rightarrow \mathcal{I}_C(c_1) \rightarrow 0$$

and let $\text{Def}_{C,\xi} : \text{Sets} \rightarrow \text{Sets}$ be the functor defined by

$$\text{Def}_{C,\xi}(R) = \left\{ \left( C_R \subseteq \mathcal{P}_R \right) \in \text{Hilb}_C(R) \text{ and } \xi_R \in \text{Ext}^1(\mathcal{I}_C(c_1),O_{\mathcal{P}_R}) \right\} / \sim$$

Two deformations $(C_R,\xi_R)$ and $(C'_R,\xi'_R)$ are equivalent if $C_R = C'_R \subseteq \mathcal{P}_R$ and if there is a commutative diagram

$$\begin{array}{ccc}
\xi'_R : 0 & \rightarrow & O_{\mathcal{P}_R} \rightarrow F_R \rightarrow \mathcal{I}_C(c_1) \rightarrow 0 \\
\downarrow \circ & \downarrow \circ & \downarrow 1 \\
\xi_R : 0 & \rightarrow & O_{\mathcal{P}_R} \rightarrow F'_R \rightarrow \mathcal{I}_C(c_1) \rightarrow 0,
\end{array}$$

both reducing to the extension $\xi$ via $(-) \otimes_R k$. In the same way we identify the given $(C,\xi)$ with any $(C',\xi')$ provided $C = C'$ and $\xi' = u\xi$ for some unit $u \in k^\times$. Note that we may in this definition of equivalence replace the identity $1$ on $\mathcal{I}_C(c_1)$ by any $O_{\mathcal{P}_R}$ linear map. See [Ma 2, 6.1] and recall $\text{Hom}(\mathcal{I}_C,\mathcal{I}_C) = k$. Now there is a forgetful map

$$\alpha : \text{Def}_{C,\xi} \rightarrow \text{Def}_{\mathcal{P},s},$$
and using (2.1i) we immediately have an inverse of \( \alpha \).

Hence \( \alpha \) is an isomorphism. Observe that we might construct
the inverse of \( \alpha(R) \) for \( R \in \text{ob} \mathcal{A} \) by considering the in-
vertible sheaf \( \det F_R \) on \( \mathbb{P}_R \). See [Ma 1, 4.2] or [G, 4.1].
In fact if \( (F_R, s_R) \) is given, there is an \( \mathbb{P}_R \) a morphism

\[
i : \mathbb{P}_R \to \det F_R \cong O_{\mathbb{P}_R}(c_1)
\]

and a complex

\[
0 \to O_{\mathbb{P}_R} \xrightarrow{s_R} F_R \xrightarrow{\text{if}((-) \wedge s_R)} \mathbb{P}_R(c_1)
\]

which after the tensorization \((-) \otimes_R k\) is exact. Hence

\[
0 \to O_{\mathbb{P}_R} \xrightarrow{s_R} F_R \to \text{coker } s_R \to 0
\]

is exact, \( \text{coker } s_R \) is \( R \)-flat and \( \text{coker } s_R \cong O_{\mathbb{P}_R}(c_1) \),
and putting this together, we can find an inverse of \( \alpha(R) \).

One should compare the isomorphism of \( \alpha \) with [H 3, 4.1]
which implies that there is a bijection between the set of
pairs \( (F, s) \) and the set of \( (C, \xi) \) moduli equivalence under
certain conditions on the pairs. Thinking of these families
of pairs as moduli spaces, [H 3, 4.1] establishes a bijection
on the \( k \)-points of these spaces while the isomorphism of \( \alpha \)
takes care of the scheme structure as well.

To be more precise we claim that there is a quasiprojective
scheme \( D \) parametrizing equivalent pairs \( (C, \xi) \) where

1) \( C \) is an equidimensional Cohen Macaulay curve and where

2) the extension \( \xi : 0 \to O_{\mathbb{P}} \to F \to I_C(c_1) \to 0 \) is

such that \( F \) is a stable reflexive sheaf.
Moreover there are projection morphisms
\[ \begin{array}{c}
\mathcal{D} \xrightarrow{q} H(d, g) \\
\text{(*)}
\end{array} \]
\[ p \big| \bigg\downarrow \bigg| \]
\[ M(c_1, c_2, c_3) \]
defined by \( p(F_K, s_K) = F_K \) and \( q(C_K, \xi_K) = C_K \) for a geometric K-point \((C_K, \xi_K)\) corresponding to \((F_K, s_K)\), such that the fibers of \( p \) and \( q \) are smooth connected schemes. Furthermore, \( p \) is smooth at \((F_K, s_K)\) provided \( H^1(F_K) = 0 \), and \( q \) is smooth at \((C_K, \xi_K)\) provided \( H^1(I_{C_K}(c_1 - 4)) = 0 \).

