ON THE BOHR TOPOLOGY

IN AMENABLE TOPOLOGICAL GROUPS

by

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Introduction.

In [1] E. M. Alfsen and P. Holm have characterized the Bohr compactification of a topological group $(G, \mathcal{T})$ as the completion of G with respect to a group topology $\mathcal{T}_B$ (the Bohr topology) which is coarser than $\mathcal{T}$. The purpose of this note is to prove that the general description of $\mathcal{T}_B$ can be simplified in amenable groups, i.e. groups admitting an invariant mean on the space of bounded left uniformly continuous functions.

In Section 1 of the present paper the existence of the Bohr compactification and how it is obtained from the Bohr topology is shown. The treatment is close to that of Alfsen and Holm, Theorem 1 is a slight improvement of Theorem 1 in [1], however. The main tool in Section 2 will be the upper and the lower mean value, and they are utilized in deriving elementary properties of invariant means. Subsets of an abelian group with positive upper or lower mean value have been characterized by P. Tomter, and his results are easily generalized to non-abelian groups.

Section 3 is a review of some properties of positive definite functions developed by R. Godement. The result we shall need states that on the linear space spanned by the positive definite functions we can define a convolution such that the convolution of two functions will be almost periodic. The last section is devoted to the description of the Bohr topology in amenable topological groups. Though stated in another way, our main result Theorem 5 has earlier been proved by E. Følner for abelian groups ([3], Theorem 1, and [4]) and his ideas are used extensively.

1.   The Bohr compactification.

From now on $(G, \mathcal{T})$ will be some fixed topological group with identity e. If ρ is a continuous homomorphism of G into a (Hausdorff) compact group $\hat{G}$, the pair $(\rho, \hat{G})$ is called the Bohr compactification of G if the following properties hold :

(a) $\rho(G)$ is dense in $\hat{G}$.

(b)   If ξ is any continuous homomorphism of G into a compact group H, then there is a continuous homomorphism $\xi': \hat{G} \to H$ such that $\xi = \xi' \circ \rho$.

Evidently, the Bohr compactification is unique up to an algebraic and topologic isomorphism.

In $[1]$ the existence of a Bohr compactification for a topological group $(G, \mathcal{T})$ is obtained by showing that $(G, \mathcal{T})$ admits a finest uniform structure $\mathcal{U}$ satisfying :

(1.1) $\mathcal{U}$ is totally bounded.

(1.2) $\mathcal{U}$ is compatible with the group structure, i.e. the group operations are uniformly continuous.

(1.3) $\mathcal{U}$ defines a topology on G coarser than $\mathcal{T}$.

In fact, we take $\hat{G}$ as the Hausdorff completion of G with respect to $\mathcal{U}$, and ρ as the canonical injection of G into $\hat{G}$. The group operations on $\rho(G)$ are well defined, and by uniform continuity they can be extended to the compact space $\hat{G}$. It is now easy to check that (a) and (b) are satisfied.

Our next observation is that if a uniform structure $\mathcal{U}$ satisfies (1.2), then $\mathcal{U}$ is completely determined by the associated group topology on G.

Lemma 1. Suppose $\mathcal{U}$ is a uniform structure on G compatible with the group structure, and let $\mathcal{T}'$ be the topology $\mathcal{U}$ induces on G. Then $(G,\mathcal{T}')$ is a topological group whose left and right uniform structures both are equal to $\mathcal{U}$.

Proof. See [1], Proposition 1.

Hence it suffices to look for a finest group topology on G satisfying the analogues of (1.1), (1.2) and (1.3).

Recall that a subset A of G is called left (right) relatively dense if there is a finite set $\{a_1,\cdots,a_n\}$ in G such that $G = \bigcup_{i=1}^{n} a_i A$ $(G = \bigcup_{i=1}^{n} A a_i)$. If A is both left and right relatively dense, A is called relatively dense. The right uniform structure of a topological group is totally bounded if and only if the left uniform structure is, and this is the case if and only if each neighbourhood of e is relatively dense. It is well known that in this case the left and the right uniform structures coincide. A proof of this fact is not so easily traced in the literature, so we include one for completeness.

Lemma 2. If $(G,\mathcal{T}')$ is a totally bounded topological group $\mathcal{U}_1$ $(\mathcal{U}_r)$ the left (right) uniform structure, then $\mathcal{U}_1 = \mathcal{U}_r$, and the group operations are uniformly continuous.

