LOCAL REACHABILITY FOR DIFFERENTIAL CONTROL SYSTEMS WITH BANACH-VALUED TRAJECTORIES

By

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Local reachability for differential control systems with Banach-valued trajectories.

**Introduction.** In systems of differential equations depending on "controls", (parameters being functions of time), it is of importance to know if nearby points of the end point of a trajectory (solution) can be reached, by small variations of the controls.

This problem arises for example when one wants to establish Pontryagin maximum principles and [1], [2], [3], [4] contain such results implicitly in their proofs. Explicitely such results have been stated for example in [5], Ch.6,[6] for $\mathbb{R}^n$-valued trajectories. Below we give results on local reachability for switching-closed systems in the case of Banach valued trajectories. At the end of the paper we indicate how this result implies a maximum principle for such systems.

**Definitions.** $X$ is a Banach space, let $J = [0,1] \subset \mathbb{R}$ have the Lebesque measure. For $p \in [1,\infty]$, $L^p(J, X)$ is the set of Lebesque-measurable functions $f(.)$ such that $\|f(.)\|^p$ is integrable, or, for $p = \infty$, $\|f(.)\|$ is essentially bounded. (Measurability in the Bochner sense, [7].) $L_p(J, X)$ for $p \in [1,\infty]$ are the corresponding quotient spaces. Their elements are written $f(.)$. The
norms in these spaces are written \( \| f \|_p \). \( \mathcal{D}(J,X) \) is the set of maps \( x(\cdot): x(t) = x + \int g , \) for \( g \in \mathcal{L}^1(J,X) \), \( x \in X \). As a subset of \( \mathcal{C}(J,X) \) (the continuous maps \( J \to X \)), it is normed by the supremumsnorm \( \| \cdot \|_\infty \). Continuity and continuous differentiability with respect to two metrics \( \alpha \) and \( \beta \) (in the domain and range spaces, resp.) is written continuity \( (\alpha, \beta) \), continuous differentiability \( (\alpha, \beta) \).

Other topological concepts involving one or both norms are written similarly (e.g. convergence \( (\alpha), \mathcal{C}: \) with respect to \( \alpha \)). In a product of two spaces the product metric is denoted by \( \alpha \times \beta \). An open neighborhood of a subset (or point) \( x \) with "radius" \( \delta \) is written \( B(x, \delta) \). An error function \( e(d) \) is an extended realvalued nonnegative function on \( (0, \infty) \) such that \( \lim_{d \to 0^+} e(d) \) exists and equals zero. If \( Z, Z' \) are normed spaces, \( \mathcal{L}(Z,Z') \) denotes the set of continuous linear maps from \( Z \) into \( Z' \). A map \( g(\cdot): Z \to Z' \) is continuously differentiable on a convex subset \( A \subseteq Z \) if there is a continuous map \( g'(\cdot): A \to \mathcal{L}(Z,Z') \), and, for each \( a_0 \) in \( A \) an error function \( e(d) \) such that

\[
\| g(a) - g(a_0) - g'(a_0)(a - a_0) \| \leq e(d) \cdot \| a - a_0 \|, \quad a \in B(a_0, d)
\]

Usual properties hold also for this definition of continuous differentiation, in particular

\( ^1 \) this set topologized by the supremumsnorm.
(1) \[ \| g(a') - g(a) - g'(a_0)[a'-a]\| \leq \sup_{s \in [a':a]} \| g'(s) - g'(a_0) \| \cdot \| a' - a \| \]

for \( a', a, a_0 \in A \). See [8] Ch.VIII.

Let \( I \) be some set. A subset \( \mathcal{F} \) of \( X^I \times J \) is said to have property (SW) iff:
\[ g \in \mathcal{F}, g' \in \mathcal{F}; M < J, M \text{ measurable } \Rightarrow g \cdot M + g'(J - M) \in \mathcal{F} \]
(we apply the symbol of a set also as a symbol of its indicator function). \( \mathcal{F} \) has property (cw) iff it is closed in \( X^I \times J \) in the invariant pseudometric given by:
\[ \sigma(f,0) = \inf \{ \text{meas}(M)/M \supset \{ t/f(i,t) \neq 0 \}, \forall i \in I \} \]
(the \( M \)'s being measurable).

