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A proof of the existence of minimal R-algebra resolutions.

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Throughout this note R denotes a local, noetherian ring with maximal ideal $\underline{m}$. It is well known that $R/\underline{m}$ as an R-module has a minimal resolution X, i.e. $dX \subset \underline{m}X$. It was shown by Tate [3, teorem 1] that $R/\underline{m}$ has a free resolution which is a differential skew - commutative algebra, briefly called an R-algebra.

In the present note we prove that $R/\underline{m}$ always has a minimal resolution which is an R-algebra. This settles a question raised by Tate, see footnote in [3, p23].

The existence of minimal R-algebra resolutions simplifies the study of the R-algebra $\mathrm{Tor}^R(R/\underline{m}, R/\underline{m})$, cf. [3, §5]. In particular one immediately obtains generalizations of known results on the Betti-numbers of R ; see [1, §§2, 4].

Notations. Let k denote the residue class field $R/\underline{m}$. The vectorspace dimensions $\dim_k \mathrm{Tor}^R(k,k)$ are called the Betti-numbers of R. They are denoted by $b_p(R)$. The Betti-series of R is the power series

$$B(R) = \sum_{p=0}^{\infty} b_p(R)Z^p$$

The term "R-algebra" will be used in the sense of [3] i.e. an associative, graded, differential, strictly skew-commutative, algebra X over R, with unit element 1, such that the homogeneous components X_q are finitely generated modules over R. We require that

$$X_o = 1 \cdot R \quad \text{and} \quad X_q = 0 \quad \text{for} \quad q < 0 .$$

R is considered as an R-algebra with trivial grading and differential.

We shall use the symbol

$$X \langle S \rangle ; dS = s$$

to denote the R-algebra obtained from an R-algebra X "by the adjunction of a variable" S of degree w which kills a cycle s of degree $w-1$. Cf. [3, §2]

Let $\langle \ldots, S_i, \ldots \rangle$ be a set of variables indexed by an initial part of the natural numbers, which may be empty or infinite. If these S_i are adjoined successively to an R-algebra X to kill cycles s_i, there results a natural direct system of R-algebras and inclusion maps. We denote the direct limit of this system by

$$X \langle \ldots, S_i, \ldots \rangle ; \; dS_i = s_i$$

The degree of a homogeneous map j or a homogeneous element x will be denoted by $\deg j$ and $\deg x$ respectively. R-algebras and elements are indexed by superscripts and subscripts respectively.

<u>Definition</u>. Let X be an R-algebra with differential d. A <u>derivation</u> j on X is an R-linear homogeneous map $j: X \to X$ satisfying

i) $dj = jd$

ii) $j(xy) = (-1)^{w \cdot q} j(x)y + xj(y)$

where $w = \deg j$ and $y \in X_q$.

<u>Lemma</u>. Let j be a derivation on an R-algebra X , and s a cycle in X . Put $Y = X\langle S\rangle$; $dS = s$. Then j can be extended to a derivation j' on Y if and only if

(1) $j(s) \in B(Y)$.

<u>Proof</u> If j can be extended, (1) is satisfied because $j(s) = j(dS) = dj'(S)$. On the other hand if (1) is satisfied, choose an element $G \in Y$ with the property

$$dG = j(s)$$

We treat the cases $\deg S$ odd and $\deg S$ even separately. If $\deg S$ is odd, we have

$$Y = X \oplus XS$$

For $x_o, x_1 \in X$ define

(2) $j'(x_o + x_1 S) = j(x_o) + (-1)^{\deg j} j(x_1)S + x_1 G$

If $\deg S$ is even, we have

$$Y = X \oplus XS \oplus XS^{(2)} \oplus \dots$$

For $x_o, \dots, x_m \in X$ define

$$(3) \qquad j' \sum_{i=o}^{m} x_i S^{(i)} = \sum_{i=o}^{m} j(x_i) S^{(i)} + \sum_{i=1}^{m} x_i S^{(i-1)} G$$

It is a straightforward matter to check that in both cases j' becomes a derivation on Y .

Theorem. Let R be a local noetherian ring with maximal ideal $\underline{m}$. There exists an R-algebra X which is an R-free resolution of $R/\underline{m}$ with the property

(i) $dX \subset \underline{m}X$

d being the differential on X .

