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# Modular Lattices and the Lebesque-Radon-Nikodym Theorem

by

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## 1. Introduction.

The aim of this note is to improve some results due to Alfsen ([1] p.453). We shall assume that the reader is familiar with this paper, in particular with the definition on pp.442-453. We shall improve Alfsen's form of the Lebesgue-Radon-Nikodym Theorem as follows. Alfsen represents valuations of $L$ by elements in a space $\mathcal{L}_1(S,\mathcal{F},m)$ and each element $x$ of $L$ is represented by elements $H_x$ and $K_x$ of $\tilde{\mathcal{F}}$ in such a way that

(1) $$\int_{H_x} f_v dm \leq v(x) \leq \int_{K_x} f_v dm$$

Moreover, he proves that this result is optimal in the sense that it can not subsist for any larger $H_x$ or any smaller $K_x$.

We shall represent each element $x$ in $L$ by an element $h_x$ in $\mathcal{L}_1(S,\mathcal{F},m)$ which yields an exact Lebesgue-Radon-Nikodym-Theorem with an equation of the form

(2) $$v(x) = \int h_x f_v dm.$$

## 2. Definitions.

Let $L$ be a fixed modular lattice with a least element $\emptyset$ and a greatest element $e$. Consider the set $V$ of valuations on $L$ which satisfy $v(\emptyset) = 0$. Let $L'$ be the set of projectivity-invariant additive interval-functions on $L$. The formula $\mu(x,y) = v(y)-v(x)$ (as well as the formula $\mu(\emptyset,x) = v(x)$) establishes a 1-1-correspondence between the sets $V$ and $L'$. We shall in this paper mainly deal with interval functions. Given an interval function $\mu \in L'$, we shall also use $\mu$ to denote the corresponding valuation in $V$; i.e. $\mu(x)$ is an abbreviation for $\mu(\emptyset,x)$. Then the equation $\mu(x,y) = \mu(y)-\mu(x)$ is also valid.

Let $L^*$ denote the greatest directed vector-subspace of $L'$ By using a Riemann-Darboux integration process we may define the lattice operation within $L^*$ by $\vee$ and $\wedge$. In this way, $L^*$ becomes a complete vector lattice. (See 1 p.447). Since $L$ has both a least element and a greatest element, $L^*$ consists of those $\mu \in L'$ which are of bounded variations. Let $\mu \in L^*$ and let $\mu \geqq 0$.

Let $\mathfrak{A}(\mu)$ be the closed ideal generated by $\mu$. Then $\mathfrak{A}(\mu)$ is an (L-)space with respect to the norm

$$N(\nu) = |\nu| (\emptyset,e) = (\nu \vee 0)(\emptyset,e)-(\nu \wedge 0)(\emptyset,e)$$

and $\mu$ is a weak order unit for $\mathfrak{A}(\mu)$.

Let $B(\mu)$ be the greatest subspace of $\mathfrak{A}(\mu)$ for which is a strong unit i.e.

$$B(\mu) = \{\nu \mid (\exists n)(-n\mu \leq \nu \leq n\mu)\}$$

Let $\mathcal{L}_1(S,\mathcal{F},m)$ be the Kakutani-representation of the (L-)space $\mathfrak{A}(\mu)$, defined on p.445 and p.452 in [1].[1] Let each $\nu \in \mathfrak{A}(\mu)$ be represented by $f_\nu \in \mathcal{L}_1(S,\mathcal{F},m)$. Then the mapping $\nu \rightarrow f_\nu$ of $\mathfrak{A}(\mu)$ into $\mathcal{L}_1(S,\mathcal{F},m)$ has the property that $f_\mu(z) = 1$ for every $z \in S$. Let $f \equiv g$, mean that $m(\{s \mid s \in S \ \& \ f(s) \neq g(s)\}) = 0$, (i.e. $f = g$ almost everywhere). Then the mapping $\nu \rightarrow f_\nu$ satisfies the condition

$$f_{\nu+\tau} \equiv f_\nu + f_\tau$$

If $\nu \in B(\mu)$ then $f_\nu \in \mathcal{L}_\infty(S,\mathcal{F},m)$ and we may choose the mapping $\nu \rightarrow f_\nu$ such that

(3) $$f_{\nu+\tau} = f_\nu + f_\tau$$

for every $\nu, \tau \in B(\mu)$. This follows from the fact that there exists a mapping $\rho$ of $\mathcal{L}_\infty(S,\mathcal{F},m)$ into $\mathcal{L}_\infty(S,\mathcal{F},m)$, satisfying the following conditions. (See [2] and references mentioned therein.)

