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AN EXISTENCE THEOREM FOR A NON-CHARACTERISTIC CAUCHY-PROBLEM

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Introduction.

Let $f(x,y,z_1,z_2,z_3)$ be a continuous real-valued function in R^5. The Peano existence theorem for ordinary differential equations has been generalized to the problem

$$(1.1) \qquad u_{xy} = f(x,y,u,u_x,u_y) \quad u(0,y) = u(x,0) = 0$$

under certain extra conditions on f. See (3) for further details. Kamke (1) has proved that if proper data are given on a non-characteristic curve and if f is Lipschitz continuous in the z-variables then it follows that

$$u_{xy} = f(x,y,u,u_x,u_y),$$

has a unique solution u satisfying the given data. Here u, u_x, u_y, u_{xy} are supposed to be continuous.

The existence part of this theorem has been generalized by Walter (5). By analogy from existence theorems for (1.1) he proved the existence of a solution of the Kamke problem when the only condition on f is the continuity and some conditions on the z_2- and z_3-variables weaker than Lipschitz continuity. Theorem 2 in section 3 below is a similar extension of theorem 1 in (4). Theorem 1,(4), is a generalization of the Kamke theorem to equations of arbitrary order and to an arbitrary number of independent variables. These generalizations only cover Lipschitz continuity in the z-variables as an extra condition. The weaker condition of Walter type is not treated.

The notation and some definitions are given in section 2. In section 3 we restate theorem 1 in (4). We also state theorem 2 mentioned above. Theorem 2 can be proved almost exactly as theorem 3 in (3), if we use the notation in (4) and the variant of exponential majorization used there. We shall not give the proof here. It should be mentioned that the simplicity of the proof in (3) is due to the merger of an idea due to L. Gårding with exponential majorization. The reader may also consult (2) as an introduction.

2. Preliminaries. Let $x = (x_1,\dots,x_n) \in R^n$ and let $z = (z_1,\dots,z_N) \in C^N$. By $= \alpha(\alpha_1,\dots,\alpha_n)$ we denote a multi-index with non-negative integers as components. If $D_x = D = (\partial/\partial_{x_1},\dots,\partial/\partial x_n)$, then we write $D^\alpha = (\partial/\partial_{x_1})^{\alpha_1}\dots(\partial/\partial x_n)^{\alpha_n}$. We also write $|x| = |x_1| +\dots+|x_n|$, and

$$\alpha \le \beta \Leftrightarrow \alpha_j \le \beta_j, \qquad 1 \le j \le n.$$

Definition. Let $u(x)$ be a complex-valued function defined in all R^n, and let β be a multi-index. If all derivatives $D^\alpha u$, $\alpha \le \beta$ exist and are continuous together with u itself then we say that u belongs to the function-class $C(\beta,R^n)$.

Definition. The function $f(x,z)$ is a complex-valued function defined in all $R^n \times C^N$. If to every subset $K \subset R^n$ there exists a constant M such that

(2.1) $$|f(x,z) - f(x,z')| \leq M|z-z'|, \quad x \in K, \ z \in C^N, \quad z' \in C^N,$$

then we say that f belongs to the function class $CL(0,R^n \times C^N)$. Here we have let $|z-z'| = |z_1-z_1'|+\ldots+|z_N-z_N'|$.

3. Theorems for Cauchy problems.

We start by restating theorem 1 in (4).

Theorem 1. Let ϕ be a continuously differentiable real-valued function in R^n with $D_j\phi(x) > 0$, $x \in R^n$, $1 \leq j \leq n$, such that to every x there exist numbers $\bar{x}_j$, such that $\phi(x_1,\ldots,x_{j-1},\bar{x}_j,x_{j+1},\ldots,x_n) = 0$, $1 \leq j \leq n$. Let f belong to $CL(0,R^n \times C^N)$. Let β and α^k, $1 \leq k \leq N$, be multiindices such that

(3.1) $$\beta \geq \alpha^k, \quad \beta \neq \alpha^k, \qquad 1 \leq k \leq N.$$

Then it follows that there exists a unique function $u \in C(\beta,R^n)$ such that

(3.2) $$D^\beta u = f(x,D^{\alpha^1}u,\ldots,D^{\alpha^N}u),$$

and

(3.3) $$D^\alpha u(x) = 0, \quad \phi(x) = 0, \qquad \alpha \leq \beta, \quad \alpha \neq \beta.$$

We also state the following theorem.

Theorem 2. Let $f(x,z)$ be a complex-valued function defined in $R^n \times C^N$ having compact support. There exists an integer N', $1 \leq N' \leq N$, and a constant $M \geq 0$ such that

$$(3.4)\quad |f(x,z_1,\dots,z_{N'},z_{N'+1},\dots,z_N)-f(x,z_1,\dots,z_{N'},\bar{z}_{N'+1},\dots,\bar{z}_N)| \leq$$

$$\leq M \sum_{N'<k\leq N} |z_k-\bar{z}_k|,\ (x,z,\dots,z_{N'},z_{N'+1},\dots,z_N) \in R^n \times C^N,$$

$$(x,z_1,\dots,z_{N'},\bar{z}_{N'+1},\dots,\bar{z}_N) \in R^n \times C^N.$$

Let β and α^k, $1 \leq k \leq N$ be multi-indices such that

$$(3.5)\qquad \alpha^k_j < \beta_j \qquad 1 \leq j \leq n, \qquad 1 \leq k \leq N',$$

and also such that

$$(3.6)\qquad \alpha^k \leq \beta, \qquad \alpha^k \neq \beta, \qquad 1 \leq k \leq N.$$

Let ϕ be a function satisfying the conditions of the function ϕ in theorem 1.

Then it follows that there exists a function $u \in C(\beta,R^n)$ such that

$$(3.7)\qquad D^\beta u = f(x, D^{\alpha'}u,\dots, D^{\alpha^N}u),$$

and

$$(3.8)\qquad D^\alpha u(x) = 0, \qquad \phi(x) = 0, \qquad \alpha \leq \beta, \quad \alpha \neq \beta.$$

As to the proof see section 1.

References

(1) Kamke, E., Differentialgleichungen reeller Funktionen, Leipzig. Akademische Verlagsgesellschaft, 1930.

(2) Persson, J., Exponential majorization and global Goursat problems. Math. Ann. 178, 271-276 (1968).

(3) — An existence theorem for a general Goursat problem. To appear in J. diff.eq.

(4) — Non characteristic Cauchy problems and generalized Goursat problems in R^n. To appear in J.Math. and Mech.

(5) Walter, W., Über die Differentialgleichung $u_{xy} = f(x,y,u,u_x,u_y,u_{xy})$, III. Math. Z., 73, 268-279 (1960).