In fact every R-algebra satisfying (ii)-(v) below has the property (i) .

(ii) $H_p(X) = 0$ for $p \neq 0$. $H_o(X) = R/\underline{m}$.

(iii) X has the form

$$X = R\langle \ldots, S_i, \ldots \rangle ; \; dS_i = s_i$$

(iv) $\deg S_{i+1} \geqslant \deg S_i$ for all $i \geqslant 1$.

(v) The cycles s_α of degree 0 form a minimal system of generators for $\underline{m}$. If $\deg s_\alpha \geqslant 1$ then s_α is not a boundary in

$$R\langle S_1, \ldots, S_{\alpha-1} \rangle ; dS_i = s_i \; .$$

Proof. In [3] Tate showed that there exists an R-algebra X satisfying (ii) -(v) above. Let X be such an R-algebra . We are going to show (i) . We assume that $\underline{m} \neq 0$, otherwise it follows from (v) that $X = R$. We also assume that the set of all adjoined variables is infinite. Only trivial modifications must be carried out if this set is finite.

Let X^0 denote the R-algebra R . Define inductively

$$X^\alpha = X^{\alpha-1}\langle S_\alpha\rangle; dS_\alpha = s_\alpha \quad \text{for } \alpha \geq 1 .$$

Let i^α denote the natural inclusion map $i^\alpha : X^{\alpha-1} \longrightarrow X^\alpha$.

We have

$$X = \varinjlim X^\alpha$$

For each $\alpha \geq 1$ define a derivation

$$j^\alpha : X^\alpha \longrightarrow X^\alpha$$

in the following way. Let $j = 0$ be the trivial derivation on $X^{\alpha-1}$. Put $G = 1$ and let j^α be the extension of j given by (2) resp. (3) . Then

$$\deg j^\alpha = - \deg S_\alpha$$

First we show that for all $\alpha \geq 1$, j^α can be extended to a derivation J^α on X which is of negative degree. By passing to a direct limit it clearly suffices to show the following: If $\alpha \leq \gamma$ and $j^{\alpha,\gamma}$ is a derivation on X^γ which is an extension of j^α , then $j^{\alpha,\gamma}$ can be extended to a derivation $j^{\alpha,\gamma+1}$ on $X^{\gamma+1}$.

Now let $j^{\alpha,\gamma}$ be a derivation on X^γ which extends j^α . We will prove that $j^{\alpha,\gamma}$ can be extended to a derivation on $X^{\gamma+1}$. By the lemma it suffices to show that

(4) $\qquad j^{\alpha,\gamma}(s_{\gamma+1}) \in B(X^\gamma)$.

To prove (4) we consider two cases. First assume that $\deg S_\alpha \neq \deg s_{\gamma+1}$. This yields

$$0 \neq \deg j^{\alpha,\gamma}(s_{\gamma+1}) < \deg s_{\gamma+1}$$

However it follows from (ii) and (iv) that

$$H_p(X^{\gamma}) = 0 \quad \text{for} \quad 0 \neq p < \deg s_{\gamma+1} .$$

Hence in this case (4) follows. Next assume that

$\deg S_{\alpha} = \deg s_{\gamma+1}$. Then $j^{\alpha,\gamma}(s_{\gamma+1}) \in X_0$. Let $S_{\mu},\dots,S_{\mu+\nu}$ be all the adjoined variables of degree $\deg s_{\gamma+1}$. Then there exist elements $x \in X^{\mu-1}$ and $r_{\mu},\dots,r_{\mu+\nu} \in R$ such that

(5) $$s_{\gamma+1} = x + \sum_{i=\mu}^{\mu+\nu} r_i S_i .$$

Differentiation yields

$$\sum_{i=\mu}^{\mu+\nu} r_i s_i \in B(X^{\mu-1})$$

It follows from (v) that $r_i \in \underline{m}$ for $i = \mu,\dots,\mu+\nu$
Since $\mu-1 < \alpha$ we have

$$j^{\alpha,\gamma}(x) = j^{\alpha}(x) = 0$$

Hence applying $j^{\alpha,\gamma}$ to (5) one deduces

$$j^{\alpha,\gamma}(s_{\gamma+1}) \in \underline{m}\, X_0$$

However, $\deg s_{\gamma+1} = \deg S_{\alpha} \geq 1$ so $\underline{m}\, X_0$ is already killed. Again (4) follows.