Proof. It is an easy established fact that the group operations are uniformly continuous if and only if $\mathcal{U}_1 = \mathcal{U}_r$, and this is the

case if and only if G admits a fundamental system of neighbour-
hoods of e whose members V are all <u>invariant</u> in the sense that
$x V x^{-1} = V$ for every x in G.

Let U be an arbitrary neighbourhood of e. Choose a
symmetric neighbourhood V of e such that $V^3 \subset U$. $G = \bigcup_{i=1}^{n} a_i V$
for some $a_1, \cdots, a_n \in G$. Let $V_1 = \bigcap_{i=1}^{n} a_i V a_i^{-1}$, and let
$W = \bigcup_{x \in G} x^{-1} V_1 x$. Then W is an invariant neighbourhood of e. If
$y \in V_1$ and x is arbitrary, we have $x \in a_i V$ for some i. Now

$$x^{-1} y x \in (a_i V)^{-1} (a_i V a_i^{-1}) a_i V = V^3 \subset U, \quad \text{so} \quad W \subset U,$$

and the lemma is proved.

The problem of finding a finest uniform structure satisfying
(1.1), (1.2) and (1.3) have now been reduced to find a finest group
topology on G coarser than the original one such that each neighbour-
hood is relatively dense. We now show the existence of such a topo-
logy by an explicit characterization.

<u>Theorem 1.</u> Every toplogical group $(G, \mathcal{T})$ admits a finest group
topology $\mathcal{T}_B$ satisfying

(1.4) $\mathcal{T}_B$ defines a totally bounded uniform structure.

(1.5) $\mathcal{T}_B$ is coarser than $\mathcal{T}$.

The left and the right uniform structures defined by $\mathcal{T}_B$ coincide,
and this uniform structure is the finest satisfying (1.1), (1.2)
and (1.3).

The neighbourhood system of e associated with $\mathcal{T}_B$ consists

of those subsets V of G which admit a sequence $\{V_n\}$ of sets such that

(1.6) $\qquad V_1{}^2 \subset V$ and $V_{n+1}{}^2 \subset V_n$ for $n = 1,2,\ldots$

(1.7) $\qquad$ Every V_n is a symmetric and relatively dense

$\qquad \mathscr{T}$-neighbourhood of e.

Proof. Let $\mathscr{V}$ be the subsets V of G which admit a sequence satisfying (1.6) and (1.7). We will show that $\mathscr{V}$ is the neighbourhood system of e for some topological group structure on G.

If U, $V \in \mathscr{V}$ with corresponding sequences $\{U_n\}$ and $\{V_n\}$, take $W = U \cap V$ and $W_n = U_n \cap V_n$. W_n contains the set $(U_{n+1}^{-1} U_{n+1}) \cap (V_{n+1}^{-1} V_{n+1})$ which is relatively dense ([1] , Proposition 3), hence $W \in \mathscr{V}$.

If $V \in \mathscr{V}$, obviously $V^{-1} \in \mathscr{V}$. Further $V_1 \in \mathscr{V}$ and $V_1{}^2 \subset V$. If $V \in \mathscr{V}$ and $a \in G$, then we see that $aVa^{-1} \in \mathscr{V}$ by taking $V_n' = aV_n a^{-1}$. Thus $\mathscr{V}$ defines a group topology $\mathscr{T}_B$ on G, $\mathscr{T}_B$ is coarser than $\mathscr{T}$ and defines a totally bounded uniform structure.

Suppose $\mathscr{T}'$ is another group topology on G satisfying (1.4) and (1.5). Every $\mathscr{T}'$-neighbourhood V of e then admits a sequence $\{V_n\}$ satisfying (1.6) and (1.7), so $\mathscr{T}' \subset \mathscr{T}_B$. By our previous remarks the uniform structure defined by $\mathscr{T}_B$ is the fin;st satisfying (1.1), (1.2) and (1.3).

The topology $\mathscr{T}_B$ is called the Bohr topology on G, and the corresponding uniform structure is denoted $\mathscr{U}_B$. We have seen that the Bohr compactification is the Hausdorff completion of G with respect to $\mathscr{U}_B$.

Let CB(G) be the set of continuous, bounded complex valued functions on G, and let CB(G) have the sup-norm-topology. The translates of a function on G is defined by

$$_af(x) = f(a^{-1}x) \qquad\qquad f_a(x) = f(xa)$$

A function f in CB(G) is called <u>almost periodic</u> if the set $\{_af : a \in G\}$ has compact closure in CB(G). We have the following important characterization of the almost periodic functions :

<u>Theorem 2.</u> For a function f in CB(G) the following are equivalent :

    (a) f is almost periodic.

