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The notion of scecant scheme for quasi-projective morphisms

by

Audun Holme
University of Oslo, Norway

Introduction. First we extend the notion of secant variety introduced by E. Lluis in [6] for a projective variety embedded in some (fixed) $\mathbb{P}_k^N$, to any quasi-projective morphism $g: X \to Y$ (relative to a fixed Y-embedding $i: X \to \mathbb{P}_Y^N$), where Y is a (not necessarily separated) quasi-compact scheme. Even the introduction of reducible schemes, projective over a field k , makes it impossible to work with generic points in the classical sense: Trivial examples show that the secant scheme may be larger than the union of the secant varieties for each irreducible component.

Even so, it turns out that the estimate for the dimension of the secant variety given by Lluis in [6] holds also in the general case: It's less than or equal to $2\dim(X) + 1 - \dim(Y)$, cf. Theorem 2.

In the classical case one obtains a stratification of the secant variety $Sc(X)$ by letting $Sc(X)_i$ denote the closure of the union of all lines with $i + 1$ or more points in common with X . In fact, if $G_k(1,N)$ denotes the Grassmanian which parametrizes $\mathbb{P}^1$'s in $\mathbb{P}_k^N$, let $\Gamma \subset \mathbb{P}_k^N \times G_k(1,N)$ denote the incidence correspondence, and let $S_i \subset G_k(1,N)$ be the closure of the set of points which correspond to lines with $i + 1$ or more points in common with X . Then

$$Sc(X)_i = pr_1(pr_2{}^{-1}(S_i) \cap \Gamma)$$

gives what we want.

In general all $Sc(X)_i$'s may be equal, for example if X is a linear subspace. On the other hand, if X is a smooth curve, projective over the field K then $Sc(X)_1 = Sc(X)_2$ implies that X is either a line, or in characteristic 2, a plane conic ([9]). Furthermore, it follows by a theorem of Lluis ([7]) that if $Sc(X)_1 = Sc(X)_2$ for a projective smooth variety X in $\mathbb{P}_k^N$ $(N = 2\dim(X) + 1)$ where k is of characteristic zero, then X is contained in some $\mathbb{P}_k^{N-1}$. It may be shown, using a theorem in [3], that this holds in all characteristics, cf. [5].

The stratification of the secant variety for a projective variety may be carried out analogously for a quawi-projective morphism.

Finally, in the classical case one studies the subvarieties $Sc(X,q)_i$, defined similarly to $Sc(X)_i$, but with $\mathbb{P}^q$'s instead of $\mathbb{P}^1$'s. In [6] it is shown that

$$\dim Sc(X,q)_i \leq (q-i)(N-q) + q + (i+1) \dim(X) .$$

This construction may also be generalized, and a similar estimate for the dimension holds.

One reason why we think this is interesting, is the following:

If X is a smooth, projective scheme over the infinite field k, of dimension n, one can show that X may be embedded in $\mathbb{P}_k^{2n+1}$ (cf. [6], [3] and [5].) If X is a variety and $n + 1 \leq m \leq 2n$, then there is a projective variety $Y \subset \mathbb{P}_k^m$, a birational $f: X \to Y$ and a descending chain $\{Y_i\}$ $i=1,\dots,n$ of closed subsets in Y, such that for all $y \in Y_{i-1} - Y_i$, the geometric number of points in $f^{-1}(y)$ is equal to i, see [7]. Moreover, $\dim(Y_i) \leq n - i(m-n)$. If $m = 2n$, one may choose f such that the geometric number of points in $f^{-1}(y)$ is at most 2, cf. [3].

From this classical point of view one may proceed in two directions: One of them leads to <u>formal</u> embedding and projection theorems, [3]. Here the link with the classical case is that one applies the formal theory to the completion of the local ring at the vertex of the affine come over a projective scheme. But another problem is to look for simultaneous embeddings of families of projective schemes, parametrized by some scheme Y : In other words, given a projective morphism $g: X \to Y$, find minimal N such that there exists a Y-embedding $i: X \to \mathbb{P}^N_Y$.

Furthermore, in this setting one should be able to prove a "projection theorem" similar to what one has in the classical case.

