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# On the Dirichlet problem of the Choquet boundary

## Summary

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The purpose of this summary is to sketch the proof of the following:

*Theorem.* *A continuous and bounded real (complex) valued function $f$ on the Choquet boundary $\partial X$ of a real (complex) sup-norm space $L$ over a compact Hausdorff space $X$ can be extended to a function in $L$ if and only if:*

(i) *There are no singular Shilov points for $f$,*

(ii) *$f$ is annihilated by every $L$-orthogonal boundary measure.*

Recall that $X$ is the smallest *representing boundary* for $L$, i.e. the smallest subset $Y$ of $X$ for which there exists a $\sigma$-field $\mathcal{F}$ containing $Y$ and all Baire sets, and a map $x \rightarrow \mu_x$ of $X$ into the probability measures on $\mathcal{F}$, such that for every $x$ $X$ :

$$(1) \qquad \mu_x(Y) = 1$$

$$(2) \qquad a(x) = \int a \, d\mu_x , \qquad \text{all } a \in L .$$

It appears that $\partial X$ is a natural set for prescribtion of boundary values.

When $Y = \partial X$, we may as well chose $\mathcal{F}$ to be the $\sigma$-field $\mathcal{F}_0$ generated by $\partial X$ and all Baire sets. A measure on $\mathcal{F}_0$ which vanishes on $\complement \partial X$, is called a *boundary measure*. A measure is said to be $L$-orthogonal if it annihilates all functions in $L$.

Recall that $\overline{\partial X}$ is the Shilov boundary for $L$. A point $x \in \overline{\partial X}$ is said to be a *singular Shilov point* for a bounded real valued function $f$ on $\partial X$ if

$$(3) \quad \sup\{a(x) \mid a|\partial X \leqq f\} < \inf\{b(x) \mid f \leqq b \mid \partial X\},$$

where a,b are in the space $L_r$ of real parts of functions in L . Similarly x is singular for $f = f_1 + i f_2$ if it is singular for either $f_1$ or $f_2$ (or both). Note that a point $x \in \partial X$ is <u>non-singular</u> for every continuous and bounded function f on $\partial X$ , and that every point $x \in \overline{\partial X} \setminus X$ is <u>singular</u> for some continuous and bounded function f on $\partial X$ .

Clearly (i), (ii) are necessary conditions that f be extendable to a function in L . If $\partial X$ is closed, then the condition (i) is automatically satisfied. In the general case it is non-redundant.

<u>Example</u>. Let $X = [0,1] \cup \{i\} \cup \{-i\}$ ,
and consider

$$L = \{ f \in C(X) \mid 2f(0) = f(i) + f(-i) \} .$$

Here $\partial X = X \setminus \{0\}$ , and there is (up to a constant factor) only one L-orthogonal measure on X , namely $\nu = 2\,\varepsilon_0 - \varepsilon_i - \varepsilon_{-i}$ . The function f which is identically zero on $]0,1]$ and is 1 on i and -i , will be uniformly continuous on $\partial X$ , but it is evidently not extendable to any function in L . Observe that 0 is in fact a singular Shilov point for f .

The proof of sufficiency is based on a general "lifting" technique. Let $M(\overline{\partial X})$ be the Banach space of (real or complex) Baire measures on $\overline{\partial X}$ and define maps

$$\begin{array}{ccccc} & & M(\overline{\partial X}) & & \\ & & \varrho\big\downarrow & \searrow^{\varphi} & \\ X & \xrightarrow{\ \psi\ } & L^* & \xrightarrow[\ \varphi'\ ]{} & \mathbb{C} \end{array}$$

as follows:

$$(4) \qquad \psi(x)(a) = a(x) , \qquad \text{all } a \in L, x \in X$$

$$(5) \qquad \varrho(\mu)(a) = \int a \, d\mu , \qquad \text{all } a \in L, \mu \in M(\overline{\partial X})$$

$$(6) \qquad \varphi(\mu) = \int_{\overline{\partial X}} \tilde{f} \, d\mu , \qquad \text{all } \mu \in M(\overline{\partial X}) ,$$

where $f$ is the continuity extension of $f$ from $\partial X$ to $\overline{\partial X}$. (Note that (i) entails uniform continuity.)

Finally:

$$(7) \qquad \varphi'(q) = \int_{\partial X} f \, dm ,$$

where $m$ is any probability boundary measure which <u>represents</u> the linear functional $q$, i.e. for which

$$(8) \qquad q(a) = \int_{\partial X} a \, dm , \qquad \text{all } a \in L .$$

Note that (i) entails $F_o$-measurability of $f$, and that the definition of $\varphi'$ is non-ambiguous by virtue of (ii).

Clearly $\psi, \varrho, \varphi$ are continuous w.r. to the given topology of $X$ and the w*-topologies on $L^*$ and $M(\overline{\partial X})$. The w*-continuity of $\varphi'$ is the crucial point. We shall derive it from the continuity of $\varphi$ and $\varrho$ after proving that the diagram is commutative.