To indicate why \(^1\) let \( \text{Sch/k} \) be the category of locally noetherian k-schemes and let \( \mathcal{D} : \text{Sch/k} \to \text{Sets} \) be the functor defined by
\[ \mathcal{D}(S) = \{(C_S, I_S, \xi_S) : C_S \in H(d, g)(S), I_S \text{ is invertible on } S \text{ and } \xi_S \in \text{Ext}^1(I_{C_S}(c_1), O_{\mathbb{P} \times S} \otimes I_S) \text{ such that } C_S \times_S \text{Spec}(K) \text{ satisfies (1) and } \xi_S \otimes_K \neq 0 \text{ for any geometric K-point of } S \} \]

Two deformations \((C_S, I_S, \xi_S)\) and \((C_S', I_S', \xi_S')\) are equivalent if \( C_S = C_S' \) and if there is an isomorphism \( \tau : I_S \to I_S' \) whose induced morphism \( \text{Ext}^1(I_{C_S}(c_1), \tau) \) maps \( \xi_S \) onto \( \xi_S' \). Now if \( U \subseteq H(d, g) \) is the open set of equidimensional Cohen Macaulay curves and if \( C_U \subseteq \mathbb{P} \times U \xrightarrow{\pi} U \) is the restricting of the universal curve to \( U \), one may prove that \( E = \text{Ext}^1(I_{C_U}(c_1), O_{\mathbb{P} \times U}) \) is a coherent \( O_{\mathbb{P} \times U} \)-module, flat over \( U \). By [EGA,III,7.7.6] there is a unique coherent \( O_U \)-module \( Q \) such that

\[ 1) \text{ For good ideas of this construction, see the appendix } [E,S], \text{ some of which appears in } [S,M,S]. \]
for any quasicoherent \( \mathcal{O}_U \)-Module \( \mathcal{E} \). If \( \mathbb{P}(Q) = \text{Proj}(\text{Sym}(Q)) \) is the projective fiber over \( U \) defined by \( Q \), we can use [EGA II, 4.2.3] to prove that

\[
\mathbb{D}(-) \cong \text{Mor}_K(-, \mathbb{P}(Q)).
\]

Now let \( D \subseteq \mathbb{P}(Q) \) be the open set whose \( k \)-points are \((C, \xi)\), \( \xi : 0 \to \mathcal{O}_F \to \mathcal{E} \to \mathcal{I}_C(c_1) \to 0 \), where \( \mathcal{E} \) is a stable reflexive sheaf. Then we have a diagram (\( * \)) where the existence of the morphism \( p \) follows from the definition [Ma 1, 5.5] of the moduli space \( M = M(c_1, c_2, c_3) \). Moreover since \( \mathbb{P}(Q) \) represents the functor \( \mathbb{D} \), the fiber of \( q : D \to H(d, g) \) at a \( K \)-point \( C_K \subseteq \mathbb{P}_K \) of \( H(d, g) \) is just \( D \cap \mathbb{P}(\text{Ext}^1(\mathcal{I}_K(c_1), \mathcal{O}_K))^\vee \) where \((-)^\vee = \text{Hom}_K(-, K)\). Moreover if we think of the fiber of \( p \) at a geometric \( K \)-point \( F_K \) of \( M \) as those sections \( s \in H^0(F_K) \) where \((s)_0\) is a curve, we understand that the fiber is an open subscheme of the linear space \( \mathbb{P}(H^0(F_K)^\vee) \). In particular the geometric fibers of \( p \) and \( q \) are smooth and connected.