The secant scheme. For later reference we first list the following well-known

Proposition 1. Let $x \in \mathbb{P}^N_k$ be a k-point, and let $\widetilde{\mathbb{P}^N_k}$ denote the blowing up with center x . If pr_1 and pr_2 are the projections, we have the commutative diagram

$$(1) \qquad \begin{tikzcd} \widetilde{\mathbb{P}^N_k} \arrow[r, hook] \arrow[d, "\pi_x"'] \arrow[drr, "\lambda_x"] & \mathbb{P}^N_k \times \mathbb{P}^{N-1}_k \arrow[dl, "pr_1"] \arrow[dr, "pr_2"] & \\ \mathbb{P}^N_k & & \mathbb{P}^{N-1}_k \end{tikzcd}$$

such that

i) λ_x is a $\mathbb{P}^1$-bundle.

ii) $\pi_x(\lambda_x^{-1}(y)) \ni x$, and is a projective line defined over $k(y)$.

iii) $y \longmapsto \pi_x(\lambda_x^{-1}(y))$ establishes a bijection between the k-points of $\mathbb{P}^{N-1}_k$ and the $\mathbb{P}^1_k$'s in $\mathbb{P}^N_k$ passing through x .

Using this proposition, it is easily verified that for any projective scheme X in $\mathbb{P}^N_k$, the closure of the union of all lines in $\mathbb{P}^N_k$ passing through $x \in X$ and at least one more point of X , is given by

$$Sc(x,X) = \pi_x(\lambda_x^{-1}(\lambda_x(\tilde{X})))$$

where $\tilde{X}$ denotes the blowing up of X with center x . Moreover, $Sc(x,X) \supseteq C(x,X)$, the tangential cone of X at x : In fact, we may assume $x = (1:0:\ldots:0)$, where $\mathbb{P}^N_k = \mathrm{Proj}(k[X_0,\ldots,X_N])$. Then π_x and λ_x gives the canonical $D_+(X_0) = \mathbb{A}^N_k \to \mathbb{P}^{N-1}_k$, mapping any cone with vertex x onto it's projectivisation. Finally $\lambda_x(\tilde{X}_x)$ is the projectivisation of $C(x,X)$, cf. [8], page 319.

Now let D be the diagonal of $\mathbb{P}^N_A \times \mathbb{P}^N_A$, and let $\underline{Z} \xrightarrow{\pi_D} \mathbb{P}^N_A \times \mathbb{P}^N_A$ be the blowing up with center D . A is any commutative ring with 1 . The basic construction is given by the following:

Proposition 2. There exists a scheme T of finite type over A, and morphisms λ and f such that the following diagram is commutative:

$$\begin{tikzcd}
 & & \underline{Z} \arrow[dl, "\pi_D"'] \arrow[dr, "\lambda"] & \\
 & \mathbb{P}^N_A \times \mathbb{P}^N_A \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & & T \arrow[dl, "f"] \\
\mathbb{P}^N_A & & \mathbb{P}^N_A &
\end{tikzcd}$$

Moreover, if A is a field and x is an A-point of $\mathbb{P}^N_A$, then the fiber of (2) via pr_2 is the diagram of Proposition 1.

Finally λ and f are projective fiber bundles with fibers $\mathbb{P}^{N-1}$ and $\mathbb{P}^1$, respectively.

Proof. Write $\mathbb{P}^N_A \times \mathbb{P}^N_A = \mathrm{Proj}(A[x_i\bar{x}_j \mid 0 \le i, j \le N]) = \mathrm{Proj}(A[x\bar{x}])$, the grading being defined by $x_i\bar{x}_j \in A[x\bar{x}]_1$. Put $U_i = D_+(\bar{x}_i)$, then $\mathbb{P}^N_A \times \mathbb{P}^N_A = \bigcup_{i=0}^{N} \mathbb{P}^N_A \times U_i$, and $D_i = D \cap (\mathbb{P}^N_A \times U_i)$ is the closed subscheme of $\mathbb{P}^N_A \times U_i = \mathrm{Proj}(A[\bar{x}_j/\bar{x}_i \mid j=0,\dots,N][x_0,\dots,x_N])$ defined by

$$x_b - z_{ib}x_i \ , \quad \text{for all} \quad b \neq i \ ,$$

where $z_{ij} = \bar{x}_j/\bar{x}_i$. If π_i denotes the blowing up with center D_i , we get the diagrm

$$\begin{array}{ccc} \underline{Z} & \hookleftarrow & \underline{Z}_i = \pi_D^{-1}(\mathbb{P}^N_A \times U_i) \\ \Big\downarrow{\scriptstyle \pi_D} & & {\scriptstyle \pi_i}\Big\downarrow \\ \mathbb{P}^N_A \times \mathbb{P}^N_A & \hookleftarrow & \mathbb{P}^N_A \times U_i \end{array}$$