I $\quad \rho(f) \equiv f$

II $\quad f \equiv g \Rightarrow \rho(f) = \rho(g)$

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[1] See note on page 18.

III $\rho(1) = 1$

IV $f \geq 0 \Longrightarrow \rho(f) \geq 0$

V $\rho(af+bg) = a\rho(f)+b\rho(g)$

VI $\rho(fg) = \rho(f)\rho(g)$.

The equation (3) is satisfied if we choose the mapping $\nu \to f_\nu$ such that $f_\nu = \rho(f_\nu)$.

Let $\varphi$ be the mapping of $\mathcal{L}_1(S,\mathcal{F},m)$ onto $\mathfrak{A}(\mu)$ satisfying the following two conditions.

1. $\varphi(f_\nu) = \nu$
2. $\varphi(f) = \varphi(g) \Longleftrightarrow f \equiv g$.

For some $f,g \in \mathcal{L}_1(S,\mathcal{F},m)$ we have that $fg \in \mathcal{L}_1(S,\mathcal{F},m)$ (where $(fg)(z) = f(z)g(z)$, for all $z \in S$).

Hence we may define the product $\nu\tau$ for some $\nu,\tau \in \mathfrak{A}(\mu)$ as follows

$$\nu\tau = \begin{cases} \varphi(f_\nu f_\tau) & \text{if } f_\nu f_\tau \in \mathcal{L}_1(S,\mathcal{F},m) \\ \text{undefined} & \text{if } f_\nu f_\tau \notin \mathcal{L}_1(S,\mathcal{F},m). \end{cases}$$

Let $\nu \in \mathfrak{A}(\mu)$. Then the product $\mu\nu$ is always defined and $\mu\nu = \nu$, since $f_\mu(z) = 1$ for every $z \in S$. Moreover, if $\tau \in B(\mu)$ then the product $\tau\nu$ is defined for all $\nu \in \mathfrak{A}(\mu)$. We shall write $\nu^2$ for $\nu\nu$. Let $\sqrt{\mathfrak{A}(\mu)}$ be defined as follows.

$$\sqrt{\mathcal{A}(\mu)} = \{\nu \mid \nu \in \mathcal{A}(\mu) \text{ and } \nu^2 \text{ is defined}\}.$$

It is easy to verify that if $\nu, \tau \in \sqrt{\mathcal{A}(\mu)}$ and if $t$ is a real number, then $|\nu|, -\nu, t\nu, \nu\wedge\tau, \nu\vee\tau, \nu+\tau \in \sqrt{\mathcal{A}(\mu)}$ and $\nu\,\tau \in \mathcal{A}(\mu)$. Moreover, if $\nu \leq \nu_1 \leq \tau$ and if $\nu, \tau \in \sqrt{\mathcal{A}(\mu)}$, then $\nu_1 \in \sqrt{\mathcal{A}(\mu)}$. Moreover, $B(\mu) \subseteq \sqrt{\mathcal{A}(\mu)}$. Hence $\sqrt{\mathcal{A}(\mu)}$ is a vector lattice and $\sqrt{\mathcal{A}(\mu)}$ becomes a Hilbert space if we define scalar product $\nu\circ\tau$ as follows

$$\nu\circ\tau = (\nu\tau)(\emptyset, e).$$

The $L_2$-norm of $\nu$ is then $\sqrt{\nu\circ\nu} = \sqrt{\nu^2(\emptyset, e)}$.