    (b) f is $\mathscr{U}_B$-uniformly continuous.

    (c) There is a continuous function $\hat{f}$ on $\hat{G}$ such that

        $f = \hat{f} \circ \rho.$ $((\rho, \hat{G})$ is the Bohr compactification of G).

<u>Proof.</u> See [1], Theorem 2.

2.   <u>Invariant means and related subsets of the group.</u>

On BR(G) (= the bounded real valued functions on G) we define the right upper mean value $\overline{M}$ by

$$\overline{M}(f) = \inf\{\sup_{x \in G} \sum_i \alpha_i f(xa_i) : a_i \in G, \alpha_i > 0, \sum_i \alpha_i = 1\}.$$

The right lower mean value $\underline{M}$ is defined by $\underline{M}(f) = -\overline{M}(-f)$.

<u>Lemma 3.</u>   The right upper mean value $\overline{M}$ has the following properties :

(2.1) $\displaystyle\inf_{x \in G} f(x) \leq \overline{M}(f) \leq \sup_{x \in G} f(x)$

(2.2) $\overline{M}(\lambda f) = \lambda \overline{M}(f)$ for $\lambda \geq 0$.

(2.3) $\overline{M}(f_a) = \overline{M}(f)$ for $a \in G$.

(2.4) $\overline{M}(f-f_a) \leq 0$

(2.5) $\overline{M}(f+g) \leq \overline{M}(f) + \overline{M}(g)$ if G is abelian.

<u>Proof.</u>   Only part (2.4) needs a proof.   Take $a_1 = a$, $a_{k+1} = a_k \cdot a$. Then :

$$\overline{M}(f-f_a) \leq \sup_{x \in G} \frac{1}{n} \sum_{i=1}^{n} (f-f_a)(xa_i) = \sup_{x \in G} \frac{1}{n}(f(xa_1)-f(xa_{n+1}))$$

$$\leq \frac{2}{n} \|f\|_\infty$$

This holds for any n,   and (2.4) follows.

If A is a subset of G and χ_A is its characteristic function, it is easy to see that A is right relatively dense if and only if $\underline{M}(\chi_A) > 0$.   Sets with positive upper mean value have been studied by P. Tomter [8] in the abelian case, and we will transfer his ideas to arbitrary groups.

<u>Definition.</u>   A subset A of G is called <u>left (right) relatively accumulating</u> if there is a positive integer n_0 such that for any

positive integer m, at least $m+1$ of any mn_o+1 left (right) translates of A have a common, non-empty intersection. If A is left and right relatively accumulating, A is called relatively accumulating.

Proposition 1. Let A be a subset of G. Then

    (a) if A is left relatively dense, then A is right relatively accumulating,

    (b) if A is right relatively accumulating, then $A^{-1}A$ is right relatively dense,

    (c) A is right relatively accumulating if and only if $\overline{M}(\chi_A) > 0$.

Proof. (a), (b) and the "if"-part of (c) are proved essentially the same way as in $[8]$, pp.26-27. To prove the "only if"-part of (c), first note that for any $f \in BR(G)$

$$\overline{M}(f) = \inf\{\sup_{x \in G} \frac{1}{n} \sum_{i=1}^{n} f(xa_i) : a_i \in G, \ n \in \mathbb{Z} \}$$

Now suppose A is right relatively accumulating with respect to the number n_o, and let $a_1, \cdots, a_n$ be arbitrary elements from G. Let m be the number satisfying $mn_o < n \leq (m+1)n_o$. Then $n \geq mn_o+1$, and at least $m+1$ of the sets $Aa_1^{-1}, \cdots, Aa_n^{-1}$ have a common non-empty intersection. Thus

$$\sup_{x \in G} \frac{1}{n} \sum_{i=1}^{n} \chi_A(xa_i) \geq \frac{m+1}{n} \geq \frac{1}{n_o}$$

We conclude that $\overline{M}(\chi_A) \geq \frac{1}{n_o} > 0$.

Remark. In connection with (a), note that a left relatively dense subset is not necessarily _left_ relatively accumulating. An example of von Neumann can be used, take G to be the free group of two generators a and b, and let A be the set of elements beginning with a or a^{-1} when written as reduced words. $G = A \cup aA$, so A is left relatively dense. But A is not left relatively accumulating, for instance any two distinct members of the collection $\{A, bA, \cdots, b^n A\}$ have empty intersection.