The proof of commutativity is based on certain norm- and order- preserving properties of the linear functionals on $L$ . Specifically, $L^*$ admits a Jordan-decomposition with bounds on the norms, and we shall have a general estimate

$$(9) \qquad |\varphi'(\varrho(\mu))| \leqq 2\sqrt{2}\,\|\mu\|\cdot\|f\| , \qquad \mu \in M(\overline{\partial X}),$$

and a more special estimate

$$(10) \qquad |\varphi'(\varrho(\mu)) - \varphi(\mu)| \leqq \sum_{j,k=1}^{2} \int_{\overline{\partial X}} (b_j - a_j)\, d|\mu_k| ,$$

where $\mu \in M(\overline{\partial X})$, $\mu = \mu_1 + i\mu_2$ , $f = f_1 + i f_2$ and $a_j, b_j$ are functions in $L_r$ such that

$$(11) \qquad a_j | \partial X \leqq f_j \leqq b_j | \partial X , \qquad j = 1,2.$$

The estimate (10) is useful if $a_j, b_j$ can be found such that $b_j - a_j$ is small on the support of $\mu$ . In the sequel we shall approximate a given measure on $\overline{\partial X}$ by a sum of measures for which this is possible. The inequality (9) will take care of the remainder term.

Let $M(\overline{\partial X})$ and $\varepsilon > 0$ be arbitrary. For every Baire subset $B$ of $\overline{\partial X}$ we define $\Phi(B)$ to be the (possibly empty) set of all quadrouples $(a_1, b_1; a_2, b_2)$ from $L_r$ satisfying (11) and

$$(12) \qquad b_j | B - a_j | B \lneqq \varepsilon , \qquad j = 1,2.$$

At this point we invoke the requirement (i) in an essential way to construct a sequence $\{B^n\}$ of mutually disjoint Baire subsets of $\overline{\partial X}$ such that $\Phi(B^n) \neq \emptyset$ and $\{B^n\}$ cover $\overline{\partial X}$ up to a $\mu$-null set. Then we can find sequences $\{a_j^n\}$, $\{b_j^n\}$, $j = 1,2$, from $L_r$ such that

(13) $a_j^n | \partial X \leqq f_j \leqq b_j^n | \partial X$ , $j = 1,2,$

(14) $b_j^n | B^n - a_j^n | B^n \leqq \mathcal{E}$ , $j = 1,2,$

and

(15) $$\mu = \sum_n \mu_n ,$$

where $\mu_n$ is the restriction of $\mu$ to $B^n$ .

We shall not go into details concerning the inductive construction of the sequences, but we observe that by (10), (13) and (14), we shall have the following inequality for every n :

(16) $$|\varphi'(\rho(\mu_n)) - \varphi(\mu_n)| \leqq \sum_{j,k=1}^{2} \int (b_j^n - a_j^n) d|\mu_k^n| \leqq 4\mathcal{E}\, |\mu|(B^n) .$$

Now choose a natural number N such that

(17) $$\sum_{n>N} |\mu|(B^n) < \mathcal{E},$$

and define $$\mu_0 = \sum_{n>N} \mu_n .$$

By (9) and (17):

$$|\varphi'(\rho(\mu)) - \varphi(\mu)| \leqq \sum_{n=0}^{N} |\varphi'(\rho(\mu_n)) - \varphi(\mu_n)| \leqq 4(\|f\| + \|\mu\|)\mathcal{E} .$$

Since $\mathcal{E} > 0$ was arbitrary, this completes the proof that the diagram is commutative.

To prove $w^*$-continuity of $\varphi'$, we consider a closed subset $F$ of $\mathbb{R}$. We observe that $\rho$ maps the unit ball $M_1(\overline{\partial X})$ onto the unit ball $L_1^*$; hence by commutativity:

$$L_1^* \cap (\varphi')^{-1}(F) = \rho(M_1(\overline{\partial X}) \cap \varphi^{-1}(F))$$

By $w^*$-continuity of $\varphi$ and $\rho$, and by $w^*$-compactness of $M_1(\overline{\partial X})$, the set $L_1^* \cap (\varphi')^{-1}(F)$ is closed. Hence $\varphi' | L_1^*$ is proved to be $w^*$-continuous.

By the Theorem of Banach- Dieudonné (or Krein- Šmulyan), $\varphi'$ is $w^*$-continuous, and so there is an $\bar{f} \in L$ such that $\varphi'(q) = q(\bar{f})$ for every $q \in L^*$.

By definition

$$\bar{f}(x) = \psi(x)(\bar{f}) = \varphi'(\psi(x)) = \int_{\partial X} f \, dm ,$$

where $m$ is any boundary measure reprecenting $\psi(x)$. If $x \in \partial X$, then we may choose $m = \varepsilon_x$ to obtain $\bar{f}(x) = f(x)$. Hence $\bar{f} \in L$ is the desired extension of $f$.

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Note that [2] contains a metrizable version of the above theorem in the "geometric" case (for affine real valued functions on a compact convex set). Note also that A. Lazar and E. Effros have proved that metrizability can be avoided in the case of a Choquet simplex [3] ,[4] .

A complete proof of the theorem is given in [1] .

References:

E. M. Alfsen, On the Dirichlet problem of the Choquet boundary. (To appear).

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