3. <u>The Lebesgue-Radon-Nikodym Theorem.</u>

Let $w$ be a mapping of $\sqrt{\mathcal{A}(\mu)} \times L$ into the real numbers as follows.

$$w(\nu, x) = (\mu-\nu)^2(\emptyset, x) + \nu^2(x, e). \tag{4}$$

We shall say that $\tau \in \sqrt{\mathcal{A}(\mu)}$ is the <u>representative</u> of $x \in L$ if the following two conditions are satisfied for every $\nu \in \sqrt{\mathcal{A}(\mu)}$.

$$w(\nu, x) \geq w(\tau, x) \tag{5}$$

$$w(\nu, x) = w(\tau, x) \Longrightarrow \nu = \tau. \tag{6}$$

Given $x \in L$ we shall denote the representative of $x$ by $\tau_x$, if there exists any. $\tau_x$ is undefined if there exists no representative of $x$. We shall first prove that $\tau_x$ exists for every $x \in L$, and that $0 \leq \tau_x \leq \mu$. (Lemma 4 and Lemma 7.) Then the following theorem is easily obtained.

<u>Theorem 1</u>. Let $L$ be a modular lattice with a least element $\phi$ and a greatest element $e$, and let $\mu$ be a fixed positive element of $L^*$. Then to each $x \in L$ there corresponds a unique $\tau_x \in L^*$, such that the following conditions are satisfied for every $\nu \in L^*$.

(7) $$0 \leq \tau_x \leq \mu$$

(8) $$\nu \in \sqrt{\mathcal{M}(\mu)} \Longrightarrow w(\nu, x) \geq w(\tau_x, x)$$

(9) $$(\nu \in \sqrt{\mathcal{M}(\mu)} \ \& \ w(\nu, x) = w(\tau_x, x)) \Longrightarrow \nu = \tau_x$$

(10) $$\nu(x) = (\tau_x \nu)(e).$$

Assuming theorem 1 we may define a representative $h_x$ of $x$ in $\mathcal{L}_1(S, \mathcal{F}, m)$ as follows. Let $\tau = \tau_x$ represent $x$ in $\sqrt{\mathcal{M}(\mu)}$, then $h_x = f_\tau$ is the representative of $x$ in $\mathcal{L}_1(S, \mathcal{F}, m)$. Then we obtain the following theorem from Theorem 1.

<u>Theorem 2</u>. Let $L$ be a modular lattice with least element $\phi$ and greatest element $e$, and let $\mu$ be a fixed positive element of $L^*$. If $\nu \in L^*$ and $\nu$ are absolutely continuous with

respect to $\mu$, then for every $x \in L$

$$(11) \qquad \nu(x) = \int h_x f_\nu \, dm.$$

It is convenient to prove some Lemmas in order to prove Theorem 1, and we also need some new definitions. Let $l(x) = \inf_{\nu \in A}(w(\nu,x))$, where $A = \sqrt{\mathcal{M}(\mu)}$.

<u>Lemma 1</u>. Let $\nu, \nu_1 \in \sqrt{\mathcal{M}(\mu)}$, $\nu \neq 0$ and let $x \in L$. Then there exists $\nu_0 \in \sqrt{\mathcal{M}(\mu)}$ and a real number $t_0$ such that the following four conditions are satisfied for every real number $t$.

$$(12) \qquad \nu_0 = \nu_1 + t_0 \nu$$

$$(13) \qquad w(\nu_1 + t\nu, x) \geq w(\nu_0, x)$$

$$(14) \qquad \nu(\phi, x) = (\nu\nu_0)(\phi, e)$$

$$(15) \qquad w(\nu_0 + t\nu, x) = t^2 \nu^2(\phi, e) + w(\nu_0, x)$$

<u>Proof</u>. Let $t$ be a real number. Since

$$w(\nu_1 + t\nu, x) = (\mu - \nu_1 - t\nu)^2(\phi, x) + (\nu_1 + t\nu)^2(x, e)$$

we easily obtain

$$w(\nu_1 + t\nu, x) = t^2 \nu^2(\phi, e) + 2t((\nu\nu_1)(\phi, e) - \nu(\phi, x)) + w(\nu_1, x).$$

Hence

$$\frac{d}{dt}w(\nu_1+t\nu,x) = 2(t\nu^2(\phi,e)+((\nu\nu_1)(\phi,e)-\nu(\phi,x)$$

and

$$\frac{d^2}{dt^2}w(\nu_1+t\nu,x) = 2\nu^2(\phi,e) > 0$$

since $\nu \neq 0$.

