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ON THE FACIAL STRUCTURE OF A COMPAKT CONVEX K AND THE IDEAL STRUCTURE OF A(K)

by

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The purpose of the present paper is to present some results on the facial structure of a compact convex subset K of a locally convex Hausdorff space, relating faces of K to order ideals of the Archimedean ordered vectorspace A(K) of continuous affine functions on K.

Chapter 1 contains the necessary background material from the theory of Archimedean ordered vectorspaces and their duals, which is the natural non-commutative (or "non-lattice") generalization of Kakutani's theory of L- and M-spaces. The results stated in this chapter, go back to R.V. Kadison ((12)), ((13)), ((14)), F.F. Bonsall ((3)), D.A. Edwards ((6)), A.J. Ellis ((11)), and others. They are included for reference purposes and imply no claim at originality.

The essential results of the present paper are stated in Theorems 2-6 of Chapters 2, 3. Here the central concept is that of a strongly Archimedean face which is a strengthening of the concept of an Archimedean face as defined by E. Störmer ((17)). It is shown that one may assign to every Archimedean face F a numerical invariant $\varrho \in [1,\infty]$, the characteristic of F, which is finite iff F is strongly Archimedean (Theorem 3), and which occurs as the best possible bound on the norms in various contexts related to the extension problem for affine functions and the representation problem for signed boundary measures (Theorems 4, 6). In particular it is proved that for Archimedean faces the "extension property" implies the "bounded extension property" (Remark following Theorem 5).

Chapter 4 contains applications and examples. Applying some recent results of D.A. Edwards ((7)), ((8)) and A. Lazar ((15)), we show that every closed face of a Choquet simplex is strongly Archimedean with characteristic 1. It is also shown by application of results of E.Effros ((9)), ((10)) and E. Störmer ((17)) that a closed face of the state space of a C^*-algebra is Archimedean iff it is invariant, in which case it is strongly Archimedean with characteristic 1. It is easily verified that every Archimedean face of a finite dimensional convex compact set is strongly Archimedean. In the infinite dimensional case there may exist faces which are Archimedean but not strongly Archimedean. A somewhat technical example to this effect is given at the end of Chapter 4.

1. Preliminaries on order unit spaces and their duals

We shall use the term ordered vectorspace to mean a (partially) ordered vectorspace over the reals. A linear map $\phi : A \longrightarrow A'$ between ordered vectorspace is said to be an order homomorphism if

$$(1.1) \qquad \phi(A^+) = \phi(A)^+.$$

A 1-1 order homomorphism is an order isomorphism. Clearly (1.1) implies positivity of ϕ , but the reverse implication is inexact. (Note that the above definition of an order homomorphism is less restrictive than that of ((17)), in which the kernel $N = \phi^{-1}(0)$ is required to be positively generated, i.e. $N = N^+ - N^-$.)

A linear subspace J of an ordered vectorspace A is an order ideal if it is "order convex" in the sence of

$$(1.2) \qquad a,b \in J; \quad c \in A; \quad a \leq c \leq b \implies c \in J.$$

Let N be a linear subspace of an ordered vectorspace A. Now it is well known (and easily verified) that the canonical image $\phi(A^+)$ of A^+ in A/N is a proper cone (i.e. a cone without straight lines) iff N is an order ideal. Hence in this case A/N is an ordered vectorspace in the quotient ordering defined by

$$(1.3) \qquad (A/N)^+ = \phi(A^+),$$

and ϕ is an order homomorphism of A onto A/N.

An arbitrary positive linear map $\psi : A \longrightarrow A'$ between ordered vectorspaces can be decomosed as $\psi = \psi' \circ \phi$ were ϕ is the canonical order homomorphism of A onto $A/\psi^{-1}(o)$, and ψ' is a 1-1 positive linear map from $A/\psi^{-1}(o)$ into A'. Here ψ is an order homomorphism iff ψ' is an order isomorphism.

The order ideal generated by a positive element a of an ordered vectorspace A, is seen to be the set.

(1.4) $J(a) = \{ b \mid \exists \alpha \in \mathbb{R}^+ : -\alpha a \leq b \leq \alpha a \}$.

An element $e \in A^+$ is said to be an <u>order unit</u> if $J(e) = A$, or in other words if every $b \in A$ is bounded above by αe for some $\alpha \geq 0$.

An ordered vectorspace is said to be <u>Archimedean</u> if the negative elements $a \in A^-$ are the only ones for which $\{\alpha a \mid \alpha \in \mathbb{R}^+\}$ has an upper bound.