Now put $y_{ib} = x_b - z_{ib}x_i$ for all $b \neq i$. Then $\mathbb{P}^N_A \times U_i = \mathrm{Proj}(A[z_{ib} \mid b \neq i][y_{ib} \mid b \neq i,\ x_i])$. The center D_i is given by

$$y_{ib} = 0 \quad \text{for all} \quad b \neq i \ .$$

Hence $\underline{Z}_i \hookrightarrow \mathrm{Proj}(A[X_{ia}Y_{ib} \mid 0 \leq a,b \leq N,\ b \neq i]) = U_i \times \mathbb{P}^N_k \times \mathbb{P}^{N-1}_k$ is defined by $X_{ia}Y_{ib} = X_{ib}Y_{ia}$ for $a \neq i$, $b \neq i$. The projection pr_{13} induces a morphism $\lambda_i\colon \underline{Z}_i \to U_i \times \mathbb{P}^{N-1}_k = T_i$. To prove is that these morphisms may be pieced together to $\lambda\colon \underline{Z} \to T$, and that $pr_1\colon U_i \times \mathbb{P}^{N-1}_k \to U_i$ may be pieced together to the morphism f . The rest of Proposition 2 is obvious.

We have $T_i = \mathrm{Proj}(k[Z_i][Y_{i,0},\dots,Y_{i,i-1},Y_{i,i+1},\dots,Y_{i,N}])$, and define/ $T_{ij} = \mathrm{Proj}(k[Z_i, 1/Z_{i,j}][Y_{i,b} \mid b \neq i]) = \mathrm{Proj}(S_{i,j})$ where $Z_{i,j} = X_j/X_i$. Note that $\lambda_i^{-1}(T_{i,j}) = \underline{Z}_i \cap \underline{Z}_j = \underline{Z}_{i,j}$ where $\underline{Z}_{i,j} = \mathrm{Proj}(s_{i,j})$ and $s_{i,j}$ is obtained from $S_{i,j}$ by replasing capital letters with small ones. The relations between $z_{i,b}$, $y_{i,b}$ and $z_{j,b}$, $y_{j,b}$ which identifies $\underline{Z}_{i,j}$ and $\underline{Z}_{j,i}$ suggests how the isomorphisms

$$g_{i,j} : T_{i,j} \to T_{j,i}$$

should be defined. In fact, let $g_{i,j}$ be induced by the isomorphisms

$$h_{i,j} : S_{j,i} \to S_{i,j}$$

given by

$$Y_{j,b} \longmapsto Y_{i,b} - (Z_{i,b}/Z_{i,j})Y_{i,j} \quad \text{for} \quad b \neq i$$

$$Y_{j,i} \longmapsto -(Y_{i,j}/Z_{i,j})$$

$$Z_{j,b} \longmapsto Z_{i,b}/Z_{i,j} \quad \text{for} \quad b \neq i$$

$$Z_{j,i} \longmapsto Z_{i,j}$$

It is easily verified that the morphisms $h_{i,j}$ satisfy the usual cocycle condition , so the schemes T_i may be glued together to a scheme T by means of the morphisms $h_{i,j}$.

Since the diagrams

$$\begin{tikzcd} \underline{Z}_{i,j} \arrow[r, "id"] \arrow[d] & \underline{Z}_{j,i} \arrow[d] \\ T_{i,j} \arrow[r, "g_{i,j}"] & T_{j,i} \end{tikzcd}$$

are commutative, it follows that the morphisms λ_i may be glued together to a morphism λ . Finally, to show that the morphisms f_i may be glued together, we need to verify that the diagrams

$$\begin{tikzcd} T_{i,j} \arrow[rr] \arrow[dr, "f_{i,j}"'] & & T_{j,i} \arrow[dl, "f_{j,i}"] \\ & \mathbb{P}^{N-1}_A & \end{tikzcd}$$

where $f_{i,j}$ is the restriction of f_i ,

are commutative. But $f_{i,i}$ is the composition of the canonical morphisms $T_{i,j} \to \mathrm{Spec}(k[Z_i, 1/Z_{i,j}]) \to \mathbb{P}^N_k = \mathrm{Proj}(k[X_0,\dots,X_N])$. This completes the proof.