Hence there exists a real number $t_o$ such that $t_o\nu^2(\phi,e)+ (\nu\nu_1)(\phi,e)-\nu(\phi,x) = 0$. By choosing $\nu_o = \nu_1+t_o\nu$ we have that (12) and (13) in Lemma 1 are satisfied.

If we replace $\nu_1$ by $\nu_o$ we obtain

(16) $$w(\nu_o+t\nu,x) = t^2\nu^2(\phi,e)+2t((\nu\nu_o)(\phi,e)-\nu(\phi,x))+w(\nu_o,x)$$

But

$$w(\nu_o+t\nu,x) = w(\nu_1+(t_o+t_1)\nu,x) \geq w(\nu_o,x)$$

which is not possible for every $t$ unless

(17) $$(\nu\nu_o)(\phi,e)-\nu(\phi,x) = 0$$

This proves (14) in Lemma 1. Finally we obtain (15) by equation (16) and (17). This completes the proof of Lemma 1.

Lemma 2. Let $\nu \in \sqrt{\mathcal{R}(\mu)}$, and let $x \leq y$, when $x, y \in L$.

$$|\nu|(x,y) \leq \sqrt{\mu(x,y) \cdot \nu^2(x,y)} \,. \tag{18}$$

Proof. Since $|\nu|^2 = \nu^2$, it is sufficient to prove that

$$|\nu(x,y)| \leq \sqrt{\mu(x,y) \cdot \nu^2(x,y)} \tag{19}$$

Since $x \leq y$, we have $\mu(x,y) \geq 0$ and $\nu^2(x,y) \geq 0$. Hence (19) is equivalent to

$$\mu(x,y)\nu^2(x,y) - (\nu(x,y))^2 \geq 0 \tag{20}$$

Let $t$ be a real number. Then

$$(\nu + t\mu)^2(x,y) = \nu^2(x,y) + 2t\nu(x,y) + t^2\mu(x,y), \tag{21}$$

since $\mu\nu = \nu$ and $\mu\mu = \mu$.

Moreover,

$$\begin{aligned} ((\nu+t\mu)(x,y))^2 &= (\nu(x,y) + t\mu(x,y))^2 \\ &= (\nu(x,y))^2 + 2t\nu(x,y)\mu(x,y) + t^2(\mu(x,y))^2. \end{aligned} \tag{22}$$

Hence

$$(23)\quad \mu(x,y)\nu^2(x,y)-(\nu(x,y))^2 =$$
$$\mu(x,y)(\nu+t\mu)^2(x,y)-((\nu+t\mu)(x,y))^2 \geq 0,$$

Since $\mu(x,y)(\nu+t\mu)^2(x,y) \geq 0$ and $(\nu+t\mu)(x,y) = 0$ for some value of $t$. This completes the proof of Lemma 2.

<u>Lemma 3</u>. Let $\varepsilon$ be a real number $\geq 0$. Suppose $w(\nu_1,x)-l(x) \leq \varepsilon$ and $w(\nu_2,x)-l(x) \leq \varepsilon$. Then $(\nu_2-\nu_1)^2(\phi,e) \leq 4\varepsilon$, and $|\nu_2-\nu_1|(\phi,e) \leq 2\sqrt{\mu(\phi,e)\ \varepsilon}$.

(Recall $l(x) = \inf(w(\nu,x))$.)

<u>Proof</u>. Let $\nu = \nu_2-\nu_1$, and choose $\nu_o$ and $t_o$ such that Lemma 1 is satisfied. Then we have the following equations

$$(24)\qquad 0 \leq w(\nu_o,x)-l(x) \leq w(\nu_1,x)-l(x) \leq \varepsilon$$

$$(25)\qquad 0 \leq w(\nu_o,x)-l(x) \leq w(\nu_2,x)-l(x) \leq \varepsilon$$

by (13) since $\nu_2 = \nu_1+\nu$ .