Definition. Now let E be some linear space of complex valued functions on G which contains the constants and is closed under complex conjugation and right translations (i.e. $f \in A$, $a \in G$ => $f_a \in E$). A linear functional m on E is called a _right invariant mean_ (RIM) if

$$(2.6) \qquad m(\bar{f}) = \overline{m(f)}$$

$$(2.7) \qquad \inf_{x \in G} f(x) \leq m(f) \leq \sup_{x \in G} f(x) \quad \text{for any real valued } f \in E.$$

$$(2.8) \qquad m(f_a) = m(f).$$

(2.7) is equivalent to

$$(2.9) \qquad m(1) = 1 \quad \text{and} \quad m(f) \geq 0 \quad \text{for} \quad f \geq 0.$$

Left invariant means are defined analogously, and if m is both left and right invariant, it is called an invariant mean. If m is a RIM, and if $f \in E$ is real valued, we have $m(f) = m(\Sigma \alpha_i f_{a_i}) \leq \sup_{x \in G} \Sigma \alpha_i f(x a_i)$ for any convex combination $\Sigma \alpha_i f_{a_i}$ of translates of f. Thus $m(f) \leq \bar{M}(f)$, and we can conclude that

(2.10) $\underline{M}(f) \leq m(f) \leq \overline{M}(f)$.

If $\overline{M}$ is subadditive on E' (= the real functions in E), the Hahn-Banach theorem implies the existence of a linear functional m satisfying $m(f) \leq \overline{M}(f)$ for $f \in E'$. Applying (2.4) we find that m is a RIM on E', and m can uniquely be extended to a RIM on E. In particular the space of all complex valued functions on an abelian group will admit an invariant mean.

Definition. A topological group G is called amenable if there is a RIM on $UCB_1(G)$ (= the left uniformly continuous bounded complex valued functions on G).

This definition seems rather incidental, but for locally compact groups it is known that the existence of a RIM on the space of left and right uniformly continuous bounded functions implies the existence of a RIM on $L^\infty(G)$. Hence, for a locally compact group the existence of a RIM on any translation invariant linear space between these two spaces implies the existence of a RIM on any other space in between. The results in section 4 are valid not only for locally compact groups, and our choice of definition of amenability is motivated only by what is needed there.

A RIM is usually not strictly positive on positive, non-zero continuous functions. This is the case if and only if G is totally bounded.

3.   Positive definite functions.

We shall give a survey of those properties of positive definite functions we shall need later on. For details we refer to [6] or [2] section (13.4), and we note that the listed properties do not depend on local compactness of the group.

A continuous complex valued function ϕ on G is called positive definite if

$$\sum_{i,j} \lambda_i \overline{\lambda}_j \phi(y_j^{-1} y_i) \geq 0 \quad \text{for any} \quad \lambda_1, \cdots, \lambda_n \in \mathbb{C} \quad \text{and}$$

$$y_1, \cdots, y_n \in G.$$

A positive definite function is bounded, and we denote by P(G) the linear subspace of CB(G) spanned by the positive definite functions. It is well known that a function ϕ is in P(G) if and only if there is a continuous unitary representation U of G on a Hilbert-space H, and vectors $\xi, \eta \in H$ such that

$$\phi(x) = \langle \xi, U_x \eta \rangle \quad \text{for all} \quad x \in G.$$

This implies that P(G) is closed under translations. Since the product of two positive definite functions is positive definite, P(G) is an algebra under pointwise multiplication. P(G) contains the constants and is closed under complex conjugation and the operation $\sim$ defined as $\tilde{f}(x) = f(x^{-1})$.

Theorem 3. Over P(G) there is a unique invariant mean M. Its value on a positive definite function ϕ is given by :

$$(3.1) \qquad M(\phi) = \inf\{ \sum_{i,j=1}^{n} a_i a_j \, \phi(s_i^{-1} s_j) : s_i \in G, \ a_i > 0, \ \sum_i a_i = 1\}.$$

Proof. The for existence of M se [6], pp. 59-61. The unique-
ness can be proved by showing that a real valued function ϕ in
P(G) satisfies $\overline{M}(\phi) \leq M(\phi) \leq \underline{M}(\phi)$. Combined with (2.10) this
shows that M is unique.

The invariant mean M gives rise to a convolution over P(G).
For $\phi, \psi \in P(G)$ we define the function $\phi * \psi$ by

$$\phi * \psi(x) = \underset{t}{M}\left[\phi(t)\psi(t^{-1}x)\right],$$

with x an arbitrary fixed element of G, and t variable in G.

