Partly Gentle Perturbation
with Application to Perturbation
by Annihilation-creation Operators

by

J.R. Høegh-Krohn
Introduction.

The notion of Partly Gentle Perturbation is due to P.A. Rejto. It is a natural extension of Friedrichs' notion of Gentle Perturbation.

For a self-adjoint operator $H_0$ with an absolutely continuous spectrum Friedrichs used the notion of gentle perturbation to prove that for small values of $\lambda$ the perturbation $H = H_0 + \lambda V$ has also an absolutely continuous spectrum and is unitarily equivalent with $H_0$, provided that $V$ satisfies certain gentleness conditions.

Rejto considers the case where $H_0$ has an absolutely continuous spectrum only in a certain interval $I$, and he gives conditions on $V$ so that $H = H_0 + \lambda V$ also have absolutely continuous spectrum in the same interval $I$, and the parts of $H$ and $H_0$ corresponding to $I$ are unitarily equivalent.

Rejto called perturbations of this type Partly Gentle Perturbation.

This paper contains without proof the main result obtained in my Ph.D. thesis at Courant Institute for Mathematical Sciences, New York University, where Friedrichs was my advisor. A new definition of Partly Gentle Perturbation is given, which in some sense is more general than Rejto's. This definition is then applied to perturbation of the Laplacian by a multiplication operator and fairly strong results are obtained.

The main application is the perturbation by annihilation-creation operator. It was this problem that led to my definition of Partly Gentle Perturbation. The perturbation by annihilation-creation is modeled on the relativistic quantum field theory, and it has some of the difficulties of relativistic quantum field theory.
We consider only none-local interactions with smooth kernels and no vacuum interaction, and in this case theorem 7, 8 and 9 show that the perturbation by annihilation-creation operators behaves nicely.

I am happy to record my gratitude to K.O. Friedrichs for introducing me to the problem and also giving me help and advice in solving it. I would also like to thank many of the other members at Courant Institute for Mathematical Sciences for helpful discussions.
CHAPTER I

The general theory of
Partly Gentle Perturbations.

1. Basic notions.

Let $H_0$ be a self-adjoint operator acting in an abstract Hilbert-space $\mathcal{H}$. We will assume that $\mathcal{H}$ is separable. The domain of $H_0$ will be denoted by $D_{H_0}$. Let $V$ be a symmetric operator with domain containing the domain of $H_0$.

We will assume that $V$ is essentially bounded by $H_0$, and by this we mean that there exist two positive real numbers $a$ and $b$ such that

$$\| V \psi \| = a \| \psi \| + b \| H_0 \psi \|$$

for all $\psi \in \mathcal{H}$.

In this case it is well known that the operator

$$H = H_0 + \lambda V$$

is self-adjoint with domain $D_{H_0}$, for $|\lambda| < b$.

Let $E(\lambda)$ and $E_0(\lambda)$ be the family of spectral projections corresponding to $H$ and $H_0$, i.e.
Let \( I \) be any interval on the real line, by \( E^i \) or \( E_\circ^i \), we shall understand the spectral projection on the interval \( I \) corresponding to \( H \) or \( H_0 \), i.e.

\[
E^i = \int_I E(\lambda) \\
E_\circ^i = \int_I E_\circ(\lambda)
\]

Let \( B_1 \) and \( B_2 \) be two separable Banach-spaces. We will say that \( B_1 \) and \( B_2 \) have the regular intersection-property iff:

(i) There is a sequence of elements \( \{ f_n \} \) in \( B_1 \cap B_2 \) such that \( \{ f_n \} \) is dense in \( B_1 \) and in \( B_2 \).

(ii) The two identity mappings \( I_1 \) and \( I_2 \)

\[
I_1 : B_1 \rightarrow B_2 \quad \text{with domain} \quad B_1 \cap B_2 \\
I_2 : B_2 \rightarrow B_1 \quad \text{with domain} \quad B_1 \cap B_2
\]

are both closed mappings.
2. The notion of Perturbations, that are Gentle on an Interval.