Moreover, $\nu_1 = \nu_o+(-t_o)\nu$ and $\nu_2 = \nu_o+(1-t_o)\nu$.

Hence

$$(26)\qquad 0 \leq w(\nu_o,x)-l(x) \leq w(\nu_o,x)-l(x)+t_o^2\nu^2(\phi,e) \leq \varepsilon$$

$$(27)\qquad 0 \leq w(\nu_o,x)-l(x) \leq w(\nu_o,x)-l(x)+(1-t_o)^2\nu^2(\phi,e) \leq \varepsilon,$$

by (24), (25) and (15).

Hence

$$0 \leq t_o^2 \nu^2(\phi,e) \leq \varepsilon \quad \text{and} \quad 0 \leq (1-t_o)^2 \nu^2(\phi,e) \leq \varepsilon .$$

Since either $t_o^2 \geq \frac{1}{4}$ or $(1-t_o)^2 \geq \frac{1}{4}$, we have that $(\nu_2-\nu_1)^2(\phi,e) = \nu^2(\phi,e) \leq 4\varepsilon$.

Hence $|\nu_2-\nu_1|(\phi,e) \leq 2\sqrt{\mu(\phi,e)\cdot\varepsilon}$ by lemma 2. This completes the proof of Lemma 3.

Lemma 4. Let $x \in L$. Then $\tau_x$ is defined, i.e. there exists a $\tau \in \sqrt{\mathcal{M}(\mu)}$ such that (5) and (6) are satisfied for every $\nu \in \sqrt{\mathcal{M}(\mu)}$.

Proof. Let $x \in L$. Choose an infinite sequence $\nu_1, \nu_2, \nu_3, \ldots$, $\nu_i \in \sqrt{\mathcal{M}(\mu)}$ such that $\lim_n w(\nu_n, x) = l(x)$.

Then there exists a $\tau \in \sqrt{\mathcal{M}(\mu)}$ such that $\lim_n \nu_n = \tau$, and $\lim_n \nu_n^2 = \tau^2$ by Lemma 3. Then $w(\tau,x) = l(x)$. Hence the inequality (5) is satisfied for every $\nu \in \sqrt{\mathcal{M}(\mu)}$. Suppose now that $w(\nu,x) = w(\tau,x) = l(x)$, then $(\tau-\nu)^2(\phi,e) = 0$ by Lemma 3. Hence $\tau = \nu$. This shows that condition (6) is satisfied. This completes the proof of Lemma 4.

Lemma 5. Let $\nu \in \sqrt{\mathcal{M}(\mu)}$ and let $x \in L$. Then

$$\nu(\phi,x) = (\nu\tau_x)(\phi,e) \tag{28}$$

Proof. Let $\tau_x = \nu_1$ in Lemma 1. Then $\nu_o = \nu_1$ and $t_o = 0$. Hence we obtain (28) by applying equation (14).

Lemma 6. Let $x \leq y$, then $\tau_x \leq \tau_y$.

Proof. Let $\nu_1$ and $\nu_2$ be the Jordan decomposition of $\tau_y - \tau_x$, i.e.

$$\tau_y - \tau_x = \nu_1 - \nu_2, \ \nu_1 \geq 0, \ \nu_2 \geq 0, \ \nu_1 \wedge \nu_2 = 0.$$

Then we have

$$0 \leq \nu_2(x,y) = \nu_2(\phi,y) - \nu_2(\phi,x) = (\tau_y\nu_2)(\phi,e) - (\tau_x\nu_2)(\phi,e)$$
$$= ((\tau_y - \tau_x)\nu_2)(\phi,e) = ((\nu_1 - \nu_2)\nu_2(\phi,e)) = -\nu_2^2(\phi,e) \leq 0,$$

since $\nu_1\nu_2 = 0$.

Hence $\nu_2^2(\phi,e) = 0$ i.e. $\nu_2 = 0$ and $\tau_y - \tau_x = \nu_1 \geq 0$. This completes the proof of Lemma 6.

Lemma 7. $\tau_\phi = 0$, $\tau_e = \mu$ and $0 \leq \tau_x \leq \mu$ for every $x \in L$.