Finally the smoothness of \( p \) and \( q \) at \((C, \xi)\) follows from (1.1ii) and (2.1ii) provided we know that the morphism 

\[
p^* : \mathcal{O}_M(F) \to \mathcal{O}_D(F, s) \text{ induced by } p : D \to M \text{ makes a commutative diagram}
\]

\[
\begin{array}{ccc}
\text{Def}_{F, s} & \simeq & \text{Mor}(\mathcal{O}_D(F, s), -) \\
\varphi \downarrow & & \downarrow \text{Mor}(p^*, -) \\
\text{Def}_{F} & \simeq & \text{Mor}(\mathcal{O}_M(F), -)
\end{array}
\]

of horizontal isomorphisms on \( 1 \). In fact the commutativity from
the definition of a moduli space [Ma1, 5.5] while the construction of \( \mathcal{M} \) implies the lower horizontal isomorphism. See [Ma2, 6.4] from which we immediately have that the morphism \( \text{Def}_F \to \text{Mor}(\mathcal{O}_M,F) \) is smooth, and since the morphism induces an isomorphism of tangent spaces, both isomorphic to \( \text{Ext}^1(\mathcal{O}_M,F) \), it must be an isomorphism.

Remark 2.4. In particular the smoothness of \( \text{Def}_F \to \text{Mor}(\mathcal{O}_M,F) \) which is a consequence of the smoothness of the morphism treated in [Ma2, 6.4], implies that \( \mathcal{O}_M,F \) is a regular local ring if and only if \( \text{Def}_F \) is a smooth functor on \( I \).

3. Non-reduced components of the moduli scheme \( \mathcal{M}(C_1,C_2,C_3) \).

One knows that the Hilbert scheme \( \mathcal{H}(d,g) \) is not always reduced. In fact if \( g \) is the largest number satisfying \( g \leq \frac{d^2-4}{8} \), we proved in [Kl, 3.2.10] that \( \mathcal{H}(d,g) \) is non-reduced for every \( d \geq 14 \), and we explicitly described a non-reduced component in terms of the Picard group of a smooth general cubic surface.

Example 3.1. (Mumford [M1]). For \( d = 14 \), we have

\[ g = \frac{d^2-4}{8} = 24, \]

and there is an open irreducible subscheme \( U \subset \mathcal{H}(14,24) \) of smooth connected curves whose closure \( \overline{U} = W \) makes a non-reduced component, such that for any \( (C \subseteq \mathbb{P}) \in U \),

\[
h^0 \left( I_C(v) \right) = \begin{cases} 0 & \text{for } v \leq 2 \\ 1 & \text{for } v = 3 \end{cases} \]

\[
h^1 \left( I_C(v) \right) = 0 \quad \text{for } v \notin \{3,4,5\}, \]

\[
h^1 \left( O_C(v) \right) = \begin{cases} 0 & \text{for } v \geq 4 \\ 1 & \text{for } v = 3. \end{cases} \]
See [Kl, (3.2.4) and (3.1.3)]. In fact with \( C \subseteq \mathbb{P} \) in \( U \), there is a global complete intersection of two surfaces of degree 3 and 6 whose corresponding linked curve is a disjoint union of two coniques.

Now let \( C \subseteq \mathbb{P} \) be a smooth connected curve satisfying

\[(*) \quad H^1(\mathcal{I}_C(c_1)) = 0, \quad H^1(\mathcal{I}_C(c_1-4)) = 0 \quad \text{and} \quad H^1(\mathcal{O}_C(c_1-4)) \neq 0\]

for some integer \( c_1 \), let \( \xi \in H^0(\mathcal{O}_C(c_1-4)) = \operatorname{Ext}^1(\mathcal{I}_C(c_1), \mathcal{O}_\mathbb{P}) \) be non-trivial, and let \( (\mathbb{F}, s) \), \( s \in H^0(\mathbb{F}) \), correspond to \( (C, \xi) \) via the usual correspondence. Then \( \mathbb{F} \) is reflexive, and it is stable (resp. semistable) if and only if \( c_1 > 0 \) (resp. \( c_1 \geq 0 \)) and \( C \) is not contained in any surface of degree \( \leq \frac{1}{2} c_1 \) (resp. \( < \frac{1}{2} c_1 \)). See [H3, 4.2]. Combining (1.1) and (2.1) with (2.4) in case \( \mathbb{F} \) is stable, we find that \( \mathcal{O}_{\mathbb{M}} \mathbb{F} \) is non-reduced iff \( \mathcal{O}_{\mathbb{H}, C} \) is non-reduced.

**Example 3.2.** Let \((C \subseteq \mathbb{P}) \in \mathbb{H}(14, 24)\) belong to the set \( U \) of (3.1) and let \( c_1 \) be an integer satisfying (\( * \)), i.e. \( c_1 \leq 2 \) or \( c_1 = 6 \).