Consider now the self-adjoint operator

\[ H = H_0 + \lambda V \]

defined on \( D_{H_0} \) for \( |\lambda| < b \), and let \( R_0(z) \) be the resolvent of \( H_0 \), i.e.

\[ R_0(z) = (z - H_0)^{-1} \]

We shall say that the perturbation

\[ H = H_0 + \lambda V \]

where \( V \) is essentially bounded by \( H_0 \), is Gentle on the Interval \( I \) iff:

There is a Banach-space \( B \), such that \( B \) and \( \mathcal{H} \) have the regular intersection property; and the following four conditions are satisfied.

I. For all \( \lambda \in I \) and all \( \varepsilon > 0 \):

\[ R_0(\lambda \pm i\varepsilon) \] maps \( B \) into its dual space \( B^* \), and the norms of these mappings are uniformly bounded in \( \lambda \) and \( \varepsilon \).

II. For all \( \lambda \in I \): there exists mappings \( R_0(\lambda^+) \) and \( R_0(\lambda^-) \) of \( B \) into \( B^* \), such that \( R_0(\lambda \pm i\varepsilon) \) converge to \( R_0(\lambda \pm) \) in the weak *
topology.
That is: for all \( f \) and \( g \) in \( B \),
\[ (f, R_0(\lambda \pm i\varepsilon)g) \]
converge to \( (f, R_0(\lambda \pm)g) \) as \( \varepsilon \)
tends to zero.

III. For all \( \lambda \in I \) and for all \( \varepsilon > 0 \):
\( VR_0(\lambda \pm i\varepsilon) \)
maps \( B \) into \( B \), and the norms of
these mappings are uniformly bounded in \( \lambda \) and \( \varepsilon \).

IV. For all \( \lambda \in I \): \( VR_0(\lambda \pm i\varepsilon) \) converge strongly
to \( VR_0(\lambda \pm) \).
That is: For all \( f \in B \)
\[ \| (VR_0(\lambda \pm i\varepsilon) - VR_0(\lambda \pm))f \|_B \]
tends to zero as \( \varepsilon \) tends to zero. \( \| \cdot \|_B \) is the
norm in \( B \).

**Theorem 1.**

If the perturbation
\[ H = H_0 + \lambda V \]
is gentle on the interval \( I \), then \( H_0 \) has an absolutely
continuous spectrum in the interval \( I \).

This theorem is in fact implied by condition I
alone. To see this we observe that condition I implies
that for a dense set of elements in \( \mathcal{H} \), namely for all
elements \( f \) in \( B \cap \mathcal{H} \)
is uniformly bounded in \( \lambda \in I \) and \( \varepsilon > 0 \). This implies that \( H_0 \) has an absolutely continuous spectrum in the interval \( I \).

**Theorem 2.**

If the perturbation

\[
H = H_0 + \lambda V
\]

is gentle on the interval \( I \), then the perturbation

\[
H_0 = H - \lambda V
\]

is also gentle on the interval \( I \).

That is: For \( |\lambda| \) small enough, condition I, II, III and IV holds with the resolvent of \( H \) replacing the resolvent of \( H_0 \) everywhere.

The main theorem or theorem III below is of course the reason why we call a perturbation satisfying condition I to IV gentle on the interval \( I \). The main theorem states that if the perturbation is gentle on the interval \( I \) then the part of the operators \( H \) and \( H_0 \) corresponding to the spectral interval \( I \) are unitary equivalent.

Now let \( \lambda_0 \) be the uniform bound of the norms in condition III, then we have:
Theorem 3.

If the Perturbation

\[ H = H_0 + V \]

is Gentle on the interval \( I \), then for all \( \lambda \), with \( |\lambda| < \min(\lambda_0, b) \), we can find a partial isometric \( U_\lambda \) of \( \mathcal{D}^0 \) into \( \mathcal{D}^0 \), such that

\[
U_\lambda^* U_\lambda = I \\
U_\lambda U_\lambda^* = E_0 \\
E_\lambda^* H = U_\lambda^* E_\lambda^* H_0 U_\lambda \\
E_0 H_0 = U_\lambda E_\lambda H U_\lambda^* \\
\]

and \( U_\lambda \) is analytic in \( \lambda \) for \( |\lambda| < \min(\lambda_0, b) \).