Linear differential equations. Let \( A(t) \in \mathcal{L}(X) = \mathcal{L}(X,X) \) for \( t \in J, A(\cdot) \in \mathcal{L}_0(J,\mathcal{L}(X)) \). Then the equation
\[ \dot{x}(t) = A(t)[x(t)] + g \text{ a.e. in } J, x(t) = x_0 + \int_0^t \dot{x}(s) \]
has a a.e. unique solution \( \dot{x}_g(\cdot) \) for each \( g \in \mathcal{L}_1(J,X) \), \( x_0 \) arbitrary in \( X \). \( g \rightarrow \dot{x}_g(\cdot) \) is continuous
\( (\| \cdot \|_1, \| \cdot \|_1) \).

The equation \( U'(t) = A(t) \circ U(t) \) has a a.e. unique solution \( \dot{C}_v(t) \in \mathcal{L}_0(J,\mathcal{L}(X)) \) on \( J \), such that \( C_v(0) = 1 \), \( v \in J \) and, if \( C_v(t)^{-1} = C(t)^{-1} \), \( x_0(1) = \int_0^1 C(s) \cdot g(s) ds \).
\( C_v(s) \) is called the "resolvent" of eq. (2). See [8]
Chp. X, (and [9] Ch. 3 Probl. 1., also valid in Banach space).

**Local reachability. Definitions.** Let $\mathcal{F}$ be a subset of $X^X \times J$. For $x(.) \in \mathcal{D}(J, X)$ define the property

(Rg): There are constants $M$ and $\delta$, both $> 0$ such that for all $f'' \in \mathcal{F}$ and all $x \in B(x(J), \delta)$, the following holds: $\| f''(x, t) \| \leq M \forall t \in J$, $f''(x, t)$ exists continuously at $x$, and $\| f''(x, t) \| \leq M$, both for all $t \in J$; and, finally, $f(x, .) \in \mathcal{D}_{C^0}(J, X)$, $f_x(x, .) \in \mathcal{L}_\infty(J, \mathcal{D}(X))$.

Let $x_0 \in X$, Call pairs $(\dot{x}(.), f) \in \mathcal{L}_1(J, X) \times \mathcal{F}$ fulfilling

$$\dot{x}(.) = f(x(.)), \quad \text{a.e. on } J, x(t) = x_0 + \int_0^t \dot{x}(.)$$

system pairs\(^1\). A point $x \in X$ is said to be reachable if there is a system pair $(\dot{x}(.), f)$ for which $x(1) = x$.

Let $(\dot{x}(.), \bar{f})$ be a fixed system pair such that (Rg) holds (for $\bar{x}(.)$). Let $A(.)$ of eq. (2) be the map $\bar{F}_x(\bar{x}(.), .)$, and denote the solutions $\dot{x}_g(.)$ of (2), for $g = f(\bar{x}(.), .) - \bar{F}(\bar{x}(.), .)$, by $\dot{q}_f(.)$. Then

$$q_f(1) = \int J C(s) \cdot (f(\bar{x}(.), .) - \bar{F}(\bar{x}(.), .))d\mu$$

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\(^1\) thus, implicitly assumed, such pairs $(\dot{x}(.), f)$ have the property that $f(x(.), .)$ is measurable.
Finally, let $cB(x, \delta) = \text{int} \{0, B(x, \delta)\}$, for balls $B(x, \delta)$ in $X$. Now we can state our main result.

**Theorem:** Let $\mathcal{F} \subset X^X \times J$ have the properties (SW) and (co) (for $I = X$), let $(\mathcal{F}(\cdot), \mathcal{F})$ be a system pair such that (Rg) holds for $\mathcal{F}(\cdot)$. Then, if $cB(p, 2\delta) \subset \text{co}\{q_f(1)/f \in \mathcal{F}\} = K$ where $\delta > 0$, $p \in X$, there is a $d > 0$ such that all points of $cB(dp, d\delta) + \mathcal{F}(1)$ are reachable.

We start the proof by considering solutions (pairs) for $f$ near $\mathcal{F}$. Let $B = B(\mathcal{F}(J), \delta)$, $B_1 = B(\mathcal{F}(J), \delta)$, $B' = B(\mathcal{F}(J), \delta)$ be $D(J, X)$, and norm $\mathcal{F} = \text{linspan} \mathcal{F}$ by $\|f\|^+ = \max\{\sup_{z \in B} \|f(z, .)\|_1, \sup_{z \in B'} \|f(z, .)\|_1\}$. By aid of the continuous differentiability in $B$, we may easily establish the following properties of the map $\mathcal{F} : B_1 \times \text{co} \mathcal{F} \rightarrow L_1(J, X)$, $F(\mathcal{F}(\cdot), f) = f(x(\cdot), \cdot), (x(t) = x_0 + \int_0^t \mathcal{F}(\cdot))$: $F$ has a partial derivative $F_f$ at $\mathcal{F}(\cdot)$ equal to the linear map $\mathcal{F}(\cdot) : f \rightarrow f(x(\cdot), \cdot); F_f(\cdot, \cdot)$ exists and is continuous $(\|\cdot\|_1 \times \|\cdot\|^+, \|\cdot\|_1)$ in $B_1 \times \text{co} \mathcal{F}$. The map $A : \mathcal{F}(\cdot) \rightarrow F(x(\cdot), \cdot) \in \text{co} \mathcal{F}$ is a partial derivative $F_{\mathcal{F}(\cdot)}$ of $F$ at $\mathcal{F}(\cdot) \in B_1 \cap \mathcal{F}$ being continuously $(\|\cdot\|_1 \times \|\cdot\|^+, \|\cdot\|_1)$ dependent on $(\mathcal{F}(\cdot), f) \in B_1 \times \text{co} \mathcal{F}$.

By eq. (2), we get that $(I-A)^{-1}$ exists as an element of $\mathcal{F}(L_1(J, X))$. Then, by the implicit function theorem (a slight extension of [8] 10.2.1), there is a convex neighborhood $N$ of $\mathcal{F}$ in $\text{co} \mathcal{F}$ and a map $f \rightarrow \mathcal{F}(\cdot)$ such that $\mathcal{F}(\cdot) = \mathcal{F}(\cdot)$ and $(\mathcal{F}(\cdot), f)$ fits in (3) for all $f \in N$. $\mathcal{F}(\cdot)$ is furthermore continuously differentiable $(\|\cdot\|^+, \|\cdot\|_1)$ in $N$, and at $\mathcal{F}$ the derivative is $(I-A)^{-1} \circ \mathcal{F}(\cdot), A = A$ evaluated at
Thus also \( y(f) = x_f(1) \) is continuously differentiable in \( N \), and its derivative \( y'(g) \) at \( \overline{f} \) is given by

\[
(5) \quad y'(g) = \int g(s)g(\overline{x}(.),.).ds = q_g(\overline{x}(.,.),)(1), \quad g \in \overline{f}
\]

Thus from (1) we get, for \( e(d) = \|
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By continuity of \( y' \), \( e(.) \) becomes an error function.

Now we need a technical result, being a sort of generalization of the nonlinear interior mapping theorem [10].

**Proposition.** \( \text{(Local reachability.)} \)

Let \( Y \) be a normed space, let \( A \) be a complete pseudometric space. Let \( e(d) \) be an errorfunction. Let \( e > 0, (e \in \mathbb{R}), a \in A \) and \( \overline{p} \in Y \). For each \( d \in \angle 0, d_0 \), \( d_0 > 0 \), let \( A_d \) be a subset of \( A \), \( \overline{a} \in A_d \) for all \( d \), and \( A_d \subset A_d' \) if \( d \leq d' \). Let \( y(.) : \overline{A}_d \rightarrow Y \), \( y'(.) : A \rightarrow Y \), define \( \overline{y}(a) = y(a) - y(\overline{a}) \). Let \( y'(\overline{a}) = 0 \) and let \( y(.) \) be continuous. Assume for all \( d \in \angle 0, d_0 \) that:

(A) \( \text{diam}(A_d) \leq M \cdot d, \quad M \text{ a constant} > 0. \)

(B) \( \text{For all } a, a'' \in A_d, k \in [0,1], \text{ there exists, for each } \varepsilon > 0, \text{ an } a' \in A_d \text{ such that} \)
\( (B_1) \quad \| k y'(a) + (1-k) y'(a) - y'(a') \| \leq \varepsilon, \quad \text{and} \)
\( (B_2) \quad d(a, a') \leq M d. \)

\( (C) \quad dy'(A) \subset \overline{y'(A_d)} \)

\( (D) \quad \| y(a') - y'(a') - y(a) - y'(a) \| \leq \varepsilon(d) \cdot d(a', a), \quad a', a \in A_d \)

Then if \( \overline{y'(A)} \supset \varepsilon B(2e, p) \), there is a \( d' \in \langle 0, d \rangle \)
such that \( \varepsilon B(2d, dp) + y(A) \subset y(A_d) \subset y(A) \) for all \( d \in \langle 0, d \rangle \).