Let Y be any quasi-compact scheme, and extend the base in (2) with $A = \mathbb{Z}$ to Y. We get the diagram

(3)
$$\begin{tikzcd} & \underline{Z}_Y \arrow[d, "\pi_Y"] \arrow[drr, "\lambda_Y"] & & \\ & \mathbb{P}^N_Y \times \mathbb{P}^N_Y \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & & T_Y \arrow[dl, "f_Y"] \\ \mathbb{P}^N_Y & & \mathbb{P}^N_Y & \end{tikzcd}$$

Further, let g be a quasi-projective morphism, and let i be a Y-embedding:

(4)
$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "g"'] & \mathbb{P}^N_Y \arrow[d] \\ & Y \end{tikzcd}$$

Then $X \times_Y X$ is a subscheme of $\mathbb{P}^N_Y \times \mathbb{P}^N_Y$. Let $\widetilde{X \times_Y X}$ denote the closure in $\underline{Z}_Y$ of $\pi_Y^{-1}(\widetilde{X \times_Y X} - D_{X/Y})$ where $D_{X/Y}$ is the diagonal. Finally, put

$$Sc(g,i) = pr_1(\lambda_Y^{-1}(\lambda_Y(\widetilde{X \times_Y X})))$$

where the images are the scheme theoretic ones, [1], (9.5). When no confusion can arise, we write $Sc(g)$. If $Y = \mathrm{Spec}(k)$ where k is a field, we write $Sc(X)$ instead of $Sc(g)$.

<u>Theorem 2</u>. i) For all $y \in Y$, $Sc(g)_y = Sc(X_y)$ and the similar relation holds for the geometric fibers

ii) $\dim(Sc(g) \leq 2\dim(X) + 1 - \dim(Y)$ (provided Y is of finite type over a field).

<u>Proof</u>. i) is clear by construction, ii) follows by the facts that $\dim(X \times_Y X) = 2\dim(X) - \dim(Y)$ and that λ_Y is a $\mathbb{P}^1$-bundle.

Now let $G_k(q,N)$ denote the Grassmanian which parametrizes $\mathbb{P}^q$'s in $\mathbb{P}^N_k$. Recall that $G_k(q,N) = \mathrm{Proj}(k[T_{i_0,\dots,i_q} \mid 0 \leq i_a \leq N]/I)$, where $K[T_{i_0,\dots,i_q} \mid 0 \leq i_a \leq N]$ is graded by $T_{\underline{i}} \in S_1$, $i = (i_0,\dots,i_q)$, and I is the homogenous ideal generated by the elements

(5)

$$\sum_{l=0}^{q+1} (-1)^l \, T_{i_1,\dots,i_q,j_l} \, T_{j_0,\dots,\hat{j_l},\dots,j_{q+1}}$$

for all $i_1,\dots,i_q$, $j_0,\dots,j_{q+1}$

$T_{\underline{i}} - \mathrm{sign}(\sigma) T_{\sigma(\underline{i})}$ for all permutations σ

$T_{\underline{j}}$ for all $\underline{j}$ such that $j_a = j_b$ for some $a \neq b$

Now define T in the same way as S, but with k replaced by $\mathbb{Z}$. Put $G_Y(q,N) = \mathrm{Proj}(T) \times_{\mathbb{Z}} Y$. Then it is clear by the above that

(6) $$G_Y(q,N)_y = G_{k(y)}(q,N)$$

and that a similar relation holds for the geometric fiber.