It is easily verified that an ordered vectorspace A with an order unit e is Archimedean iff

(1.5) $a \leq \beta e$ for all $\beta > 0 \Rightarrow a \leq 0$.

Note that A/J may be non-Archimedean even if A is Archimedean. In fact let A be the 3-dimensional Euclidean space with an Archimedean ordering defined by a cone A^+ possessing a base K which is a 2-dimensional convex body with a non-exposed extreme point x. (For example, K may be the convex hull of a plane disk and some point in the plane of the disk). If J is the 1-dimensional space spanned by x, then J is seen to be an order ideal for which A/J is order isomorphic to $\mathbb{R}^2$ with lexicographic ordering. Hence A/J is non-Archimedean in this case.

<u>Proposition 1. An Archimedean ordered vectorspace A with order unit e admits a norm</u>

(1.6) $\|a\| = \inf\{\lambda \geq 0 \mid -\lambda e \leq a \leq \lambda e\}$,

<u>satisfying</u>

(1.7) $-\|a\| e \leq a \leq \|a\| e$.

<u>Proof</u>. The relation $a \leq \alpha e$ where $\alpha = \inf\{\lambda \mid a \leq \lambda e\}$, follows by an easy application of (1.5). This result and its dual yield (1.7), and now it is straightforward to verify the norm-properties of $\|a\|$.

Henceforth we shall use the term <u>order unit space</u>, and the notation (A,e), to denote an Archimedean ordered vectorspace A with a distinguished order unit e, regarded as a normed vectorspace in the order unit norm (1.6). We observe that if B is a vectorsubspace of A and $e \in B$, then (B,e) is an order unit space in the induced ordering and norm.

<u>Proposition 2</u>. <u>Let $\psi : (A,e) \to (A',e')$ be a linear map between order unit spaces, which maps e into e'. Then ψ is positive iff ψ is bounded with $\|\psi\| = 1$.</u>

<u>Proof</u>. 1) By the definition of the norm, positivity of ψ implies $\|\psi\| = 1$.

2) Assume $\|\psi\| = 1$, and consider an element $a \in A^+$. Without lack of generality we assume $\|a\| \leq 1$, i.e. $0 \leq a \leq e$. Hence also $0 \leq e - a \leq e$, and so $\|e-x\| \leq 1$. This gives

$$\|\psi(e-a)\| \leq 1.$$

By the definition of the norm, $\psi(e-a) \leq \psi(e')$, and so $\psi(a) \geq 0$, q.e.d.

<u>Corollary</u>. <u>Let $\psi : (A,e) \longrightarrow (A',e')$ be a 1-1 linear map between order unit spaces, which maps e into e'. Then ψ is an order isomorphism iff ψ is an isometry</u>.

<u>Proof</u>. Assume (without lack of generality) that ψ is onto, and apply Proposition 2 to ψ and ψ^{-1}.

<u>Remark</u>: A positive linear map $\psi : (A,e) \longrightarrow (A',e')$ with $\psi(e) = e'$, may be an order homomorphism, or in other words the induced map $\psi' : A/\psi^{-1}(0) \longrightarrow A'$ may be an order isomorphism, even if ψ' is no isometry, since the quotient norm of $A/\psi^{-1}(0)$ need not be an order unit norm. Necessary and sufficient conditions for this will be given in the sequel.

Corollary 2. *A linear functional p on an order unit space (A,e) is positive iff P is bounded and $\|p\| = p(e)$.*

Proof. Application of Proposition 2 with $\psi = p(e)^{-1}p$.

A linear functional p on an order unit space (A,e) is a *state* if it is positive and if $p(e) = 1$, or equivalently if

(1.8) $$p(e) = \|p\| = 1.$$

The set of states is seen to be a w^*-compact convex subset of A^*. It will be termed the *state space* of A, and it will be denoted by $S(A)$. The extreme points of $S(A)$ are called *extreme states* (or "pure states").

Proposition 3. *If (A,e) is an order unit space and B is a linear subspace containing e, then every state on (B,e) can be extended to a state on (A,e).*

Proof. Clearly e is an interior point of A^+, and so the theorem on extension of positive linear functionals can be applied.