Proof. $(\mu - 0)^2(\phi,\phi) + 0^2(\phi,e) = 0$.

Hence $\tau_\phi = 0$. In the same way

$$(\mu - \mu)^2(\phi,e) + \mu^2(e,e) = 0.$$

Hence $\tau_e = \mu$. Since $\phi \leq x \leq e$, we have that $\tau_\phi \leq \tau_x \leq \tau_e$ by Lemma 6. This completes the proof of Lemma 7.

We shall now complete the proof of Theorem 1. Theorem 1 follows from Lemma 4 and Lemma 7 except for equation 4; i.e. it remains to prove

$$\nu(\phi,x)=(\nu\tau_x)(\phi,e)$$

in the case in which $\nu \in \mathfrak{M}(\nu)$, but $\nu \notin \bigvee \mathfrak{M}(\mu)$.

Assume first that $\nu \geq 0$. Then $\nu\tau_x$ is defined since $0 \leq \tau_x \leq \mu$, and

$$\nu \in \mathfrak{M}(\mu) \Longleftrightarrow \nu = \sup_n(\nu \wedge n\mu)$$

(see [1] p.448). Let $\varepsilon$ be a positive number. Then

(29) $$0 \leq (\nu-(\nu \wedge n\mu))(\phi,e) \leq \varepsilon$$

and

(30) $$0 \leq (\nu-(\nu \wedge \mu n))(\phi,x) \leq \varepsilon$$

for some n. But $0 \leq \tau_x \leq \mu$ and $(\nu-(\nu \wedge n\mu)) \geq 0$.
Hence

$$0 \leq (\tau_x(\nu-(\nu \wedge n\mu)))(\phi,e) \leq (\mu(\nu-(\nu \wedge n\mu)))(\phi,e)$$
$$\leq (\nu-(\nu \wedge n\mu))(\phi,e) \leq \varepsilon.$$

But $(\nu \wedge n\mu) \in \bigvee \mathfrak{M}(\mu)$. Hence

$$0 \leq (\tau_x(\nu-(\nu \wedge n\mu)))(\phi,e) = (\tau_x\nu)(\phi,e)-(\tau_x(\nu \wedge n\mu))(\phi,e)$$
$$= (\tau_x\nu)(\phi,e)-(\nu \wedge n\mu)(\phi,x) \leq \varepsilon.$$

Hence by (30) we have

$$0 \leq (\tau_x\nu)(\phi,e)-\nu(\phi,x) \leq \varepsilon$$

This completes the proof of Theorem 1.

Theorem 3. The mapping $x \to \tau_x$ of $L$ into $L^x$ satisfies the following equation.

$$(31) \qquad \tau_x+\tau_y = \tau_{(x \wedge y)} + \tau_{(x \vee y)} \; .$$

Proof.

$$(\nu(\tau_x-\tau_{x \wedge y}))(\phi,e) = (\nu\tau_x)(\phi,e)-\nu\tau_{x \wedge y}(\phi,e)$$

$$=\nu(\phi,x)-\nu(\phi,x \wedge y) = \nu(x \wedge y,x) = \nu(y,x \vee y)$$

$$= (\nu\tau_{x \vee y}-\nu\tau_y)(\phi,e) = (\nu(\tau_{x \vee y}-\tau_y))(\phi,e).$$

Hence

$$\nu\,(\tau_x-\tau_{x \wedge y}-\tau_{x \vee y}+\tau_y)(\phi,e) = 0$$

By setting $\nu = \tau_x+\tau_y-\tau_{x \wedge y}-\tau_{x \vee y}$ we obtain

$$(\tau_x+\tau_y-\tau_{x \wedge y}-\tau_{x \vee y})^2(\phi,e) = 0$$

This completes the proof of Theorem 3.