Further, put

$$\begin{aligned} \mathbb{P}_{\mathbb{Z}} &= \mathbb{P}^N_{\mathbb{Z}} \times_{\mathbb{Z}} \dots \times_{\mathbb{Z}} \mathbb{P}^N_{\mathbb{Z}} \times_{\mathbb{Z}} G_{\mathbb{Z}}(q,N) \\ &= \mathrm{Proj}(\mathbb{Z}[X_{0,j_0} \dots X_{i,j_i} T_{\underline{i}} \mid j_b = 0,\dots,N]) \\ &= \mathrm{Proj}(T^*) \end{aligned}$$

where T^* is graded by

$$X_{0,j_0} \dots X_{i,j_i} T_{\underline{i}} \in (T^*)_1 ,$$

and where the $X_{0,j_0} \dots X_{i,j_i} T_{\underline{i}}$'s satisfy the relations induced by (5). Assume from now on that $0 < i \leq q < N$.

Finally, let $\Gamma_{\mathbb{Z}}(i,q,N)$ be the closed subscheme of $\mathbb{P}_{\mathbb{Z}}$ defined by the ideal generated by the elements

$$\sum_{\delta=0}^{q+1}(-1)^{\delta}X_{0,j_0}\cdots X_{l,i_\delta}\cdots X_{i,j_i}T_{i_0,\ldots,\hat{i}_\delta,\ldots,i_{q+1}}$$

for all $0 \leq l \leq i$ and all $0 \leq i_a , j_b \leq N$.

As before, put $\mathbb{P}_Y = \mathbb{P}_{\mathbb{Z}} \times_{\mathbb{Z}} Y$ and $\Gamma_Y(i,q,N) = \Gamma_{\mathbb{Z}}(i,q,N)\times_{\mathbb{Z}} Y$. Then it's clear that for all $y \in Y$, $\Gamma_Y(i,q,N)_y = \Gamma_{k(y)}(i,q,N)$ is the incidence correspondence in $\mathbb{P}^N_{k(y)} \times\ldots\times \mathbb{P}^N_{k(y)} \times G_{k(y)}(q,N)$: a $k(y)$-point p is in $\Gamma_{k(y)}(q,N)$ if and only if the linear subspace of $\mathbb{P}^N_{k(y)}$ which corresponds to $pr_{i+2}(p)$ contains $pr_1(p),\ldots,pr_{i+1}(p)$. (For a proof, see for instance [3], Proposition (1.3).)

Now let $g\colon X \to Y$ be a quasi-projective morphism (where as before, it is understood that we fix a Y-embedding i).

Let $U = V\times_Y G_Y(q,N)$, where $V \subset X \times_Y \ldots \times_Y X$ denotes the commpoement of the diagonals.

Put $G_Y(g,q)_i = pr_{i+1}\overline{(U \cap \Gamma_Y(i,q,N))}$. Then, for all $y \in Y$, $(G_Y(g,q)_i)_y$ is the subset of $G_{k(y)}(q,N)$ consisting of the points which correspond to $\mathbb{P}^q$'s with $i+1$ or more points in common with X_y .

Let

$$\begin{array}{ccc} & \Gamma_Y(0,q,N) & \\ \swarrow^{p_1} & & \searrow^{p_2} \\ \mathbb{P}^N_Y & & G_Y(q,N) \end{array}$$

be the morphisms induced by the projections.

Then define

$$Sc(g,q)_i = p_1(p_2^{-1}(G_Y(g,q))$$

<u>Theorem 2</u>. i) $(Sc(g,q)_i)_y = Sc(X_y,q)_i$ and similar for the geometric fiber. Furthermore, if Y is an irreducible scheme of finite type over a field, and there exists a non empty open subset Y' of Y such that for all y in Y', $X_y\otimes\overline{k(y)}$ is not contained in

any $\Gamma^{q}_{\cancel{k(y)}}$ in $\Gamma^{N}_{\overline{k(y)}}$, then the following hold:

ii) $\dim(G_Y(g,q)_i) \leq (q-i)(N-q) + (i+1)\dim(X) - i\dim(Y)$

iii) $\dim(Sc(g,q)_i) \leq (q-i)(N-q) + q + (i+1)\dim(X) - i\dim(Y)$.