(i) Let \( c_1 = 6 \). By virtue of (1.1) and (2.1) the hull of \( \operatorname{Def}_{\mathbb{F}} \) is non-reduced. Moreover \( \mathbb{F} \) is semistable with Chern classes \((c_1, c_2, c_3) = (6, 14, 18)\), and the normalized sheaf \( \mathbb{F}(-3) \) has Chern classes \((c_1', c_2', c_3') = (0, 5, 18)\).

(ii) Let \( c_1 = 2 \). The corresponding reflexive sheaf is stable and must belong to at least one non-reduced component of \( \mathbb{M}(2, 14, 74) \), i.e. of \( \mathbb{M}(0, 13, 74) \).

(iii) With \( c_1 = 1 \) we find at least one non-reduced component of \( \mathbb{M}(1, 14, 88) \cong \mathbb{M}(-1, 14, 88) \).
Combining the discussion after (2.3) and in particular the irreducibility of the morphism \( q \) with the irreducibility of the set \( U \) of (3.1), we see that we obtain precisely one non-reduced component of \( M(0,13,74) \) and \( M(-1,14,88) \) in this way.

We will give one more example of a non-reduced component and include a discussion to better understand (1.1) and (2.1). In fact recall [Kl,2.3.6] that if an equidimensional Cohen Macaulay curve \( (C \subseteq \mathbb{P}) \in H(d,g) \) is contained in a complete intersection \( V(F_1,F_2) \) of two surfaces of degree \( f_1 = \deg F_1 \) and \( f_2 = \deg F_2 \) with

\[
H^1(I_C(f_1)) = 0 \quad \text{and} \quad H^1(I_C(f_1-4)) = 0
\]

for \( i = 1,2 \), and if \( (C' \subseteq \mathbb{P}) \in H'(d',g') \) is the linked curve, then \( 0_{H',C'} \) is reduced iff \( 0_{H',C} \) is reduced. Since any curve \( (C \subseteq \mathbb{P}) \in U \) of (3.1) is contained in a complete intersection \( V(F_1,F_2) \) of two surfaces of degree \( f_1 = f_2 = 6 \), the linked curves \( C' \subseteq \mathbb{P} \) must belong to at least one (and one may prove to exactly one) non-reduced component \( W \subseteq H(22,56) \) of dimension 88. See [Kl,2.3.9]. One may see that \( W \) contains smooth connected curves. Moreover using the fact that \( \omega_C(4-f_1-f_2) \) and \( \omega_C(4-f_1-f_2) \) are the sheaves of ideals which define the closed subschemes \( C' \subseteq V(F_1,F_2) \) and \( C \subseteq V(F_1,F_2) \) respectively, one proves easily that

\[
H^0(I_C,(4)) = 0, \quad H^1(I_C,(v)) = 0 \quad \text{for} \quad v \notin \{3,4,5\} \quad \text{and} \quad H^1(O_C,(5)) \neq 0.
\]

See [S,P] and [Kl,2.3.3].

1) The condition \( H^1(I_C(f_1-4)) = 0 \) implies also that the linked curves \( C' \subseteq \mathbb{P} \) form an open subset of \( H' \).
Example 3.3. Let \((C', \mathbb{P}) \in W \subseteq H(22, 56)\) be as above with \(C'\) smooth and connected. If \(c_1\) is chosen among \(1 \leq c_1 \leq 9\), then \(C' \subseteq \mathbb{P}\) defines a stable reflexive sheaf \(\mathcal{F}'\) and in fact a vector bundle if \(c_1 = 9\) by the usual correspondence. Using (1.1) and (2.1) we find that \(\mathcal{F}'\) belongs to a non-reduced component of \(M(c_1, c_2, c_3)\) for the choices \(1 \leq c_1 \leq 2\) or \(c_1 = 6\). In particular there exists a non-reduced component of \(M(6, 22, 66) \simeq M(0, 13, 66)\). Moreover we obtain precisely one non-reduced component in this way if we make use of the discussion after (2.3). If \(c_1 = 9\), we find a reflexive sheaf \(\mathcal{F}' \in M(9, 22, 0)\), and the normalized one is \(\mathcal{F}'(-5) \in M(-1, 2, 0)\), but we cannot conclude that \(M(-1, 2, 0)\) is non-reduced, even though \(H(22, 56)\) is, because the condition \(H^1(I_{C'(c_1 - 4)}) = 0\) of (2.1, ii) is not satisfied. In fact one knows that \(M(-1, 2, 0)\) is a smooth scheme. See [H, S] or [S, M, S].