Proposition 4. *Let (A,e) be an order unit space with unit ball $A_1 = [-e,e]$. Then the unit ball of A^* is given by*

(1.9) $$(A^*)_1 = \mathrm{conv}\ (S(A) \cup -S(A)),$$

and for $a \in A$ we shall have

(1.10) $$\|a\| = \sup \{ |p(a)| \mid p \in S(A) \}$$

and

(1.11) $$a \geq 0 \iff p(a) \geq 0 \quad \text{for all} \quad p \in S(A).$$

Proof. We define $M = \{a \mid a \leq e\}$, $L = \{a \mid -e \leq a\}$, and we claim that the polar sets are given by

(1.12) $$M^o = \mathrm{conv}\ (\{0\} \cup S(A)),\ L^o = \mathrm{conv}\ (\{0\} \cup -S(A)).$$

Let $q \in M^o$. To prove q to be positive, we assume $a \leq 0$. Then $\alpha a \in M$, and hence $\alpha q(a) \leq 1$, for all $\alpha > 0$. It follows that $q(a) \leq 0$, and so a is positive. Clearly $q(e) \leq 1$, and so $q \in \operatorname{conv}(\{0\} \cup S(A))$. The reverse implication is trivial, and the corresponding verifications for L are similar to those for M.

Now (1.9) follows, since

$$(A^*)_1 = A_1{}^o = (M \cap L)^o = \operatorname{conv}(M^o \cup L^o) = \operatorname{conv}(S(A) \cup -S(A))$$

The equality (1.10) follows by the Hahn-Banach Theorem.

To prove (1.11), we first observe that M is closed. In fact if $a \in M$, then

$$\inf \{\alpha \geq 0 \mid a \leq \alpha e\} = 1 + \beta, \; \beta > 0,$$

and it is easily verified that

$$\{b \mid \|a-b\| < \tfrac{\beta}{2}\} \cap M = \emptyset.$$

If $a \not\geq 0$, then $e-a \notin M$, and so there is a $q \in M^o$ such that $q(e-a) > 1$. Thus

$$q(a) < q(e) - 1 \leq 0.$$

By (1.12) q is a positive linear functional, and the non-trivial part of (1.11) follows.

Let (A,e) be an order unit space and X some locally compact Hausdorff space. We shall use the term <u>functional representation of</u> (A,e) <u>over</u> X to denote an isometric, order isomorphism ρ of A onto a point-separating subspace of $C_R(X)$ such that $\rho(e) = 1$. (Note that the specifications of ρ are redundant by Cor. 1 to Prop. 2). A functional representation (ρ, X) of (A,e) will be said to be <u>larger</u> than another functional representation (σ, Y) of (A,e) if there exists a homeomorphism φ of Y into X such that $\sigma = \varphi^* \circ \rho$, φ^* being the conjugate map of $C_R(X)$ into $C_R(Y)$.

Let (ρ, X) be a functional representation of (A,e), and assume for a moment that Y is a closed subspace of X with canonical injection $\varphi : Y \longrightarrow X$, and such that Y is a max-boundary for $\rho(A)$, i.e.

$$(1.13) \qquad \|a\| = \sup_{q \in Y} \left| [\rho a](q) \right| , \quad \text{all } a \in A.$$

In this case the restriction map $\varphi^* : C_R(X) \longrightarrow C_R(Y)$, is an isometry of $\rho(A)$ into $C_R(Y)$ which maps the constant 1 on X to the constant 1 on Y. Hence $(\varphi^* \circ \rho, Y)$ is a functional representation of (A,e), which we shall call the restriction of (ρ, X) to Y.

Theorem 1. Every order unit space (A,e) admits a largest functional representation $(\rho, S(A))$, where $\rho(a) = \hat{a}$ and

$$(1.14) \qquad \hat{a}(p) = p(a), \qquad \text{for all } a \in A,\ p \in S(A).$$

The representing function space $\rho(A)$ consists of all restrictions to $S(A)$ of w^*-continuous linear functionals on A^*, and it comprizes all w^*-continuous affine functions on $S(A)$ iff A is complete (in order unit norm). Also (A,e) admits a smallest functional representation, namely the restriction of $(\rho, S(A))$ to the closure of the set of extreme states.

Proof. 1) By Proposition 4, $(\rho, S(A))$ is a functional representation of (A,e), and it follows from an elementary theorem on weakly continuous linear functionals on a dual space that $\rho(A)$ consists of all $S(A)$-restrictions of w^*-continuous linear functionals on A^*.