Proof. i) is immediate by the construction. For ii), we may assume that $Y = Y'$. It suffices to show that

$$\dim(\overline{U \cap \Gamma_Y(i,q,N)}) \leq (q-i)(N-q) + (i+1)(\dim(X) - \dim(Y))$$

Moreover,

$$\dim(\overline{U \cap \Gamma_Y(i,q,N)}) = \dim(U \cap \Gamma_Y(i,q,N))$$

since Y is of finite type over a field. Finally, if $y \in Y$ is a closed point, then

$$U \cap \Gamma_Y(i,q,N)_y \otimes_{k(y)} \overline{k(y)} = U_y \cap \Gamma_{\overline{k(y)}}(i,q,N)$$

where $U_y = V_y \times_{\overline{k(y)}} G_{\overline{k(y)}}(q,N)$ and $U_y \subset \overline{X}_y \times \ldots \times \overline{X}_y$ $(\overline{X}_y = X_y \otimes \overline{k(y)})$ denotes the complement of the diagonals, as before.

To complete the proof, it suffices to prove that

(5) $\dim(U_y \cap \Gamma_{\overline{k(y)}}(i,q,N)) \leq (q-i)(N-q) + (i+1)\dim(X_y)$

Indeed, $\dim(X_y) \geq \dim(X) - \dim(Y)$ and equality holds for all y in an open dense subset W_1 of Y. ([1],IV (11.1.1) and (6.1.2). Together with (5) this implies

$$\dim(U_y \cap \Gamma_{\overline{k(y)}}(i,q,N)) \leq (q-i)(N-q) + (i+1)(\dim(X)-\dim(Y))$$

for all $y \in W_1$. As before, there is an open dense subset W_2 of Y such that for all $y \in W_2$,

$$\dim(U \cap \Gamma_Y(i,q,N) = \dim(U \cap \Gamma_Y(i,q,N) - \dim(Y)$$

Hence taking $y \in W_1 \cap W_2$, we get

$$\dim(U \cap \Gamma_Y)(i,q,N))$$
$$= \dim(U \cap \Gamma_Y(i,q,N)_y) + \dim(Y)$$
$$\leq (q-i)(N-q) + (i+1)\dim(X) - i \dim(Y) .$$

To prove (5), it suffices to show the following: There exists an open dense subset W of $\overline{X_y} \times \ldots \times \overline{X_y}$, such that if $a = (a_0, \ldots, a_i) \in W$ is a closed point, then

(6) $\qquad \dim(U_y \cap \Gamma_{\overline{k(y)}}(i,q,N)_a) \leq (q-i)(N-q)$.

For this, we let W be the subset of $\overline{X_y} \times \ldots \times \overline{X_y}$ defined by

$(a_0, \ldots, a_i) \in W \iff$ the linear subspace of $\mathbb{P}^N_{\overline{k(y)}}$ spanned by $a_0, \ldots, a_i$ is of dimension i .

Clearly W is open and dense, and (6) follows by the well known fact that if $\mathbb{P}^i_K \subset \mathbb{P}^N_K$, then the set of points in $G_K(q,N)$ which correspond to $\mathbb{P}^q_K$'s containing $\mathbb{P}^i_K$, is closed, and isomorphic to $G_K(q-i-1, N-i-1)$ which is of dimension $((q-i-1)+1)(N-i-1(q-i-1)) = (q-i)(N-q)$ (cf. [2]).

iii) follows from ii) by means of the easily verified fact that p_2 is a $\mathbb{P}^q$ - bundle.

Q.E.D.

References

[1] Grothendieck, A.: "Élémentes de Géometrie Algébrique", Publ.Math. de l'Institut des Hautes Études Scientifiques.

[2] Hodge and Pedoe: "Methods in Algebraic Geometry".

[3] Holme, A.: "Formal embedding and projection theorems". To appeare. Preprints available from Department of Mathematics, University of Oslo.

[4] Holme, A.: "On the minimal dimension of the ambient space of a projective scheme", Matematisk Seminar / University of Oslo.

[5] Holme, A.: "Projective varieties where all secants are tripple". (Forthcoming)

[6] Lluis, E:: "Sur l'immersion des variétés algébriques", Annals of Math. 1955.

[7] Lluis, E.: "De las singularidacles que aparecen al proyectar variedacles algebraicas", Bol.Soc.Mat.Mexicana (1956).

[8] Mumford, D:: "Introduction to algebraic geometry".

[9] Samuel, P.: "Old and new results on algebraic curves". Tata Institude, Bombay.

CORRECTION.

The first line on page 10 should read as follows:

any $\mathbb{P}^q_{\overline{k(y)}}$ in $\mathbb{P}^N_{\overline{k(y)}}$, then the following holds: