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1. Introduction. An algebraic curve C , say, of order n is given by an equation $F(x_1,x_2,x_3) = 0$ where $F(x_1,x_2,x_3)$ is a homogeneous polynomial of degree n in the homogeneous coordinates x_1,x_2 and x_3 of points in a plane.

A point P , say, whose coordinates satisfy the equations

$$\frac{\partial^{r-1} F}{\partial x_1^{K} \partial x_2^{l} \partial x_3^{m}} = 0$$

where $k + l + m = r-1$, $k,l,m \leq r-1$ is said to be a singular point of multiplicity r . (r is the highest possible number).

The number of different points of intersection between C and a straight line is in general n . If the straight line passes through P , however, P has to be counted at least r times if the total number of intersections should be n . To estimate the exact number of intersections at P between C and another algebraic curve passing through P , we may use the development of Puiseaux

$$y = ax + a_1 x^{\frac{v+v}{v}} + a_2 x^{\frac{v+v'+v''}{v}} + \dots + a_s x^{\frac{v+v'+\dots+v^{(s)}}{v}} + \dots$$

Another way is to use the concept of neighbour points which will be treated here. The neighbour points are introduced by means of quadratic transformations and are very important for birational geometry (a geometry treating properties of algebraic curves which are invariant under birational transformations or cremona transformations). As some properties of the cremona transformations will be used a short note on them will be given

2. A note on the cremona transformations. Consider three homogeneous polynomials of the same degree n $\varphi_1(x_1,x_2,x_3)$, $\varphi_2(x_1,x_2,x_3)$ and $\varphi_3(x_1,x_2,x_3)$ where x_1, x_2, and x_3 are homogeneous coordinates for points in a plane, σ. The relations

$$x_1' = \varphi_1(x_1,x_2,x_3)$$
$$x_2' = \varphi_2(x_1,x_2,x_3)$$
$$x_3' = \varphi_3(x_1,x_2,x_3)$$

establish a transformation from σ to a new plane, σ' in which the homogeneous coordinates are x_1', x_2', x_3'. Points of exception in σ are those making all the $\varphi_i(x_1,x_2,x_3)(i=1,2,3)$ zero. These are named the fundamental points or, briefly, F-points. The F-points are the base-points in the net

$$\lambda_1\varphi_1 + \lambda_2\varphi_2 + \lambda_3\varphi_3 = 0 \tag{2}$$

It may be assumed that this net is irreducible and that the curves φ_1, φ_2 and φ_3 are linearly independent.

A straight line, $\alpha_1 x_1' + \alpha_2 x_2' + \alpha_3 x_3' = 0$, say, in σ' will by the inverse transformation be mapped into the curve

$$\alpha_1\varphi_1 + \alpha_2\varphi_2 + \alpha_3\varphi_3 = 0$$

which is a member of the net (2). A second straight line in σ'

$$\beta_1 x_1' + \beta_2 x_2' + \beta_3 x_3' = 0$$

will in the similar manner be mapped into a second curve

$$\beta_1\varphi_1 + \beta_2\varphi_2 + \beta_3\varphi_3 = 0$$

in the net (2) in σ. A point P' of intersection between the two straight lines in σ' will then be transformed by the inverse transformation T^{-1} of T into the points of intersections between the two curves in (2) . If then the least number of intersections between two curves in the net (2) absorbed at the F-points is n^2-1 , the point P' will in general be mapped by T^{-1} into one point P only. The transformation is then one to one and is named a cremona transformation.

The reasoning breaks down if any two curves in (2) passing through P have a fixed part in common. The locus of such points is called the collection of fundamental lines of T or breifly the f-lines .

3. On the neighbourpoints. Concider an algebraic plane curve denoted by C , of order n having a point P as a singular point of multiplicity r and a quadratic transformation, T , say, having P as a fundamental point and whose fundamental lines through P do not coincide with any of the tangents of C at P . Let the transformed curve of C by T be denoted by C_1 , the inverse transformation of T by T^{-1} and the fundamental line of T^{-1} answering to P by p . Let P_1, P_2 , ... ,P_i be the points of intersection between C_1 and p different from the F-points of T^{-1} . If P_t (t=1,2...,i) are points of multiplicities r_t , the curve C is said to have points of multiplicities r_t in the first neighbourhood of P . By using a new quadratic-transformation T_q , having one of these points P_q , say, as F-point and having no f-lines coincciding with tangents to C_1 in P_q , C_1 may be transformed into a new curve C_2 meeting the f-line p_q of the inverse transformation T_q^{-1} answering to P_q in the points $P_{q,1}$,$P_{q,2}$, ... different from the F-points of T_q^{-1} . If the multiplicities of $P_{q,1}$,$P_{q,2}$, ... are $r_{q,1}$,$r_{q,2}$, ... respectively by C_2 , the

curve C is said to have points of multiplicities $r_{q,s}(s=1.2,\dots)$ in the second neighbourhood of P. As one of the points $P_{q,1}, P_{q,2}, \dots$ may be situated on the transformed line of p it is to be noticed that a point may be both in the first and second neighbourhood of P. Again a new q-transformation $T_{q,s}(s=1,2,\dots)$ may be used having $P_{q,s}$ as F-point and with no f-lines coinciding with any of the tangents to C_2 in $P_{q,s}$. By $T_{q,s}$ C_2 is mapped into a new curve C_3. If C_3 meets the f-line $p_{q,s}$ of the inverse transformation $T^{-1}_{q,s}$ in the points $P_{q,s,1}, P_{q,s,2}, \dots$ different from the F-points of $T^{-1}_{q,s}$ having the multiplicities $r_{q,s,1}, r_{q,s,2}, \dots$, these points are said to be in the third neighbourhood of P. Thus we may continue. After the series $T, T_q, T_{q,s}, \dots, T_{q,s}$ where $T_{q,s,\dots,1,k}$ is a q-transformation having the point $P_{q,s,\dots,1,k}$ situated on the f-line $p_{q,s,\dots,1}$ as F-point, the curve C is transformed successively into the curves $C_1, C_2, \dots, C_{t-1}, C_t$. If C_t meet the f-line $p_{q,s,\dots,1,k}$ of $T^{-1}{}_{q,s,\dots,1,k}$ answering to $P_{q,s,\dots,1,k}$ in the points $P_{q,s,\dots,l,k,1}$ $P_{q,s,\dots,1,k,2}, \dots$ different from the F-points of $T^{-1}_{q,s,\dots,1,k}$ these points are said to be in the neighbourhood of order t of P.

The handy concept of the neighbour points is as well known due to M. Noether   The connection between the concept of the neighbourpoints and the development of Puiseaux is treated in the books [1] and [2] of Eneriques and Chisini and Van der Waerden. O. Zariski has in [3] treated the singular points by means of modern algebra. The work of Zariski has been carried on and generalized by Northcott [4] and Hironaka [5]. We have also works of Pierre Samuel on the matter [6]. The simplest description of the singular points, at least by plane algebraic curves, is by means of the neighbourpoints. A problem arising in this method is whether we get another description by changing the series of q-transformations. Professor Aubert has pointed out to me that the proof of the invariance of this descriptoin is not regarded as obvious and a simple and elementary proof seems to be lacking. The aim of this paper is to give a such proof. The proof is independent of the development of **Puiseaux** and belongs strictly to the berational geometry. It is interesting, therefore, not only in connection with algebra.

The proof may be carried out by means of induction. Consider therefore the points in the first neighbourhood of a point P_o having the multiplicity r on an algebraic curve C_o in a plane ρ. By a q-transformation $T_{o,1}$, say, having P_o as F-point but with no f-lines coinciding with tangents of C_o in P_o, the curve C_o is transformed into the curve $C_{1,1}$ in the plane $\rho_{1,1}$. Let the f-line of $T^{-1}_{o,1}$ answering to the point P_o be denoted by $A_{1,1} B_{1,1}$ where $A_{1,1}$ and $B_{1,1}$ are the F-points. The dist**inct points** of intersection between $C_{1,1}$ and this line different from $A_{1,1}$ and $B_{1,1}$ may be denoted by $P^1_{1,1}$, $P^1_{1,2}$, $P^1_{1,3}$, ... and their multiplicities by $r^1_{1,1}$, $r^1_{1,2}$, $r_{1,3}$, ..., respectivily.

Let another q-transformation having P_o as a F-point but with no f-lines coinciding with the tangents to C_o at P_o, be denoted by $T_{o,2}$. It may at first be assumed that no F-points of $T_{o,2}$ are situated on any f-lines of $T_{o,1}$. The curve C_o is mapped by $T_{o,2}$ into a curve $C_{1,2}$ situated in a plane $\rho_{1,2}$. The f-line of $T^{-1}_{o,2}$ answering to P_o may be denoted by $A_{1,2}B_{1,2}$ where $A_{1,2}$ and $B_{1,2}$ are the F-points. The distinct points of intersection between this line and $C_{1,2}$ which are different from the F-points may be denoted by $P^2_{1,1}$, $P^2_{1,2}$, $P^2_{1,3}$,...

and their multiplicities by $r^2_{1,1}, r^2_{1,2}, r^2_{1,3}, \ldots$, respectively.

It will at first be proved that the lines $A_{1,2}\ B_{1,2}$ and $A_{1,1}B_{1,1}$ are corresponding lines in a cremonatransformation. Consider therefore the transformation S_1^{-1} established between the points in $\rho_{1,2}$ and $\rho_{1,1}$ by the transformations $T^{-1}_{o,2}$ and $T_{o,1}$ put together, in that order. S_1^{-1} is a cremonatransformation. The F-points of the inverse transformation S_1 are therefore the base-points of the net of curves which are the transforms of the straight lines in $\rho_{1,2}$. Now, these lines are transformed by $T^{-1}_{o,2}$ into conics in ρ having P_o as basepoint, but not the other F-point of $T_{o,1}$. This system of conics is therefore transformed by $T_{o,1}$ into a system (φ_1) of curves of order 3 in the plane $\rho_{1,1}$ where the third F-point, $D_{1,1}$ say, of $T^{-1}_{o,1}$ is a basepoint of multiplicity 2. The two other F-points $A_{1,1}$ and $B_{1,1}$ will be base-points of multiplicity 1. This net defines, then, the cremonatransformation S_1. In similar manner it is seen that the transformation S_1^{-1} is defined by a net (φ_1') having $A_{1,2}$ and $B_{1,2}$ as base-points of multiplicity 1, the third F-point $D_{1,2}$ say, of $T^{-1}_{o,2}$ as base-point of multiplicity 2 and the two transformed F-points distinct from P_o of $T_{o,1}$ by $T_{o,2}$ as base-points of multiplicity 1.

The straight line $A_{1,1}B_{1,1}$ has one point of intersection with the curves in (φ_1) different from the F-points of S_1. Points on this line will therefore be transfomed into a straight line in $\rho_{1,2}$. In order to find which line, we may consider a point on this line different from the F-points and not situated on the f-lines of S_1^{-1}. Such a point, not situated on $A_{1,2}B_{1,2}$ will be mapped by $T^{-1}_{o,2}$ into a point in ρ different from the F-point and the f-lines of $T_{o,1}$. By $T_{o,1}$ it will therefore be transformed into a point not situated on $A_{1,1}B_{1,1}$. The only possible line is then $A_{1,2}B_{1,2}$ as $A_{1,2}D_{1,2}$ and $B_{1,2}D_{1,2}$ are f-lines of S_1^{-1}. The straight line $A_{1,1}B_{1,1}$ is then transformed into the line $A_{1,2}B_{1,2}$ in $\rho_{1,2}$. Also, as the curve $C_{1,1}$ is transformed into the curve $C_{1,2}$, the points $P^1_{1,1}, P^1_{1,2}, \ldots$ are transformed into the points $P^2_{1,1}, P^2_{1,2}, \ldots$. If not, some of the first mentioned points also are situated on a f-line of S_1. Because of the restriction on the f-lines through P_o this is not the case, however. As corresponding points in a cremonatransformations have equal multiplicities on corresponding curves the integers $r^1_{1,1}, r^1_{1,2}, r^1_{1,3}, \ldots$ are equal to the integers $r^2_{1,1}, r^2_{1,2}, r^2_{1,3}, \ldots$ respectively if the latter are arranged in a certain order.

If the two q-transformations $T_{0,1}$ and $T_{o,2}$ have F-points situated on the other f-lines, the equality of the multiplicities and the cremonian relation between the distinct points of intersection appearing on the f-lines answering to P_o when using $T_{o,1}$ and $T_{o,2}$ may be seen in this way:

Let T denote a q-transformation having P_o as F-point but no other on the f-lines of $T_{o,1}$ or $T_{o,2}$. From the preceding it is known that the statement is true when using the transformations $T_{o,1}$ and T and also when using $T_{o,2}$ and T. The statement is therefore also true when using $T_{o,1}$ and $T_{o,2}$ as two cremonatransformations put togeher is a new one.

To prove the general case, consider the two series of q-transformations with distinct F-points $T_{o,i}, T_{1,i}, \ldots, T_{t-1.i}$ $(i = 1,2)$ mapping C_o into the series of curves $C_{1,i}, C_{2,i}, \ldots, C_{t,i}$ in the different planes $\rho_{1,i}, \rho_{2,i}, \ldots, \rho_{t,i}$ respectively. $T_{s,i}$ has a F-point $P_{s,i}$ which is a point of intersection between $A_{s,i}B_{s,i}$ and $C_{s-1,i}$. $A_{s.i}B_{s,i}$ is the f-line of $T^{-1}_{s-1,i}$ answering to the point $P_{s-1,i}$. If $A_{s,i}$ and $B_{s,i}$ are F-points of $T^{-1}_{s-1,i}$, $P_{s,i}$ are supposed to be different from these points. The third F-point of $T^{-1}_{s-1,i}$ may be denoted by $D_{s,i}$. The f-lines of $T_{s,i}$ are not supposed to coinside with any of the tangents to $C_{s,i}$ in $P_{s,i}$.