The state space $S(A)$ is located on a hyperplane off the origin of A^*. Hence $\rho(A)$ is also equal to the set of all $S(A)$-restrictions of w^*-continuous affine functions on A^*, and this set is known to be uniformly dense in the (uniformly closed) space of all w*-continuous affine functions on $S(A)$ (cf. e.g. ((16, Ch.4))). By the isometry of ρ , $\rho(A)$ is equal to the space of all w*-continuous affine functions on $S(A)$ iff A is complete.

The closure of the set of extreme states, i.e. the set $\overline{\partial_e S(A)}$, is a max-boundary for the space of w*-continuous affine functions on A^* (by virtue of the Krein Milman Theorem), and so $(\rho, S(A))$ admits a restriction to $\overline{\partial_e S(A)}$.

2) To prove maximality and minimality, we consider an arbitrary functional representation (σ, Y) of (A,e). To every point $q \in Y$, we assign a state $\tilde{q} = \varphi(q)$ defined by

$$\tilde{q}(a) = [\sigma a](q). \tag{1.15}$$

By the continuity of the functions σa, the map $\varphi : Y \longrightarrow S(A)$ is continuous. By assumption, $\sigma(A)$ separates the points of Y, and so φ is 1-1. Hence φ is an homëomorphism of Y onto $S(A)$.

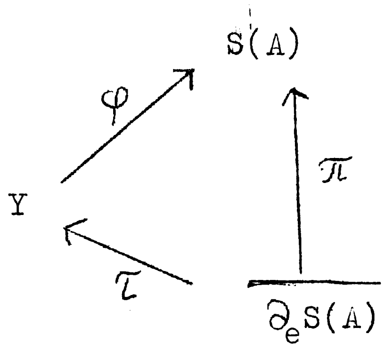
By (1.15)

$$\sup_{q \in Y} |\hat{a}(\varphi(q))| = \sup_{q \in Y} |[\sigma a](q)| = \|a\| =$$

$$= \sup_{p \in S(A)} |\hat{a}(p)| .$$

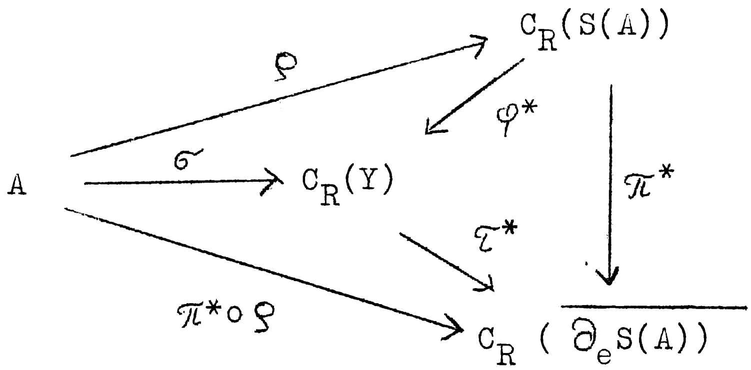
Hence $\varphi(Y)$ is a closed max-boundary for the space of w*-continuous affine functions on $S(A)$. Hence $\varphi(Y)$ contains $\overline{\partial_e S(A)}$ which in the smallest closed max-boundary for this space. (This is an elementary consequence of the Milman Theorem).

Let π be the canonical injection of $\overline{\partial_e S(A)}$ into $S(A)$. Then there is a homëomorphism τ of $\overline{\partial_e S(A)}$ into Y such that the diagram



is commutative.

Passing to the conjugate maps, and making use of the definition (1.15) which may be restated as $\varphi^* \circ \rho = \sigma$, we arrive at the following commutative diagram:



It is seen that $(\rho, S(A))$ is a larger functional representation than (σ, Y) which in turn is larger than $(\pi^* \circ \rho, \overline{\partial_e S(A)})$, and the proof is complete.

The maximal functional representation $(\rho, S(A))$ of an order unit space (A, e) will be called the <u>canonical representation of (A,e) over the state space</u>.

Theorem 1 is essentially due to Kadison. For the existence of the canonical representation cf. ((12)), for the surjectivity of ρ in the complete case cf. ((1)), ((14, Lem.

4,3)), ((18)), and for the maximality and minimality properties cf. ((13, p. 328)).

We reeall that a directed vectorspace is an ordered vectorspace E such that $E = E^+ - E^+$. The following sesult is due to D.Edwards ((6, Th.4)).

Proposition 5. If E is a vectorspace for which the cone E^+ has a base K such that $S = \mathrm{conv}(K \cup -K)$ is radially compact, then the gage functional

$$\| x \| = \inf \{ \lambda \mid x \in \lambda S \} \qquad (1.16)$$

is a norm, and the closed unit ball of this norm is equal to S.

If K is compact in some locally convex Hausdorff topology $\mathcal{T}$ on E, then E is complete in the norm (1.16).

Proof. 1) The set S is absorbing by the directedness of E, and so the gage functional is well defined. It is a norm by the radial boundedness of S, and its colsed unit ball is equal to S since S is radially closed.

2) Assume K, and hence S, to be $\mathcal{T}$-compact, and consider a Cauchy sequence $\{x_n\}$ in the norm (1.16). We may assume $\| x_n \| < \lambda$ for $n=1,2,\ldots$, and by the $\mathcal{T}$-compactness of λS, there is a $\mathcal{T}$-condensation point y of $\{x_n\}$ in λS.

Let $\varepsilon > 0$ be arbitrary and determine n_o such that

$$\| x_n - x_n \| < \varepsilon , \qquad \text{for } n,m \geq n_o.$$

In particular $x_n \in x_{n_o} + \lambda S$ for $n \geq n_o$. Since εS is $\mathcal{T}$-closed, we shall have $y \in x_{n_o} + \varepsilon S$, or equivalently

$$\| y - x_{n_o} \| < \varepsilon .$$

Hence for every $n \geq n_o$

$$\| y - x_n \| \leqq \| y - x_{n_o} \| + \| x_{n_o} - x_n \| .$$

Thus, $\{x_n\}$ converges in the norm to y, and the proof in complete.

We shall use the term base norm space, and the notation (E,K) to denote a directed vectorspace E such that E^+ has a base K for which conv $(K \cup -K)$ is radially conpact, considered as a normed space in the norm (1.16).

The next three propositions are due to Ellis ((11)).

Proposition 6. The norm of a base norm space (E,K) is additive on E^+, in fact $\|x\| = e(x)$ for $x \geq 0$, e being the linear functional which carries the base, i.e. $K \subset e^{-1}(1)$.

Proof. Let $x \in E^*$, i.e. $x = \rho x_0$ with $\rho \geq 0, x_0 \in K$. Then $x \in \rho S$, and so $\rho \geq \|x\|$. Applying e to the equation $x = \rho x_0$, we obtain $e(x) = \rho \geqq \|x\|$.

Conversely $\|x\|^{-1} x \in S$, and so there are elements $y, z \in K$ and scalars $\lambda, \mu \geq 0$ such that

$$\|x\|^{-1} x = \lambda y - \mu z, \qquad \lambda + \mu = 1.$$

Applying the functional e, one obtains

$$\|x\|^{-1} e(x) = \lambda - \mu \leq 1.$$

Hence $e(x) \leq \|x\|$, and the proof is complete.

Proposition 7. Every element x of a base norm space E admits a decomposition $x = y - z$, where $y, z \geq 0$ and $\|x\| = \|y\| + \|z\|$.

Proof. From $\|x\|^{-1} x \in S$ it follows that

$$\|x\|^{-1} x = \lambda y_1 - \mu z_1,$$

where $y_1, z_1 \in K$ and $\lambda, \mu \geq 0$, $\lambda + \mu = 1$. Now the proof is complete with $y = \|x\| \lambda y_1$, $z = \|x\| \mu z_1$.

Proposition 8. If (A,e) is an order unit space, then $(A^*, S(A))$ is a base norm space whos norm is identical with the standard norm of A^*. Conversely, if (E,K) is a base norm space and e is the functional carrying K (cf. Prop. 6), then (E^*,e) is an order unit space whose norm is identical with the standard norm of E^*.

Proof. 1) The first part of the proposition follows directly from Proposition 4.

2) Clearly E^* is Archimedean in the natural ordering. It follows from Proposition 6 and from the decomposition of Proposition 7, that for any $x \in E$:

$$|e(x)| = |e(y) - e(z)| \leq \|y\| + \|z\| = \|x\| .$$

Hence e is bounded with norm 1.

Let $a \in E^*$, and let ϱ be the functional norm of a. Since $S = \mathrm{conv}\,(K \cup -K)$, we shall have

$$\begin{aligned} &= \sup\{\,|a(x)| \mid x \in K\,\} \\ &= \inf\{\,\lambda \geq 0 \mid -\lambda \leq a(x) \leq \lambda \text{ ,all } x \in K\,\} \\ &= \inf\{\,\lambda \geq 0 \mid -\lambda e \leq a \leq \lambda e\,\}. \end{aligned}$$

It follows that e is an order unit, and that the two norms coincide, q.e.d.

