A proof of the existence of minimal R-algebra resolutions.

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Throughout this note \( R \) denotes a local, noetherian ring with maximal ideal \( \mathfrak{m} \). It is well known that \( R/\mathfrak{m} \) as an \( R \)-module has a minimal resolution \( X \), i.e. \( dX \subset \mathfrak{m}X \). It was shown by Tate \([3, theorem 1]\) that \( R/\mathfrak{m} \) has a free resolution which is a differential skew-commutative algebra, briefly called an \( R \)-algebra.

In the present note we prove that \( R/\mathfrak{m} \) always has a minimal resolution which is an \( R \)-algebra. This settles a question raised by Tate, see footnote in \([3, p23]\).

The existence of minimal \( R \)-algebra resolutions simplifies the study of the \( R \)-algebra \( \text{Tor}^{R}(R/\mathfrak{m}, R/\mathfrak{m}) \), cf. \([3, \S5]\). In particular one immediately obtains generalizations of known results on the Betti-numbers of \( R \); see \([1, \S\S2, 4]\).

Notations. Let \( k \) denote the residue class field \( R/\mathfrak{m} \). The vector space dimensions \( \dim_{k}\text{Tor}^{R}(k,k) \) are called the Betti-numbers of \( R \). They are denoted by \( b_{p}(R) \).

The Betti-series of \( R \) is the power series

\[
B(R) = \sum_{p=0}^{\infty} b_{p}(R) z^{p}
\]


The term "R-algebra" will be used in the sense of [3] i.e. an associative, graded, differential, strictly skew-commutative, algebra $X$ over $R$, with unit element 1, such that the homogeneous components $X_q$ are finitely generated modules over $R$. We require that

$$X_0 = 1 \cdot R \quad \text{and} \quad X_q = 0 \quad \text{for} \quad q < 0.$$ 

$R$ is considered as an $R$-algebra with trivial grading and differential.

We shall use the symbol

$$X <S>; dS = s$$

to denote the $R$-algebra obtained from an $R$-algebra $X$ "by the adjunction of a variable" $S$ of degree $w$ which kills a cycle $s$ of degree $w-1$. Cf. [3, §2]

Let $<\ldots, S_i, \ldots>$ be a set of variables indexed by an initial part of the natural numbers, which may be empty or infinite. If these $S_i$ are adjoined successively to an $R$-algebra $X$ to kill cycles $s_i$, there results a natural direct system of $R$-algebras and inclusion maps. We denote the direct limit of this system by

$$X <\ldots, S_i, \ldots>; dS_i = s_i$$

The degree of a homogeneous map $j$ or a homogeneous element $x$ will be denoted by $\deg j$ and $\deg x$ respectively. $R$-algebras and elements are indexed by superscripts and subscripts respectively.
Definition. Let $X$ be an $R$-algebra with differential $d$. A derivation $j$ on $X$ is an $R$-linear homogeneous map $j: X \to X$ satisfying

i) $dj = jd$

ii) $j(xy) = (-1)^{w \cdot q} j(x)y + xj(y)$

where $w = \deg j$ and $y \in X_q$.

Lemma. Let $j$ be a derivation on an $R$-algebra $X$, and $s$ a cycle in $X$. Put $Y = X \langle S \rangle$; $dS = s$.

Then $j$ can be extended to a derivation $j'$ on $Y$ if and only if

$$(1) \quad j(s) \in B(Y).$$

Proof. If $j$ can be extended, (1) is satisfied because $j(s) = j(dS) = dj'(S)$. On the other hand if (1) is satisfied, choose an element $G \in Y$ with the property

$$dG = j(s).$$

We treat the cases $\deg S$ odd and $\deg S$ even separately.

If $\deg S$ is odd, we have $Y = X \oplus XS$.

For $x_0, x_1 \in X$ define

$$(2) \quad j'(x_0 + x_1S) = j(x_0) + (-1)^{\deg j(x_1)} x_1 S + x_1 G$$

If $\deg S$ is even, we have $Y = X \oplus XS \oplus XS^{(2)} \oplus \ldots$.

For $x_0, \ldots, x_m \in X$ define
(3) \[ j' \sum_{i=0}^{m} x_i s(i) = \sum_{i=0}^{m} j(x_i) s(i) + \sum_{i=1}^{m} x_i s(i-1) g \]

It is a straightforward matter to check that in both cases \( j' \) becomes a derivation on \( Y \).