Proof. Choose \( d' \) so that for all \( d \in \langle 0, d' \rangle \)
\( \varepsilon(d) \cdot M < \varepsilon / 4, \quad e = \varepsilon / 3. \) By \( (C) \) for each \( p \in B(\varepsilon, p) \) and \( d \in \langle 0, 1 \rangle \):

\( (C'): \quad \varepsilon B(d \varepsilon, dp) \subset \varepsilon B(d 2 \varepsilon, dp) \subset d ly(A) \subset il y(A_d) \)

If we prove that \( d \in y(A_d) \) for each \( d \in \langle 0, d' \rangle \)
and each \( p \in B(\varepsilon, p) \), then \( d \) \( d p \) is element of \( y(A_d) \subset y(A_d) \) for
\( k \in \langle 0, 1 \rangle \), and hence \( \varepsilon B(d \varepsilon, dp) \subset y(A_d) \).

We shall now prove that \( d \in y(A_d) \) for all
\( d \in \langle 0, d' \rangle, \quad p \in B(\varepsilon, p), \) and to this end we shall apply an
induction process of successive "convex" approximations.

In the induction step we shall use the following

Sublemma. Define \( \tilde{y}(a) = d \tilde{p} + y'(a) - y(a) \).
Let \( d(z) \) for \( z \in Y \)
mean the distance from \( z \) to \( \varepsilon B(\varepsilon, dp) \).
We then have:

For each \( a \in A_d \), for which \( d(y'(a)) \) and \( d(\tilde{y}(a)) \)
both are \( > \| d p - \tilde{y}(a) \| \), there exists an \( a' \in A_d \) such that
\( d(y'(a')) > \| d p - \tilde{y}(a') \|, \quad d(\tilde{y}(a')) > \| d p - \tilde{y}(a') \|, \)
\( \| d p - \tilde{y}(a') \| \leq \frac{1}{2} \| d p - \tilde{y}(a) \| \) \quad and \( d(a', a) \leq M \cdot \| d p - \tilde{y}(a) \| / 2e. \)
Proof. As \( y'(a) \) and \( \dot{y}(a) \) are in \( B(ed, dp) \), then
\[ u = ||y'(a) - \dot{y}(a)|| \leq 2 \cdot ed, \]
thus
\[ (2ed/u)(y'(a) - \dot{y}(a)) + y'(a) = \bar{h} \in B(3ed, dp), \]
and there is a point \( y'(a'), a'' \in A_d \), by ( ), such that \( ||y'(a') - \bar{h}|| \leq \frac{ed}{8} \). Now let \( a' \in A_d \) have the properties of (B) for
\[ k = \frac{u}{2ed} \] and \( \xi = \frac{ed}{8} \).

As \( kh + (1-k)y'(a) = \dot{y}(a) \), \( ky'(a') + (1-k)y'(a) \) is at a
distance \( \leq k \cdot ed/8 \) from \( y(a) \), thus:

\[ (b_1) \quad ||y'(a') - \dot{y}(a)|| \leq 2 \cdot \frac{ed}{8} = \frac{u}{8} \]

Now (D) and \( (B_2) \) implies that

\[ (b_2) \quad ||y(a') - y'(a') - \dot{y}(a) - y'(a)|| \leq e(d) \cdot Md \]

and \( e(d) \cdot Md \leq \frac{u}{8} \). Then
\[ ||y(a') - y'(a') - \dot{y}(a) - y'(a) - (\dot{y}(a) - y'(a'))|| = ||y(a') - dp|| \]
\( \leq \frac{u}{4} \leq \frac{u}{2} \). Next, \( (b_1) \) implies that \( dy'(a') \geq \frac{7u}{8} > \frac{u}{2} \),
and thus \( d(\dot{y}(a')) > \frac{u}{2} \) since \( ||\dot{y}(a') - y'(a')|| = ||\dot{y}(a') - dp|| \leq \frac{u}{4} \). By \( (B_2) \) \( d(a', a) \leq \frac{Mu}{2e} \), and the proof
of the sublemma is finished.