Chapter II

Perturbation of the negative Laplacian by a multiplication operator.

1. Laplacian in three dimensions.

In this chapter we apply the general theory
developed in the first chapter to the case where
\( H_0 = -\Delta \); where \( \Delta \) is the Laplacian in three dimensions.
The perturbing operator \( V \) will be a multiplication operator given by the multiplication by a real function \( V(x) \).

So let the Hilbert-space \( \mathcal{H} \) be the space \( L_2(\mathbb{R}^3) \), and in this space we consider the following perturbation

\[
H = -\Delta + \lambda V
\]

where \( -\Delta \) is regarded as a self-adjoint operator in \( L_2(\mathbb{R}^3) \) on its natural domain of definition. \( V \) is the operator of multiplication by the real function \( V(x) \).

We prove the following theorem.

**Theorem 4.**

If \( V(x) \) is in \( L^{3/2}(\mathbb{R}^3) \) then the perturbation

\[
H = -\Delta + \lambda V
\]

is gentle on the whole real line.

By theorem 3 this implies that for \( V(x) \) in \( L^{3/2}(\mathbb{R}^3) \), then

\[
H = -\Delta + \lambda V \quad \text{and} \quad H_0 = -\Delta
\]

are unitary equivalent for \( |\lambda| < \lambda_0 \) where \( \lambda_0 \) is given by a constant times the \( L^{3/2} \) norm of \( V(x) \). So that for
\(|\lambda| < \lambda_0\) there is a unitary operator mapping \(L_2(R^3)\) onto itself such that

\[H_0 = U_\lambda H U_\lambda^*\]

and \(U_\lambda\) is analytic in \(\lambda\) for \(|\lambda| < \lambda_0\). This result was recently obtained independently by Kato by other methods.

We prove theorem 4 by giving the Banach-space \(B\) and verifying condition I to IV in chapter I.

For the Banach-space \(B\) we take \(L_{6/5}(R^3)\), and recalling that the resolvent of \(-\Delta\) is given by the kernel

\[R_0(z) = \frac{e^{-\sqrt{-z}|x-y|}}{4\pi|x-y|}, \text{Re} \sqrt{-z} \geq 0\]

Sobolev's inequality gives us that the kernel

given by

\[
\frac{1}{4\pi} \frac{1}{|x-y|}
\]

maps \(L_{6/5}(R^3)\) into \(L_6(R^3)\), which is the dual space of \(L_{6/5}(R^3)\).

Since \(\frac{1}{4\pi} \frac{1}{|x-y|}\) is larger than the absolute value of the kernel of \(R_0(z)\), this verifies condition I, and using dominated convergence we also get condition II.

Let \(q = 6/5, q' = 6\), and let \(V \in L_p\). By
Hölder's inequality we have that $V$ maps $L_{q'}$ into $L_q$ if

$$\frac{1}{p} + \frac{1}{q'} = \frac{1}{q}$$

and this gives us $p = \frac{3}{2}$. So if $V(x) \in L_{3/2}(R^3)$ then $V$ maps $B^\infty$ into $B$ with bounded norm, and this gives us condition III. Condition IV follows again by dominated convergence.

2. Laplacian in $n$ dimensions, $n > 3$.

Let now $\Delta$ be the Laplacian in $n$ dimensions, $n > 3$. Theorem 4 generalizes to this case under somewhat stronger conditions on the function $V(x)$.

**Theorem 5.**

If the real function $V(x)$ is bounded and integrable then the perturbation

$$H = -\Delta + \lambda V$$

is gentle on the whole real line.

So again by theorem 3, we have that for $V(x)$ bounded and integrable, there is a $\lambda_0$ depending on the $L_\infty$ and the $L_1$ norm of $V(x)$, such that for $\lambda \leq \lambda_0$ we have that
\[ H = -\Delta + \lambda V \text{ and } H_0 = -\Delta \] are unitary equivalent, i.e.
\[ H_0 = U_\lambda H U_\lambda^* \]
where \( U_\lambda \) is analytic in \( \lambda \) for \( |\lambda| \leq \lambda_0 \) and unitary.