**Theorem.** Let \( R \) be a local noetherian ring with maximal ideal \( m \). There exists an \( R \)-algebra \( X \) which is an \( R \)-free resolution of \( R/m \) with the property

(i) \( dX \subseteq mX \)

\( d \) being the differential on \( X \).

In fact every \( R \)-algebra satisfying (ii)-(v) below has the property (i).

(ii) \( H_p(X) = 0 \) for \( p \neq 0 \). \( H_0(X) = R/m \).

(iii) \( X \) has the form

\[ X = R \langle \ldots, S_i, \ldots \rangle ; dS_i = s_i \]

(iv) \( \deg S_{i+1} \geq \deg S_i \) for all \( i \geq 1 \).

(v) The cycles \( s_\infty \) of degree 0 form a minimal system of generators for \( m \). If \( \deg s_\infty \geq 1 \) then \( s_\infty \) is not a boundary in

\[ R \langle S_i, \ldots, S_{\infty-1} \rangle ; dS_i = s_i . \]

**Proof.** In [3] Tate showed that there exists an \( R \)-algebra \( X \) satisfying (ii)-(v) above. Let \( X \) be such an \( R \)-algebra. We are going to show (i). We assume that \( m \neq 0 \), otherwise it follows from (v) that \( X = R \).

We also assume that the set of all adjoined variables is infinite. Only trivial modifications must be carried out if this set is finite.
Let \( X^0 \) denote the \( \mathbb{R} \)-algebra \( \mathbb{R} \). Define inductively
\[
X^\alpha = X^{\alpha-1} \langle S_\alpha \rangle; ds_\alpha = s_\alpha \quad \text{for} \quad \alpha \geq 1.
\]
Let \( i^\alpha \) denote the natural inclusion map \( i^\alpha : X^{\alpha-1} \rightarrow X^\alpha \).

We have
\[
X = \varprojlim X^\alpha
\]
For each \( \alpha \geq 1 \) define a derivation
\[
j^\alpha : X^\alpha \rightarrow X^\alpha
\]
in the following way. Let \( j = 0 \) be the trivial derivation on \( X^{\alpha-1} \). Put \( G = 1 \) and let \( j^\alpha \) be the extension of \( j \) given by (2) resp. (3). Then
\[
\deg j^\alpha = - \deg S_\alpha
\]

First we show that for all \( \alpha \geq 1 \), \( j^\alpha \) can be extended to a derivation \( j^\alpha \) on \( X \) which is of negative degree. By passing to a direct limit it clearly suffices to show the following: If \( \alpha \leq 1 \) and \( j^\alpha, \gamma \) is a derivation on \( X_\gamma \) which is an extension of \( j^\alpha \), then \( j^\alpha, \gamma \) can be extended to a derivation \( j^\alpha, \gamma+1 \) on \( X_{\gamma+1} \).

Now let \( j^\alpha, \gamma \) be a derivation on \( X_\gamma \) which extends \( j^\alpha \). We will prove that \( j^\alpha, \gamma \) can be extended to a derivation on \( X_{\gamma+1} \). By the lemma it suffices to show that
\[
(4) \quad j^\alpha, \gamma (s_{\gamma+1}) \in B(X_\gamma).
\]
To prove (4) we consider two cases. First assume that \( \deg S_\alpha \neq \deg s_{\gamma+1} \). This yields
\[
0 \neq \deg j^\alpha, \gamma (s_{\gamma+1}) < \deg s_{\gamma+1}
\]
However it follows from (ii) and (iv) that

\[ H_p(x^y) = 0 \text{ for } 0 \neq p < \deg s_{y+1}. \]

Hence in this case (4) follows. Next assume that \( \deg S_\alpha = \deg s_{y+1}. \) Then \( j^{\alpha,y}(s_{y+1}) \in X_0. \) Let \( S_{\mu}, \ldots, S_{\mu+\nu} \) be all the adjoined variables of degree \( \deg s_{y+1}. \) Then there exist elements \( x \in X^{\mu-1} \) and \( r_\mu, \ldots, r_{\mu+\nu} \in R \) such that

\[ s_{y+1} = x + \sum_{i=\mu}^{\mu+\nu} r_i S_i. \]

Differentiation yields

\[ \sum_{i=\mu}^{\mu+\nu} r_i s_i \in B(x^{\mu-1}) \]

It follows from (v) that \( r_\mu \in M \) for \( \mu = \mu, \ldots, \mu+\nu \)

Since \( \mu-1 < \alpha \) we have

\[ j^{\alpha,y}(x) = j^\alpha(x) = 0 \]

Hence applying \( j^{\alpha,y} \) to (5) one deduces

\[ j^{\alpha,y}(s_{y+1}) \in m X_0 \]

However, \( \deg s_{y+1} = \deg S_\alpha \geq 1 \) so \( m X_0 \) is already killed. Again (4) follows.