Suppose now that $P_{s,1}$ and $P_{s,2}$ $(s=1,2,\dots,t-1)$ are corresponding points in cremonatransformations T_s defined by a net (φ'_s) between the planes $\rho_{s,1}$ and $\rho_{s,2}$ mapping $C_{s,1}$ into $C_{s,2}$. In this case we are going to prove that the distinct points of intersection between $A_{t,1}B_{t,1}$ and $C_{t,1}$ and between $A_{t,2}B_{t,2}$ and $C_{t,2}$ are corresponding points in a new cremonatransformation mapping $C_{t,1}$ into $C_{t,2}$.

Consider the transformation T^{-1}_t obtained by putting $T^{-1}_{t-1,2}$, T^{-1}_{t-1} and $T_{t-1,1}$ together. T^{-1}_t is a cremonatransformation and it is therefore possible to find its order by considering a general straight line in $\rho_{t,1}$. This line is by $T^{-1}_{t-1,1}$ mapped into a conic passing through $P_{t-1,1}$, and the two other F-points of $T_{t-1,1}$. If the order of T_{t-1} is denoted by n_{t-1} and the multiplicities of (φ'_{t-1}) at the F-points of $T_{t-1,1}$ different from $P_{t-1,1}$ by a_{t-1} and b_{t-1} , the conic is by T_{t-1} transformed into a curve K of order $2n_{t-1} - a_{t-1} - b_{t-1}=n_k$ This curve is passing through $P_{t-1,2}$. If the multiplicities of K by the two F-points of $T_{t-1,2}$ different from $P_{t-1,2}$ are denoted by t_1 and t_2 , K is by $T_{t-1,2}$ transformed into a curve φ^2_t of order $n_t = 2n_k - t_1 - t_2 - 1$. The multiplicity of the curve at $A_{t,2}$ will be $n_k - 1 - t_1$ and at $B_{t,2}$ the multiplicity will be $n_k - 1 - t_2$. The net (φ^2_t) of algebraic curves defining T^{-1}_t has then the same order as (φ^2_t) and $A_{t,2}$

and $B_{t,2}$ are base-points (F-points of the transformation) of multiplicities $n_k - 1 - t_1$ and $n_k - 1 - t_2$. The curves in (φ_t^2) will therefore intersect the line $A_{t,2}B_{t,2}$ in one point different from the base-points of (φ_t^2). Hence $A_{t,2}B_{t,2}$ cannot be a f-line in T_t^{-1} and according to the usual ruels of cremona-transformations it will be mapped into a straight line in $\rho_{t,1}$.

In the same manner it is seen that T_t is defined by a net (φ_t^1) where the curves intersect the line $A_{t,1}B_{t,1}$ in one point different from the base-points. $A_{t,1}B_{t,1}$ is therefore mapped into a straight line. To find which line, assume that this line is different from $A_{t,2}B_{t,2}$. Such a line will have a point not situated on $A_{t,2}B_{t,2}$ or on any f-lines of T_t^{-1}. From following this point through the transformations $T_{t-1,2}^{-1}$, T_{t-1}^{-1} and $T_{t-1,1}$ it is seen, however, that this point is transformed by T_t^{-1} into a point not situated on $A_{t,1}B_{t,1}$. As $A_{t,2}D_{t,2}$ and $B_{t,2}D_{t,2}$ are f-lines of T_t^{-1} if $t_1 = t_2 = 0$ it follows that $A_{t,1}B_{t,1}$ is transformed into $A_{t,2}B_{t,2}$. As also $C_{t,1}$ and $C_{t,2}$ are corresponding curves in the same transformation, it follows that the points of intersection between $A_{t,1}B_{t,1}$ and $C_{t,1}$ correspond to those between $A_{t,2}B_{t,2}$ and $C_{t,2}$, $A_{t,1}$, $A_{t,2}$, $B_{t,1}$ and $B_{t,2}$ are excluded. The points have also then the same multiplicities and our statement is proved. If t_1 and t_2 are different from zero the same conclusion follows by inserting an auxiliary q - transformation.

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