3. The scattering matrix.

Let \( H \) and \( H_0 \) be as above, and define
\[ U_\lambda^+ = \text{strong lim } e^{iHt} - e^{-iH_0t} \]
\[ U_\lambda^- = \text{strong lim } e^{-iHt} - e^{iH_0t} \]
whenever the strong limits exist. If \( U_\lambda^+ \) and \( U_\lambda^- \) exist we define the scattering operator by
\[ S_\lambda = U_\lambda^-(U_\lambda^+)^* \]

**Theorem 6.**

Let \( H_0 = -\Delta \) in \( n \) dimensions \( n \geq 3 \), and let \( H = -\Delta + \lambda V \), where \( V \) is the multiplication operator by a function \( V(x) \). If \( V(x) \) is bounded and integrable then there is a \( \lambda_0 > 0 \), depending only on the \( L_\infty \) and the \( L_1 \) norm of \( V(x) \). For \( |\lambda| < \lambda_0 \) \( U_\lambda^+ \) and \( U_\lambda^- \) exist.
and are unitary operators depending analyticly on $\lambda$. Hence the scattering operator $S_\lambda$ exists for $|\lambda|<\lambda_0$ and depend analyticly on $\lambda$.

Chapter III

Partly Gentle Perturbation by Annihilation-Creation Operators.

In this chapter the undisturbed operator $H_0$ will be of the same type as the free energy operator in Relativistic Quantum Field Theory and the disturbance $V$ is built up in terms of Annihilation-Creation Operators.

The disturbing operator $V$ in a relativistic quantum field theory has the following characteristic properties: it is Lorentz invariant, it is local and it is built up in terms of annihilation-creation operators. We will consider the perturbation problems where the disturbing operator $V$ has only the last property, namely that it is built up in terms of annihilation-creation operators. Moreover we will assume that the kernels which express $V$ in terms of annihilation-creation operators have certain smoothness properties.

In order to simplify the exposition, we will consider only the case where there is one Fermion field
which interact with itself.

1. The Hilbert-space $\mathcal{H}$ and its particle Representation.

The undisturbed operator $H_0$ will be given with respect to a specific representation of the Hilbert space $\mathcal{H}$; the so called particle representation.

An element $f$ in $\mathcal{H}$ is represented by a sequence of complex valued functions

$$f = \{ f_0, f_1, f_2, \ldots \}$$

where $f_0$ is just a complex constant and $f_n$ for $n \geq 1$ are complex valued antisymmetric functions $f_n(v_1, \ldots, v_n)$ of $n$ real variables $v_1, \ldots, v_n$, where the domain of each $v_i$ is the interval $[\alpha, \infty)$.

By saying that the function $f_n(v_1, \ldots, v_n)$ is antisymmetric we mean that

$$f_n(v_1, \ldots, v_n) = \frac{1}{n!} \sum_{\sigma} (-1)^{\sigma} f(v_{\sigma(1)}, \ldots, v_{\sigma(n)})$$

where the sum on the right hand side is over all permutation $\sigma$ of the numbers $1, \ldots, n$.

We shall assume that $\alpha$ is positive

$$\alpha > 0$$

$$\alpha = m_0^2$$, where $m_0$ is the restmass of the free particle.
The inner product in $\mathcal{H}$ is given by

$$(f,g) = \sum_{n=0}^{\infty} n! (f_n, g_n)$$

where

$$(f_n, g_n) = \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} f_n(v_1, \ldots, v_n) g_n(v_1, \ldots, v_n) dv_1, \ldots, dv_n$$

The undisturbed operator $H_0$ is given by

$$(H_0 f)_n(v_1, \ldots, v_n) = (\sum_{i=1}^{n} v_i) \cdot f_n(v_1, \ldots, v_n)$$

$$(H_0 f)_0 = 0$$

$H_0$ is obviously self-adjoint on its natural domain of definition.