Now by (\( C \)), there is a \( y'(a_0), a_0 \in A_d \) such that
\( ||dp - y'(a_0)|| < \frac{ed}{4} \), that is, \( d(y'(a_0)) > \frac{3ed}{4} \). If we let
\( a' = \bar{a}, a_0 = a \) in (D) we get, by (A)

\[ (a) \quad ||\dot{y}(a_0) - y'(a_0)|| \leq \varepsilon(d) \cdot Md \]

and \( \varepsilon(d) \cdot Md \leq \frac{ed}{4} \). This gives that
\[ ||\dot{y}(a_0) - dp|| = ||y'(a_0) - \dot{y}(a)|| \leq \frac{ed}{4}, \] and
\[ \| dp-\bar{y}(a_0) \| < \varepsilon /2, \quad \text{(as } dp-\bar{y}(a_0) = dp-y'(a_0) + y'(a_0) - \bar{y}(a_0)\text{)}). \]

This implies that both \( d(y'(a_0)) \) and \( d(\bar{y}(a_0)) \) are

\[ \| dp-\bar{y}(a_0) \|. \]

By the sublemma, we may now find by induction a sequence \( a_0, a_1, a_2, \ldots \), such that for each

\[ n \geq 1 : \| dp-\bar{y}(a_n) \| = \frac{1}{2} \| dp-\bar{y}(a_{n-1}) \| , \quad d(a_n, a_{n-1}) \leq \| dp-\bar{y}(a_{n-1}) \| \cdot M/2\varepsilon; \]

and both \( d(y'(a_n)) \) and \( d(\bar{y}(a_n)) \) are

\[ \| dp-\bar{y}(a_n) \| , \]

such that the process of induction may be continued indefinitely. As \( \| dp-\bar{y}(a_n) \| = (\frac{1}{2})^n \| dp-\bar{y}(a_0) \| \leq \varepsilon /2^n \) we see that \( \{ a_n \} \) is a Cauchy sequence. Let

\[ a_n \rightarrow a \in \bar{A}. \]

Then \( \bar{y}(a) = dp \) by continuity. \( \quad \text{q.e.d.} \)

**The convexity property of switching.** Let \( J' = [0,1] \). Let \( \mathcal{A} \) be the set of finite unions of disjoint intervals of type \([a,b)\) in \( J' \). If \( h, h' \in \mathcal{L}_1(J,X), k \in [0,1] \), there

is, for each \( \varepsilon > 0 \) a set \( C_k \in \mathcal{A} \) such that

\[ \text{meas}(C_k) = k \]

and

\[ (7) \quad \| \int_{J'} kh + (1-k)h' \, d\mu - \int_{J'} h - C_k + h' \cdot (1-C_k) \, d\mu \| \leq \varepsilon \]

hence

\[ (8) \quad \| \int_{J'} h \cdot C_k + h! (1-C_k) \| \leq \varepsilon + \| \int_{J'} kh + (1-k)h' \| \]

and, likewise, if \( (h_1, h'_1), \ldots (h_n, h'_n) \) is a finite collection of pairs, we may find one \( C_k \) such that \( (a) \) is fulfilled for all indices \( i = 1, \ldots, n \). \( \text{(8) is easily seen to hold for piecewise constant functions, even for } \varepsilon = 0. \)
The general case is proved by approximating \( h \) and \( h' \) by piecewise constant functions. Compare [11] Sec.II lemma 1.)

**Proof of the theorem.** Observe that \( ||f' - f||^+ \leq M6(f', f) \).

If \( d_0 > 0 \) is so chosen that \( B(\bar{f}, d_0) \cap \co \bar{F} \subset N \), \( y(f) \) is defined for \( f \in B(\bar{f}, d_0) \cap \co \bar{F} \). Hence, we shall prove that the system \( (\bar{F}, \sigma), \bar{F}_d = B(\bar{f}, d) \cap \bar{F}, d \in \langle 0, d_0 \rangle \), \( \bar{F}, y(f) \) and \( y'(f) \) fulfill the conditions of the Proposition. The above observation gives that (6) implies (D), for \( \bar{e}(d) = e(d) \cdot M \), and continuity of \( y(f) \). To establish completeness it suffices to consider Cauchy sequences \( \{f_n\} \) of the type \( G(f_n, f_{n+1}) < \frac{1}{2^n+1} \). Then there exist sets \( C_{n+1} \) such that \( f_{n+1}(., t) \) differ from \( f_n(., t) \) only for \( t \in C_{n+1} \), and \( \text{meas}(C_{n+1}) < 1/2^{n+1} \). If \( B_n = \bigcup \{C_m/m \geq n+1\} \), we see that \( f_n \) differs from \( f_m \), \( m \equiv n+1 \) only on \( B_n \), and \( \text{meas}(B_n) < 1/2^n \). We may obviously find a \( f \in X^X \times J \) such that for all \( n, f = f_n \) on \( B_n^\prime \). Hence \( f_n \rightarrow f \); by (c) \( f \in \bar{F} \).