As a starting point of these final considerations, we will suppose as known that there is an open smooth connected subscheme \(U_M \subseteq M(-1, 2, 0)\) of stable reflexive sheaves \(\mathcal{F}\) for which there exists a global section \(s \in H^0(\mathcal{F}(2))\) whose corresponding scheme of zero's \(C' = (s)_0\) is a disjoint union of two coniques. Moreover \(\dim U_M = 11\). In fact [H, S] proves even more. We then have an exact sequence

\[0 \to \mathcal{O}_M \to \mathcal{F}(2) \to I_{C'}(3) \to 0\]

for \(\mathcal{F} \in U_M\), and since the dimension of the cohomology groups \(H^i(I_{C'}(v))\) is easily found in case \(C'\) consists of two disjoint
coniques, we get
\[ h^0(I_C(1)) = h^0(I_C(2)) = 1 \]
and
\[ h^1(I_C(v)) = h^1(I_C(v+1)) = \begin{cases} 1 & \text{for } v = -1, 1 \\ 2 & \text{for } v = 0 \\ 0 & \text{for } v \notin \{-1, 0, 1\} \end{cases} \]

By \( \dim U_M = 11 \), \( \Ext^2_{O_P}(I_C(I_C)) = 0 \). (The reader who is more familiar with the Hilbert scheme may prove our assumptions on \( U_M \) by first proving that there is an open smooth connected subscheme \( U \subseteq H(4, -1) \) of disjoint coniques \( C' \) and that \( \dim U = 16 \). This is in fact a very special case of [K1, (3.1.10 i) ]. See also [K1, (3.1.4) and (2.3.18)]. With \( c_1 = 3 \), we have \( H^1(I_C(c_1)) = H^1(I_C(c_1-4)) = 0 \), and by the discussion after (2.3), there exists an open smooth connected subscheme of \( M(3, 4, 0) \rightarrow M(-1, 2, 0) \) defined by \( U_M = i(p(q^{-1}(U))). \) Moreover \( \dim U_M = 11 \) because \( \dim U_M + h^0(I_C(2)) = \dim U + h^0(w_C^{4-c_1}) \).

Fix an integer \( v \geq 1 \), and let \( U(v) \) be the subset of \( H(d, g) \) obtained by varying \( C \subseteq U_M \subseteq (M(-1, 2, 0) \) and by varying the sections \( s \in H^0(I_C(v)) \) so that \( C = (s) \) is a curve, i.e. let \( U(v) = q(p^{-1}(U_M)) \) and regard \( U_M \) as a subscheme of \( M(c_1, c_2, 0) \) with \( c_1 = 2v-1, \ c_2 = 2-v+v^2, \ d = c_2 \) and \( g = 1 + \frac{1}{2} c_2 (c_1-4) \).

Recall that \( p \) and \( q \) are projection morphisms
\[
\begin{array}{ccc}
D & \xrightarrow{f} & H(d, g) \\
\downarrow & & \downarrow p \\
M(c_1, c_2, 0) & \xrightarrow{q} & M(c_1, c_2, 0)
\end{array}
\]
For \((C \subseteq \mathbb{P}) \in U(v)\), there is an exact sequence

\[0 \rightarrow O_{\mathbb{P}} \rightarrow \mathbb{P}(v) \rightarrow I_C(2v-1) \rightarrow 0\]

some \(\mathbb{P}(v) \in U_M\). Now (1.1.ii) and (2.1.ii) apply for \(v = 2\) and all \(v \geq 6\), and it follows that \(H(d,g)\) is smooth at any \((C \subseteq \mathbb{P})\) in the open subset \(U(v) \subseteq H(d,g)\). Moreover by the irreducibility of \(p\), \(U(v)\) is an open smooth connected subscheme of \(H(d,g)\).