The annihilation operator $a(\omega)$ is defined by

$$(a(\omega)f)_n(v_1, \ldots, v_n) = (n+1)f_{n+1}(\omega, v_1, \ldots, v_n)$$

The creation operator $a^+(\omega)$ is defined as the adjoint of $a(\omega)$.

Actually $a(\omega)$ and $a^+(\omega)$ are improper operators. Introducing for any testfunction $h$ in $L_2(\mathbb{R}, \mathbb{R})$

$$a(h) = \int_{\mathbb{R}} a(\omega) h(\omega) d\omega$$

$$a^+(h) = \int_{\mathbb{R}} a(\omega) h(\omega) d\omega$$
we get well defined operators and
\[ a(h)^* = a^+(\overline{h}) \]

The fact that \( f_n(v_1, \ldots, v_n) \) is antisymmetric implies the following identity

\[ a(h)a^+(\overline{h}) + a^+(\overline{h})a(h) = \|h\|_2^2 \]

And this again implies that
\[ \|a(h)\| \leq \|h\|_2 \]

So for \( h \) in \( L_2 \), \( a(h) \) and \( a^+(h) \) are bound operators.

2. The disturbing operator \( V \) and the Perturbation Problem.

We will consider disturbing operators of the following form

\[ V = \sum_{1 \leq j, k \leq N} V_{jk} \]

where \( N \) may be any number.

The operators \( V_{jk} \) are expressed in terms of annihilation - creation operators in the following manner
\[ V_{jk} = \int \cdots \int_{\alpha} dv_1 \cdots dv_j d\omega_1 \cdots d\omega_k V_{jk}(v_1, \ldots, v_j | \omega_1, \ldots, \omega_k) \]

\[ \cdot a^+(v_1) \cdots a^+(v_j) a(\omega_1) \cdots a(\omega_k) \]

where the kernel \( V_{jk}(v_1, \ldots, v_j | \omega_1, \ldots, \omega_k) \) is a complex valued function that is antisymmetric in the variables \( v_1, \ldots, v_j \) and in \( \omega_1, \ldots, \omega_k \).

Using the definition of \( a^+(\omega) \) and of \( a(\omega) \) we can give the action of the operator \( V_{jk} \) directly in terms of its kernel

\[ V_{jk}(v_1, \ldots, v_j | \omega_1, \ldots, \omega_k) \]

Let

\[ m = j-k+n \]

then

\[(V_{jk}f)_m(v_1, \ldots, v_m) =
\]

\[ \binom{n}{k} k! \text{Asym} \int \cdots \int d\omega_1 \cdots d\omega_k V_{jk}(v_1, \ldots, v_j | \omega_1, \ldots, \omega_k) \]

\[ \cdot f_n(\omega_1, \ldots, \omega_k, v_j+1, \ldots, v_m) \]

Here we have introduced the notion

\[ \text{Asym } g_m(v_1, \ldots, v_m) = \frac{1}{m!} \sum_{\sigma} (-1)^{\sigma} g_m(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) \]

where \( g_m \) is a function of \( m \) real variables, and the summation on the right hand side is taken over all
permutations of the numbers $1,\ldots,m$.

Let us now introduce the Fourier transform of the kernels $V_{jk}(v_1,\ldots,v_j|\omega_1,\ldots,\omega_{j-k})$. To be specific we will take the Fourier transform with respect to the variables $v_1,\ldots,v_j$, and the inverse Fourier transform with respect to the variables $\omega_1,\ldots,\omega_{j-k}$.

$$\hat{V}_{jk}(s_1,\ldots,s_j|t_1,\ldots,t_k)$$

$$= \frac{i+k}{2} \int \cdots \int dv_1,\ldots,dv_j d\omega_1,\ldots,d\omega_{j-k}$$

$$e^{i(\sum v se - \sum \omega e)} V_{jk}(v_1,\ldots,v_j|\omega_1,\ldots,\omega_{j-k})$$

We will say that the kernel $V_{jk}(v_1,\ldots,v_j|\omega_1,\ldots,\omega_{j-k})$ is smooth if the four following expressions are all finite.