In the case of a lattice ordering the theory of order unit- and base norm spaces reduces to Kakutani's theory of L- and M-spaces ((11, Th. 10)).

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2. Archimedean ideals.

We shall study the interrelationship between ideals of an order unit space (A,e) and faces of its state space $S(A)$, and we shall use the notations :

$$(2.1) \qquad N^{\perp} = \{ p \in S(A) \mid p(a) = 0, \text{all } a \in N \}, \ N \subset A.$$

$$(2.2) \qquad F_{\perp} = \{ a \in A \mid p(a) = 0, \text{all } p \in F \}, \ F \subset S(A).$$

It will be necessary also to work in the duality of A and A^*, where the following notations will be applied :

$$(2.3) \qquad N^{o} = \{ q \in A^* \mid q(a) = 0, \text{all } a \in N \}, \quad N \subset A.$$

$$(2.4) \qquad M_{o} = \{ a \in A \mid q(a) = 0, \text{all } a \in M \}, \quad M \subset A^*.$$

Proposition 9. If N is any subset of an order unit space (A,e), then N is a w^* - closed convex subset of $S(A)$ and N is a w^*-closed vector subspace of A^*. If N is positively generated, i.e. $N \subset \text{lin}(N^+)$, then N is a face of $S(A)$ and N is an order ideal of A^*. Conversely if M is any subset of A^*, then M is a (norm and w -) closed vector subspace of A. If M is positively generated, then M is an order ideal. In particular, F is a closed order ideal of (A,e) for every subset F of $S(A)$. The proof is a straight forward verification.

An order ideal J of an order unit space (A,e) is said to be Archimedean (Störmer ((14))) if

(i) J is closed.

(ii) A/J is Archimedean.

(iii) J is positively generated.

If in addition :

(iV) <u>J^o is positively generated,</u>

then we shall say that <u>J is strongly Archimedean</u>.

It is not entirely abvious that there exist order ideals that are Archimedean, but not strongly Archimedean. We shall see that this in fact is impossible in finite dimensional spaces. However it may occur in the in finite dimension case. A somewhat technical example to this effect is presented in section 4 (Prop.9).

<u>Proposition 10.</u> <u>A positively generated order ideal J of an order unit space (A,e) is Archimedean iff it is the kernel of an order homomorphism ψ into an order unit space (A', e') such that $\psi(e)=e'$</u>

<u>Proof</u>. If J is Archimedean, then A/J is an order unit space with unit $\phi(e)$, $\phi: A \to A/J$ being the canonical homomorphism. Conversely, if $\psi: (A,e) \to (A', e')$ is an order homomorphism with kernel J and $\psi(e) = e'$, then A/J is order isomorphic to the Archimedean ordered vectorspace $\psi(A)$. It remains to be proved that J is closed. To this end assume $a \notin J$. Then $\psi(a) \neq 0$, and by Proposition 4, there is a $q \in S(A')$ such that $q(\psi(a)) \neq 0$. It is easily verified that $q \circ \psi$ is a state on (A,e). In particular $q \circ \psi$ is continuous, and so $\bar{J} \subset (q \circ \psi)^{-1}(o)$, $a \notin (q \circ \psi)^{-1}(0)$ which completes the proof.

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<u>Proposition 11.</u> <u>If J is an Archimedean ideal of an order unit space (A,e) and $\varphi: A \to A/J$ is the canonical map. then $(A/J, \varphi(e))$ is an order unit space and $\varphi^*: S(A/J) \to J^\perp$ is a 1-1 affine isomorphism and A w^* - homeomorphism onto.</u>

The proof is a straight forward verification. Recall that $\varphi^* q = q \circ \varphi$ for $q \in S(A/J)$.

By virtue of Proposition 11, we shall identify $S(A/J)$ with $J^\perp$, and for $p \in J^\perp$ we shall write

(2.5) $$p(\varphi(a)) = p(a), \qquad a \in A.$$

The following theorem is a characterization of Archimedean order ideals of (A,e) in terms of their annihilator faces on $S(A)$; first proved by E. Störmer in a slightly redundant form ((17)).

Theorem 2. Assume (A,e) to be an order unit space. If J is an Archimedean ideal of A, then $F = J^\perp$ is a w^*-closed face of $S(A)$ such that

(2.6) $$a \in A,\ p(a) \geq o \text{ all } p \in F \implies c \in A^+ : c \geq a,\ p(c) = p(a) \text{ all } p \in F,$$

and $J = (J^\perp)_\perp$. Conversely, if F is a w^* - closed face of $S(A)$ satisfying (2.6), then $F_\perp$ is an Archimedean ideal of (A,e) and $F = (F_\perp)^\perp$.