Theorem 4. P(G) is closed under the convolution defined above,
and $\phi * \psi$ is almost periodic for any ϕ, ψ in P(G).

Proof. Each function in P(G) is bounded and uniformly continuous,
and hence $\phi * \psi$ will be continuous. Define an involution $*$ by
$\phi^*(x) = \overline{\phi(x^{-1})}$, then

$$4 \phi * \psi^* = (\phi+\psi)*(\phi+\psi)^* - (\phi-\psi)*(\phi-\psi)^* + i(\phi+i\psi)*(\phi+i\psi)^*$$

$$-i(\phi-i\psi)*(\phi-i\psi)^*$$

$\phi*\phi^*$ is positive definite for any $\phi \in P(G)$, and this proves the first
part. For the proof that $\phi*\psi$ is almost periodic, see [6],
Théorème 15.

4. The Bohr topology in amenable topological groups.

We are now going to show that the characterization of the Bohr neighbourhoods given in Theorem 1 can be improved in amenable groups, in fact we shall prove that it suffices to have a finite chain of subsets of the sort described.

The following fundamental construction is due to Følner ([3]).

Lemma 4. Let G be a topological group, V a neighbourhood of e, and E a subset of G. Suppose that (A) and (A'), or (B) and (B') of the following conditions are satisfied :

(A) The right upper mean value $\bar{M}$ is subadditive over the space of real valued functions in $UCB_1(G)$.

(A') E is right relatively accumulating.

(B) There is a right invariant mean on $UCB_1(G)$, i.e. G is amenable.

(B') E is right relatively dense.

In both cases there is a non-zero, almost periodic and positive definite function $\psi: G \to [0,1]$ such that

$$\psi(e) > 0 \quad \text{and} \quad \psi(x) = 0 \quad \text{for} \quad x \notin (V^{-1}E^{-1}EV)^2.$$

Proof. There is a left uniformly continuous function $j: G \to [0,1]$ with $j(x) = 1$ for $x \in E$ and $j(x) = 0$ for $x \notin EV$. (This is proved in a way similar to the proof that a topological group is completely regular, see for instance [7],(8.2)).

If (A) is satisfied, the subadditivity of $\bar{M}$ implies (via

the Hahn-Banach-theorem) that there is a right invariant mean m on $UCB_1(G)$, and m can be chosen such that $m(j) = \overline{M}(j)$, (or any other number in the interval $\left[\underline{M}(j), \overline{M}(j)\right]$, cf. the considerations done in connection with the definition of invariant means). Together with (A') this gives

$$m(j) = \overline{M}(j) \geq \overline{M}(X_E) > 0.$$

If (B) and (B') are satisfied, we have

$$m(j) \geq \underline{M}(j) \geq \underline{M}(X_E) > 0.$$

Hence in both cases we have a right invariant mean m on $UCB_1(G)$ with $m(j) > 0$.

A function ϕ is defined by

$$\phi(x) = m(j_x \cdot j) = m_t\left[j(tx)j(t)\right].$$

The left uniform continuity of j implies that ϕ is continuous, and straight forward calculations show that ϕ is positive definite. $\phi(x) \geq 0$ for any x, and $\phi(x) = 0$ for $x \notin V^{-1}E^{-1}EV$. If M is the invariant mean on $P(G)$, we want to show that $M(\phi) > 0$. (Of course $M(\phi) = m(\phi)$). To this end we utilize the expression (3.1). If $\{a_i\}_1^n$ are positive numbers with $\sum_1^n a_i = 1$ and $\{s_i\}_1^n$ are elements from G, then by the right invariance of m we find that

$$\sum_{i,j} a_i a_j \phi(s_i^{-1} s_j) = m_t\left[(\sum_i a_i j(ts_i))^2\right] \geq (m_t\left[\sum a_i j(ts_i)\right])^2$$

$$= m(j)^2.$$

Thus $M(\phi) \geq m(j)^2 > 0$.

Now let $\psi(x) = \phi * \phi(x) = M_t\left[\phi(t)\phi(t^{-1}x)\right]$. We find that ψ is

positive definite and almost periodic by Theorem 4. $\psi(x) \geq 0$ for all x, and $\psi(x) = 0$ for $x \notin (V^{-1}E^{-1}EV)^2$. $\psi(e) = M(|\phi|^2) \geq |M(\phi)|^2 > 0$.

Theorem 5 A. Let $(G, \mathcal{T})$ be a topological group satisfying condition (A) in Lemma 4, W a subset of G, and let $\mathcal{T}_B$ be the Bohr topology on G. The following are equivalent:

    (i) W is a $\mathcal{T}_B$-neighbourhood of e.