$$\sup_{s_1,\ldots,s_j} \left\{ \cdots \int dt_1,\ldots,dt_k |\hat{V}_{jk}(s_1,\ldots,s_j|t_1,\ldots,t_k) \right\}$$

$$\sup_{t_1,\ldots,t_k} \left\{ \cdots \int ds_1,\ldots,ds_j |\hat{V}_{jk}(s_1,\ldots,s_j|t_1,\ldots,t_k) \right\}$$

$$\sup_{s_1,\ldots,s_j} \left\{ \cdots \int dt_1,\ldots,dt_k d\tau |\hat{V}_{jk}(s_1+\tau,\ldots,s_j+\tau|t_1,\ldots,t_k) \right\}$$

$$\sup_{t_1,\ldots,t_k} \left\{ \cdots \int ds_1,\ldots,ds_j d\sigma |\hat{V}_{jk}(s_1,\ldots,s_j|t_1+\sigma,\ldots,t_k+\sigma) \right\}$$

The two last of these expressions were used by Friedrichs as gentleness norms in his book "Perturbation of Spectra in Hilbertspace".
Let \( H_0 \) be the subspace of our Hilbert-space that is orthogonal to the vacuum state. That is, \( H_0 \) is the space spent by all elements \( f \) in \( H \) such that

\[
f_0 = 0
\]

Since \( V \) contains no terms of the form \( V_{j0} \) or \( V_{0k} \), our perturbation problem is actually a perturbation problem in \( H_0 \).

This reduction of the problem to a perturbation problem in \( H_0 \) has the advantage that \( H_0 \) has an absolutely continuous spectrum in \( H_0 \).

We are now in position to state the theorem about perturbations by annihilation-creation operators.

**Theorem 7.**

If \( V \) is symmetric and essentially bounded by \( H_0 \), and if \( V \) has the form

\[
V = \sum_{1 \leq j, k \leq N} V_{jk}
\]

where the kernels \( V_{jk}(v_1, \ldots, v_j | \omega_1, \ldots, \omega_k) \) are all smooth, then the perturbation

\[
H = H_0 + \lambda V
\]

in \( H_0 \), is gentle on any bounded interval \( I \) on the real axis.
The proof of this theorem is rather involved and will be given in a later publication.

Together with theorem 3 this theorem gives us that for any bounded interval $I$

$$E^I H \text{ and } E^I_0 H_0$$

are unitarily equivalent for $|\lambda| \leq \lambda_0$ where $\lambda_0 > 0$ and depend on the interval $I$.

Let us introduce the scattering operator $S_\lambda$ defined as in chapter II, by means of the two operators

$$\ U_\lambda^\pm = \text{strong lim } e^{i t H - i t H_0} t \to \pm \infty$$

We have then the following theorem.

**Theorem 8.**

Under the conditions of theorem 7, the strong limits above exists for all values of $\lambda$ and defines operators $U_\lambda^\pm$ that will be isometries in $\mathcal{H}$.

This theorem was proved under slightly different hypothesis by Chestjakov and his proof carries over to our case with only small adjustments.

From theorem 8 it follows that the scattering operator $S_\lambda$ is defined for all values of $\lambda$, and from
the definition of $S_\lambda$ it is easy to see that it must comute with $H_0$. This gives us that $S_\lambda$ must have the following form.

$$S = \int S_\lambda(\omega) dE_0(\omega)$$

where $E_0(\omega)$ is the spectral family of projections corresponding to $H_0$.

$S_\lambda(\omega)$ is called the scattering matrix corresponding to the energy $\omega$.

We now have the following theorem.

**Theorem 9.**

There is a positive function $\lambda_0(\omega), \lambda_0(\omega) > 0$ for all $\omega$, such that for any fixed energy $\omega$; $S_\lambda(\omega)$ is a unitary operator for $|\lambda| \leq \lambda_0(\omega)$.

$$S_\lambda(\omega)S_\lambda^*(\omega) = S_\lambda^*(\omega)S_\lambda(\omega) = 1$$

Moreover $S_\lambda(\omega)$ is analytic in $\lambda$ for $|\lambda| = \lambda_0(\omega)$.

The proof of this theorem follow closely the proof of theorem 7.