Proof. 1.) Let J be an Archimedean ideal and assume $a \in A$, $p(a) \geq o$ for all $p \in J^\perp$. By Propositions 10, 11, this means that every state on the order unit space $(A/J, \varphi(e))$ takes a pos tive value at $\varphi(a)$. By Proposition 4, $\varphi(a) \geq o$.
By the definition of quotient ordering, there is a $b_1 \in A^+$ such that $\varphi(b_1) = \varphi(a)$. It follows that $a - b_1 \in J$. Since J is positively

generated, there is a $b_2 \in J^+$, such that $a - b_1 \leq b_2$. Now define $c = b_1 + b_2$. Then $c \in A^+ + J^+ \subset A^+$ and $a \leq c$. Moreover

$$p(c) = p(b_1) = p(a) \quad \text{all } p \in J^\perp,$$

since $b_2 \in J$ and $a - b_1 \in J$. Hence we have proved (2.6). Trivially $J \subset (J^\perp)_\perp$. To prove the converse, we consider an element $a \notin J$. Then $\varphi(a)$, and so there is a state on A/J not vanishing at $\varphi(a)$. By Proposition 11, there is a $p \in J^\perp$ such that $p(a) \neq 0$. Hence $a \notin (J^\perp)_\perp$, and the first part of the proof is complete.

2.) Let F be a w^* - closed face of $S(A)$ with the property (2.6). To prove the closed order ideal F to be positively generated, we consider an element $a \in F$. Then $p(a) = 0$ for all $p \in F$, and by (2.6) there is a $c \in A^+$ such that $c \geq a$ and $p(c) = p(a) =$ for all $p \in F$. In other words $c \in (F_\perp)^\perp$ and $c \geq a$. Hence we have proved that $F_\perp$ is positively generated.

Next we assume φ to be the canonical map of A onto A/F. To prove $A/F_\perp$ Archimedean we assume $a \in A$, and

$$\varphi(a) \leq \frac{1}{n}\varphi(e), \qquad n = 1, 2, \cdots.$$

By definition of quotient ordering, there exists $b_n \in F_\perp$ such that

$$a \leq \frac{1}{n} e + b_n, \qquad n = 1, 2, \cdots.$$

Now for every $p \in F$.

$$p(a) \leq \frac{1}{n}, \qquad n = 1, 2 \cdots.$$

Thus $p(a) \leq 0$ for all $p \in F$. By (2.6) there is a $b \in A^+$, $b \geq -a$ such that $p(b) = -p(a)$ for all $p \in F$. Writing $c = b + a$, we shall have $a \leq c$, and

$$p(c) = p(b) + p(a) = 0 \text{ , all } p \in F.$$

It follows that $c \in F_{\perp}$, and since φ is an order preserving map with kernel $F_{\perp}$, we shall have

$$\varphi(a) \leq \varphi(c) = 0 .$$

Hence we have proved the Archimedicity of A/F.

Again the inclusion $F \subset (F_{\perp})^{\perp}$ is trivial. To prove the converse, we consider an element $q \in S(A) \setminus F$. By the Hahn Banach Theorem, there exists a w^* - continuois linear functional on A^*, i.e. an element a of A such that

$$q(a) < 0 \leq p(a), \quad \text{all } p \in F.$$

By (2.6) there is a $c \in A^+$ such that $p(a) = p(c)$ for all $p \in F$. It follows that $a - c \in F$, whereas

$$q(a-c) = q(a) - p(c) .$$

Hence $q \notin (F_{\perp})^{\perp}$ and the proof is complete.

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<u>Corollary. The map $J \rightsquigarrow J^{\perp}$ is an order reversing bijection of the Archimedean ideals of the order unit space (A, e) onto the set of w^* - closed faces F of $S(A)$ satisfying (2.6) ; the inverse map being $F \rightsquigarrow F_{\perp}$.</u>

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If J is an Archimedean ideal of an order unit space (A,e), then we shall denote the norm of the order unit space $(A/J, \varphi(e))$ by $\|\varphi(a)\|_0$, and we shall denote the quotient norm of A/J induced from the (order unit) norm of A by $\|\varphi(a)\|_q$. The duals of A/J in these two norms will be denoted by $(A/J)_0^*$ and $(A/J)_q^*$, respectively.