Furthermore

\[
\dim U(v) = 4d + \frac{1}{6}v(v-5)(2v-5) \quad \text{for} \quad v \geq 6
\]

(resp. = 4d for \(v = 2\)) which asymptotically is \(\sim 4d + \frac{1}{3}d^{3/2}\) for \(v \gg 0\). To find the dimension of \(U(v)\), we use the fact that \(p\) and \(q\) are smooth morphisms of relative dimension \(h^0(\mathbb{P}(v))-1\) and \(h^0(\omega_C(4-c_1))-1\) respectively. This gives

\[
\dim U_M + h^0(\mathbb{P}(v)) = \dim U(v) + h^0(\omega_C(4-c_1))
\]

for \(v = 2\) and \(v \geq 6\), and since \(h^0(\omega_C(4-c_1)) = h^1(\mathcal{O}_C(c_1-4)) = 1\) for \(v \geq 6\) (resp. = 2 for \(v = 2\)), we get

\[
\dim U(v) = 10 + h^0(\mathbb{P}(v)) \quad \text{for} \quad v \geq 6
\]

(resp. = 9 + h^0(\mathbb{P}(v)) for \(v = 2\)). The reader may verify that

\[
h^0(\mathbb{P}(v)) = \chi(\mathbb{P}(v)) = \frac{1}{6}(v-1)(2v+3)(v+4) = 4d + \frac{1}{6}(v-5)(2v-5)v - 10
\]

for any \(v \geq 2\), and the conclusion follows.

We will now discuss the cases \(3 \leq v \leq 5\) where we can not guarantee the smoothness of \(q\) since (2.1.ii) does not apply. If \(v = 5\), then the closure of \(U(5)\) in \(H(22,56)\) makes a non-reduced component by (3.3). For \(v = 3\) or 4, we claim that \(H(d,g)\) is smooth along \(U(v)\) and the codimension
where \( W \) is the irreducible component of \( H(d,g) \) which contains \( U(\nu) \). To see this it suffices to prove \( H^1(\mathcal{N}_C) = 0 \) and \( \text{Ext}^2(\mathcal{O}_C(c_1), \mathcal{I}_C) = 0 \) for any \( (C \subseteq \mathbb{P}) \in U(\nu) \) because these conditions imply that the scheme \( D \) and \( H(d,g) \) are non-singular at any \( (C, \xi) \) with \( \xi \in H^0(\omega_C(4-c_1)) \) and \( (C \subseteq \mathbb{P}) \in H(d,g) \) respectively. See (1.1.i). Moreover if these "obstruction groups" vanish, we find

\[
\dim W - \dim U(\nu) = \dim W - \dim q^{-1}(U(\nu)) = h^0(\mathcal{N}_C) - \dim \text{Ext}^1(\mathcal{O}_C(c_1), \mathcal{I}_C) = h^1(\mathcal{O}_C(c_1 - 4))
\]

where \( \dim U(\nu) = \dim q^{-1}(U(\nu)) \) because of \( h^0(\omega_C(4-c_1)) = 1 \), and where the equality to the right follows from the long exact sequence of (2.2). Now to prove \( \text{Ext}^2(\mathcal{O}_C(c_1), \mathcal{I}_C) = 0 \) we use the long exact sequence (*) in the proof of (1.1.i) combined with \( H^1(\mathcal{I}_C) = 0 \) and \( \text{Ext}^2(\mathcal{I}_C, \mathcal{I}_C) = 0 \), and to prove \( H^1(\mathcal{N}_C) = 0 \) we use the long exact sequence of (2.2) combined with \( \text{Ext}^2(\mathcal{O}_C(c_1), \mathcal{I}_C) = 0 \) and \( \text{Ext}^3(\mathcal{O}_C(c_1), \mathcal{I}_C) \approx H^0(\mathcal{O}_C(c_1 - 4)) \vee = H^0(\mathcal{I}_C(c_1 - 4)) \vee = 0 \) for \( \nu = 3 \) or \( \nu = 4 \), and we are done.

Computing numbers, we find for \( \nu = 3 \) that \( U(3) \) is a locally closed subset of \( H(8,5) \) of codimension 1, and any smooth connected curve \( (C \subseteq \mathbb{P}) \in U(3) \) is a canonical curve, i.e. \( \omega_C \approx \mathcal{O}_C(1) \). For \( \nu = 4 \), \( U(4) \) is of codimension 2 in \( H(14,22) \) and \( \omega_C \approx \mathcal{O}_C(2) \) for any \( (C \subseteq \mathbb{P}) \in U(4) \).
Bibliography.