In the rest of the proof we consider the underlying complexes of the respective R-algebras. For each $\alpha \geq 1$, j^{α} leads to an exact sequence of complexes

(6) $$0 \longrightarrow X^{\alpha-1} \xrightarrow{\ i^{\alpha}\ } X^{\alpha} \xrightarrow{\ j^{\alpha}\ } X^{\alpha}$$

which splits as a sequence of R-modules, cf.[3,p.17-18].

Consider the functor $X \rightsquigarrow \overline{X}$, where $\overline{X} = X/\underline{m}X$. For $\alpha \geqslant 0$ let I^{α} denote the natural inclusion map $I^{\alpha}: X^{\alpha} \to X$. It follows that I^{α} is direct, hence we may identify $\overline{X}^{\alpha}$ with its image in $\overline{X}$. From (6) we deduce a commutative diagram

(7) $$\begin{array}{ccccccc} 0 & \to & \overline{X}^{\alpha-1} & \longrightarrow & \overline{X}^{\alpha} & \longrightarrow & \overline{X}^{\alpha} \\ & & & & \downarrow \overline{I}^{\alpha} & & \downarrow \overline{I}^{\alpha} \\ & & & & \overline{X} & \xrightarrow{\overline{J}^{\alpha}} & \overline{X} \end{array} \qquad \alpha \geqslant 1$$

in which the upper row is exact. This yields

(8) $$\bigcap_{\gamma \geqslant 1} \ker \overline{J}^{\gamma} \subset \overline{X}^{0}$$

Indeed let $x \in \bigcap_{\gamma \geqslant 1} \ker \overline{J}^{\gamma}$. Let $\alpha \geqslant 1$ and suppose that $x \in \overline{X}^{\alpha}$. It follows from (7) that $x \in \overline{X}^{\alpha-1}$. Repeating this it follows that $x \in \overline{X}^{0}$.

By induction on q we are going to show that

(9) $$B_q(\overline{X}) = 0$$

For $q = 0$ this is clear by (v). Let $r \geqslant 1$ and assume that (9) has been established for $q < r$. For every $\gamma \geqslant 1$, $\overline{J}^{\gamma}$ is of negative degree and commutes with the differential on $\overline{X}$. Hence

$$\overline{J}^{\gamma}(B_r(\overline{X})) \subset \coprod_{q<r} B_q(\overline{X}) = 0 \quad \text{for } \gamma \geqslant 1 .$$

It follows from (8) that $B_r(\overline{X}) \subset B_r(\overline{X}) \cap \overline{X}^{0} = 0$

Since $B(\overline{X}) = 0$ we have $B(X) \subset \underline{m}X$.

Q.E.D.

Let X be a minimal R-algebra resolution of k as described in the theorem (ii)-(v) . There is an isomorphism of R-algebras , cf. [3, §5] :

$$\mathrm{Tor}^R(k,k) \approx H(X \otimes_R k) = X \otimes_R k$$

This yields the following generalization of a result due to Assmus [1, §4] :

Corollary 1 The Betti-series of R may be written in the form

$$(9) \qquad B(R) = \frac{(1+Z)^{n_1}(1+Z)^{n_3}\ldots}{(1-Z^2)^{n_2}(1-Z^4)^{n_4}\ldots}$$

where n_q q=1,2,... is the number of adjoined variables of degree q in a minimal R-algebra resolution.

Corollary 2 The Betti-numbers $\{b_p(R)\}$ of a non-regular local ring R form a non-decreasing sequence. Cf. [2] .

Proof In the above notation we have

$$n_1 = \dim_k \underline{m}/\underline{m}^2$$

and

$$n_2 = \dim_k H_1(X^{n_1}) \ .$$

Let R be non-regular. It follows from the Eilenberg characterization of regularity that $n_2 \neq 0$, cf. [3, lemma 5] . Since also $n_1 \neq 0$, B(R) contains a factor $\frac{1}{1-Z}$. Hence $\{b_p(R)\}$ is non-decreasing.

Q.E.D.

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