Theorem 4. The mapping $x \longrightarrow h_x$ of $L$ into $\mathcal{L}_1(S,\mathcal{F},m)$ satisfies the following equation.

$$h_x+h_y = h_{x\wedge y}+h_{x\vee y} \tag{32}$$

Proof. Since $0 \leq \tau_x \leq \mu$ and $0 \leq \tau_y \leq \mu$ we have that $\tau_x, \tau_y, \tau_{x\wedge y}, \tau_{x\vee y} \in B(\mu)$. Hence Theorem 4 follows from Theorem 3 and from the fact that the mapping $\nu \longrightarrow f_\nu$ is defined in such a way that

$$\nu, \tau \in B(\mu) \Longrightarrow f_{\nu+\tau} = f_\nu + f_\tau.$$

Theorem 5. Let $L$ be a modular lattice with least element $\phi$ and greatest element $e$. Then there exist a space $(S,\mathcal{F},m)$ and two mappings $x \longrightarrow h_x$ and $\nu \longrightarrow f_\nu$ such that the following conditions are satisfied.

I $x \longrightarrow h_x$ is a mapping of $L$ into $\mathcal{L}_\infty(S,\mathcal{F},m)$

II $f$ is a 1-1 mapping of $L^*$ into $\mathcal{L}_1(S,\mathcal{F},m)$.

III $h_\phi = 0$, $h_e = 1$, $0 \leq h_x \leq 1$ for all $x \in L$.

IV $\nu(x) = \int h_x f_\nu \, dm$ for every $x \in L$ and $\nu \in L^*$

V $h_x+h_y = h_{x\wedge y}+h_{x\vee y}$ .

VI $L^*$ is isomorphic to $L_1(S,\mathcal{F},m)$ by the mapping $\nu \to f_\nu$.[1)]

We shall sketch a proof. By using Zorn's Lemma it is easy to prove that there exists a subset $K$ of $L^*$ satisfying the following three conditions.

1) see note on page 18.

1. $\mu \in K \Longrightarrow \mu > 0$

2. $(\mu_1 \in K \;\&\; \mu_2 \in K \;\&\; \mu_1 \neq \mu_2) \Rightarrow \mu_1 \wedge \mu_2 = 0$

3. $(\nu \in L^{*} \;\&\; \nu > 0) \Longrightarrow$ there exists a $\mu \in K$ such that $\mu \wedge \nu \neq 0$.

Let

$$K = \{\mu_\alpha \mid \alpha \in I\}$$

in such a way that $\mu_\alpha \neq \mu_\beta$ if $\alpha \neq \beta$ and $\alpha, \beta \in I$. Let $\nu \in L^{*}$. Then there exist mappings $\nu \to \nu_\alpha$ for every $\alpha \in I$ such that $\nu_\alpha \in \mathfrak{M}(\mu_\alpha)$ and $|(\nu - \nu_\alpha)| \wedge \mu_\alpha = 0$. Moreover, $\nu_\alpha = 0$ except for a countable subset $I_\nu$ of $I$. It is easy to prove that

$$\nu = \sum_{\alpha \in I} \nu_\alpha. \tag{33}$$

Let the space $(S_\alpha, \mathcal{F}_\alpha, m_\alpha)$ be the Kakutani-representation of $\mathfrak{M}(\mu_\alpha)$, $\alpha \in I$. Let $S_\alpha \cap S_\beta = \emptyset$ if $\alpha \neq \beta, \alpha, \beta \in I$. Let

$$S = \bigcup_{\alpha \in I} S_\alpha$$

$$\mathcal{F} = \{X \mid X \subseteq S \;\&\; (\forall \alpha)(X \cap S_\alpha \in \mathcal{F}_\alpha)\}$$

$$m(X) = \sum_{\alpha \in I} m_\alpha(X \cap S_\alpha).$$

Moreover, for every $\nu \in L^{*}$ and $\alpha \in I$, there corresponds a representative $f_{\nu,\alpha}$ in

Note that our notation is somewhat different from that used by Alfsen. He identifies functions which are egnal almost everywhere in the space $\mathcal{L}_1(S, \mathcal{F}, m)$, and we do not. We shall use the notaticn $L_1(S, \mathcal{F}, m)$ to denote the quotient space obtained from $\mathcal{L}_1(S, \mathcal{F}, m)$ by identifying functions which are equal almost everywhere.

Strictly speaking, we should have used $L_1(S, \mathcal{F}, m)$ to denote the Kakutani-representation.

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