    (ii) There is a right relatively accumulating subset E of G and a $\mathcal{T}$-neighbourhood V of e such that $(V^{-1}E^{-1}EV)^2 \subset W$.

Theorem 5 B. Suppose $(G, \mathcal{T})$ is an amenable topological group, (i.e. condition (B) of Lemma 4 is satisfied) and W a subset of G. The following are equivalent:

    (i) W is a $\mathcal{T}_B$-neighbourhood of e.

    (ii) There is a right relatively dense subset E of G and a $\mathcal{T}$-neighbourhood V of e such that $(V^{-1}E^{-1}EV)^2 \subset W$.

Proofs. If part (ii) is satisfied, in both cases there is an almost periodic function ψ with the properties in Lemma 4. Take

$$W_0 = \{x \in G: |\psi(x) - \psi(e)| < \psi(e)\}.$$

By Theorem 2 W_0 is a $\mathcal{T}_B$-neighbourhood of e, $W_0 \subset (V^{-1}E^{-1}EV)^2 \subset W$, so W is a $\mathcal{T}_B$-neighbourhood of e.

If W is a $\mathcal{T}_B$-neighbourhood, there is by Theorem 1 a symmetric, relatively dense $\mathcal{T}$-neighbourhood V with $V^8 \subset W$. Then take $E = V$, and E will also be relatively accumulating.

Theorem 5 B can be given in a weaker form which makes it clear that it is an improvement of Theorem 1.

Corollary 1. If $(G, \mathcal{T})$ is an amenable topological group, then a subset W is a $\mathcal{T}_B$-neighbourhood of e if and only if there is a symmetric, relatively dense $\mathcal{T}$-neighbourhood V of e with $V^7 \subset W$.

Proof. The "only if"-part was proved in the last part of the previous proof. If $V^7 \subset W$, take E = V and let U be a $\mathcal{T}$-neighbourhood of e satisfying $UU^{-1} \subset V$. Apply Theorem 5 B with E and U, and the conclusion follows from

$$(U^{-1}E^{-1}EU)^2 \subset U^{-1}E^{-1}EVE^{-1}EU \subset V^7 \subset W.$$

In abelian groups we can simplify even more, and since condition (A) always holds in this case, we have :

Corollary 2. If $(G, \mathcal{T})$ is an abelian topological group, a subset W is a $\mathcal{T}_B$-neighbourhood of 0 if and only if there is a symmetric, relatively accumulating $\mathcal{T}$-neighbourhood U of 0 such that $U^5 \subset W$.

Proof. Again, the "only if"-part is obvious. Conversely, if U satisfies the condition, let V be a symmetric neighbourhood of 0 with $V^4 \subset U$, and take E = U in Theorem 5 A.

If G is a discrete group, we may take V = {e} in Theorem 5 A. In this case the conditions (A) and (B) of Lemma 4 are equivalent ([5] Theorem 1), and we have

<u>Corollary 3.</u>   If G is an amenable discrete group, then a subset W is a $\mathcal{T}_B$-neighbourhood if and only if there is a right relatively accumulating subset E of G with $E^{-1}EE^{-1}E \subset W$.

For an amenable topological group let n be the minimal number such that V^n is a Bohr neighbourhood whenever V is a symmetric, relatively dense neighbourhood of e. We have seen that in general $n \leq 7$, $n \leq 5$ for abelian groups and $n \leq 4$ for discrete groups. A natural question is whether this number can be reduced for some special groups. The following example shows that in general we have $n \geq 1$.

Take the discrete group of integers $\mathbb{Z}$, and let V = {0, ±1, ±3, ±5,•••}, this set is symmetric and relatively dense. Since the characters on a group are almost periodic, the subset

$$U = \{n \in \mathbb{Z}: |e^{n\pi i} - 1| < 1\} = \{0, \pm 2, \pm 4, \pm 6, \cdots\}$$

is a Bohr neighbourhood of 0. $U \cap V = \{0\}$, thus V is not a Bohr neighbourhood. Hence for $\mathbb{Z}$ we have $2 \leq n \leq 4$. For the real numbers with the usual topology a similar argument shows that $2 \leq n \leq 5$.

Another question naturally arises, if G is not amenable, will then such an n exist,   or perhaps the finite chain characterization of the Bohr neighbourhoods (at least for locally compact groups) is equivalent to amenability. The answers to these questions are not known to the author.

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