(B) Let \( f''', f \in \bar{F}_d, k \in \{0, 1\}, h = ky'(f'') + (1-k)y'(f) \). By formulas (5) and (7) there is a \( C_k \) such that for the element \( f' = f'' \cdot C_k + f \cdot (1-C_k) \), \( ||h - y'(f')|| \leq \varepsilon \). There are sets \( C'' \) and \( C \) with \( \text{meas}(C'') \) and \( \text{meas}(C) < d \), such that \( f'' \) (resp. \( f \)) differ from \( \bar{F} \) only on \( C'' \), (resp. \( C \)). Hence \( f' \) differ from \( \bar{F} \) only on \( C'' \cdot C_k + C \cdot (J-C_k) \) and by (8) we may choose \( C_k \) so that the measure of the former set is \( < d \), that is, \( f' \in \bar{F}_d^\prime \); (\( f' \) is element of \( \bar{F} \), by (SW)). Finally
\[ \mathcal{S}(f', f) \leq \sum_{i=1}^{1} (C'' + C) \cdot C_k \] and by (8) we may choose \( C_k \) also so that \( \mathcal{S}(f', f) \leq 2kd \), and (B) is proved.

(C). For any \( k \) slightly less than \( d \), (B,1) says that
\[ ky'(f') + (1-k)y'(\tilde{f}) = ky'(f') \] may be approximated as closely as wanted by an element \( f' = f'' \cdot C_k + \tilde{f} \cdot (1-C_k) \in \tilde{F} \). As \( \text{meas}(C_k) = k < d \), \( f' \in \tilde{F}_d \). Thus \( ky'(f') \in y'(\overline{\tilde{F}_d}) \), hence also \( dy'(f') \) is, for any \( f'' \) in \( \tilde{F} \), and (C) follows. This ends the proof of the theorem.

Remark. Let \( L' \) be a line through the origin in \( X \), let \( l \in X \), and \( \varphi \in X^\mathbb{K} \) be nonzero on \( L' \). Define \( L = L' + l \).
Let an admissible pair mean a system pair \((\hat{x}(.), f)\) such that \( x(1) \in L \). Suppose \((\hat{x}(.), \hat{f})\) is optimal in the problem of minimizing \( \varphi(x(1)) \) as function of pairs in the set of all admissible pairs. Assume now:

\[(\text{body}) \quad \text{int} \ K \neq \emptyset, \ K = \overline{\text{co}} \{ q_f(1)/f \in \tilde{F} \} .\]

If we assume by contradiction that \( L'^{-} = \{ x/x \in L'_1, \varphi(x) \leq 0 \} \) has points in common with \( \text{int} \ K \), the above theorem implies that a point on \( \overline{x(1)+L'^{-}} \subset L \) is reachable, contradicting optimality. Thus \( L'^{-} \) has to be weakly separated from \( \text{int} \ K \), thus also from \( K \). If \( p^\mathbb{K} \in X^\mathbb{K}, p^\mathbb{K} \neq 0 \), is so chosen that \( p^\mathbb{K}(K) \leq p^\mathbb{K}(L'^{-}) \) we get the following

Maximum principle. If \((\hat{x}(.), \hat{f})\) is optimal in the sense above and (body) holds, there is a \( p^\mathbb{K} \in X^\mathbb{K}, p^\mathbb{K} \neq 0 \), \( p^\mathbb{K} = \alpha \varphi \) on \( L' \), \( \alpha \leq 0 \), and \((\text{max}'): p^\mathbb{K}(K) \leq 0 \).
The property \( (\text{max}') \) may be rewritten, as is wellknown, in the following way

\[
(\text{max}) \quad \sup_{f \in F} \int f(\overline{x}(.),.) \, p(.) \, \text{d}\mu = \int \langle \overline{f}(\overline{x}(.),.) , p(.) \rangle \, \text{d}\mu
\]

where \( p(.) \) is the solution of

\[
\dot{p}(.) = -f^*_X(\overline{x}(.),.)[p(.)] \quad \text{a.e.} \quad p(1) = p^*.
\]

\( (p(.) \in \mathcal{L}_1(J,X^*), \ f^*_X(\overline{x}(t),t) \) meaning the adjoint of \( f_X(\overline{x}(t),t) \) \) (compare \([12]\) Ch.18, p.377).

Details of the arguments in this paper, and various generalizations may be found in \([13]\).
Literature.