In the rest of the proof we consider the underlying complexes of the respective \( R \)-algebras. For each \( \alpha > 1, \)
\( j^\alpha \) leads to an exact sequence of complexes

\[ 0 \to x^{\alpha-1} \xrightarrow{i^\alpha} x^\alpha \xrightarrow{j^\alpha} x^\alpha \]

which splits as a sequence of \( R \)-modules, cf.\[3,p.17-18\].
Consider the functor $X \mapsto \overline{X}$, where $\overline{X} = X/\mathfrak{m}X$. For $\alpha > 0$ let $I^{\alpha}$ denote the natural inclusion map $I^{\alpha}: X^{\alpha} \to X$.

It follows that $I^{\alpha}$ is direct, hence we may identify $X^{\alpha}$ with its image in $\overline{X}$. From (6) we deduce a commutative diagram

\[
0 \rightarrow \overline{X}^{\alpha-1} \rightarrow \overline{X}^{\alpha} \rightarrow \overline{X}^{\alpha} \rightarrow \overline{X}^{\alpha} \rightarrow \overline{X}
\]

in which the upper row is exact. This yields

\[
\bigcap_{\gamma \geq 1} \ker \overline{J}^{\gamma} \subset \overline{X}^{\gamma}
\]

Indeed let $x \in \bigcap_{\gamma \geq 1} \ker \overline{J}^{\gamma}$. Let $\alpha > 1$ and suppose that $x \in X^{\alpha}$. It follows from (7) that $x \in \overline{X}^{\alpha-1}$.

Repeating this it follows that $x \in \overline{X}^{\gamma}$.

By induction on $\gamma$ we are going to show that

\[
B_{q}(\overline{X}) = 0
\]

For $\gamma = 0$ this is clear by (v). Let $r > 1$ and assume that (9) has been established for $\gamma < r$. For every $\gamma > 1$, $\overline{J}^{\gamma}$ is of negative degree and commutes with the differential on $\overline{X}$. Hence

\[
\bigcap_{\gamma < r} B_{\gamma}(\overline{X}) \subseteq B_{q}(\overline{X}) = 0 \text{ for } \gamma > 1.
\]

It follows from (6) that $B_{r}(\overline{X}) \subseteq B_{r}(\overline{X}) \cap \overline{X}^{\gamma} = 0$.

Since $B(\overline{X}) = 0$ we have $B(\overline{X}) \subseteq \mathfrak{m}X$.

Q.E.D.
Let $X$ be a minimal $R$-algebra resolution of $k$ as described in the theorem (ii)-(v). There is an isomorphism of $R$-algebras, cf. [3, §5]:

$$\text{Tor}_R^R(k, k) \cong H(X \otimes k) = X \otimes_R k$$

This yields the following generalization of a result due to Assmus [1, §4]:

**Corollary 1** The Betti-series of $R$ may be written in the form

\[(9) \quad B(R) = \frac{(1 + z)^{n_1}(1 + z)^{n_2} \cdots}{(1 - z^2)^{n_2}(1 - z^4)^{n_4} \cdots}\]

where $n_q$ q=1,2,... is the number of adjoined variables of degree $q$ in a minimal $R$-algebra resolution.

**Corollary 2** The Betti-numbers \{\(b_p(R)\)\} of a non-regular local ring $R$ form a non-decreasing sequence. Cf. [2].

**Proof** In the above notation we have

\[n_1 = \dim_k \frac{\mathfrak{m}}{\mathfrak{m}^2}\]

and

\[n_2 = \dim_k H_1(X^{n_1}).\]

Let $R$ be non-regular. It follows from the Eilenberg characterization of regularity that $n_2 \neq 0$, cf. [3, lemma 5]. Since also $n_1 \neq 0$, $B(R)$ contains a factor $\frac{1}{1-z}$. Hence \{\(b_p(R)\)\} is non-decreasing.

Q.E.D.
References