Note that if $F = J^{\perp}$, then $\mathrm{lin}(F) \subset J^{\circ}$, and $\mathrm{lin}(F)$ is a <u>base norm space</u> in the gage functional g_F of $\mathrm{conv}(F \cup -F)$.

<u>Proposition 12.</u> <u>If J is an Archimedean ideal of an order unit space (A,e), if $\varphi: A \to A/J$ is the canonical map and $F = J^{\perp}$, then</u>

(2.7) $\quad \|\varphi(a)\|_o \leq \|\varphi(a)\|_q, \quad$ all $a \in A$,

<u>$(A/J)_o^*$ is a w^* - dense subset of $(A/J)^*$, the map $\varphi^*: (A/J)_q^* \to A^*$ is an isometry onto J° provided with the norm included from A, and the restriction of φ^* to $(A/J)_o^*$ is an isometry onto lin(F) provided with the base norm g_F.</u> -

<u>Proof</u> 1.) If $a \in A$ and $b \in J$, then $-\|a+b\| e \leq a+b \leq \|a+b\| e$.

Hence

$$-\|a+b\| \varphi(e) \leq \varphi(a) \leq \|a+b\| \varphi(e),$$

and so

$$\|\varphi(a)\|_o \leq \|a+b\|$$

Since $b \in J$ was arbitrary, this implies $\|\varphi(a)\|_o \leq \|\varphi(a)\|_q$.

2.) It follows from (2.7) that $(A/J)_o^* \subset (A/J)_q^*$.
The state space of A/J is contained in $(A/J)^*$. Hence $(A/J)^*$ separates the points of A/J, and so $(A/J)_o^*$ is w^* - dense in $(A/J)_q^*$.

3.) The isometry of $\varphi^*: (A/J)^* \to J^{\circ}$ is standard. (Cf. ((4,Ch4, §5, no 4))).

4.) By Proposition 8, $(A/J)_o^*$ is a base norm space, and by Proposition 7, every $q \in (A/J)_o^*$ may be decomposed as follows.

$$q = \rho_1 q_2 - \rho_2 q_2 ,$$

where $\rho_1 + \rho_2 = \|q\|$; $\rho_1, \rho_2 \geq 0$, and q_1, q_2 are states on $(A/J, \varphi(e))$. By Proposition 11, $\varphi^* q_i \in F$ for i = 1,2 Hence

$$\varphi^* q \in \|q\| \text{ conv } (F \cup -F),$$

and so $\varphi^* q \in \text{lin } (F)$ and $g_F (\varphi^* q) \leq \|q\|$. Thus φ^* is a norm decreasing map of $(A/J)^*_0$ into F.

By Part 3 of the proof, φ^* is 1 - 1. Hence it only remains to be proved that φ^* is onto and that the inverse map is norm decreasing. To this end consider an element $p \in \text{lin } (F)$. One may decompose.

$$p = \lambda_1 p_1 - \lambda_2 p_2 ,$$

where $\lambda_1 + \lambda_2 = g_F (p)$; $\lambda_1 \lambda_2$ and p_1, $p_2 \in F$. By Proposition 11, there are unique states q_1, q_2 of $(A/J, \varphi(e))$ such that $p_i = \varphi^* q_i$, i= 1,2. Writing $q = \lambda_1 q_1 - \lambda_2 q_2$, we shall have $q \in (A/J)^*_0$, $\|q\| \leq g_F (p)$, and $\varphi^* q = p$. This completes the proof.

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To every Archimedean ideal J of an order unit space (A,e) we shall assign a number $\rho_J \in [1,\infty]$, the <u>characteristic</u> of J, indicating to what extent the norm-preserving decomposition of Proposition 7 fails for the subspace J° of A^*. Specifically :

$$(2.8) \quad \rho_J = \begin{cases} +\infty, & \text{if } J^\circ \neq (J^\circ)^+ - (J^\circ)^+ \\ \sup_{q \in J^\circ} \inf \left\{ \dfrac{\|q_1\| + \|q_2\|}{\|q\|} \;\middle|\; q = q_1 - q_2 ; q_1 q_2 \in (J^\circ)^+ \right\}, & \text{otherwise} \end{cases}$$

Proposition 13. If J is an Archimedean ideal of finite characteristic in an order unit space (A,e) and $F = J^{\perp}$, then $J^{o} =$ lin F,

$$(2.9) \qquad \rho_J = \sup_{q \in J^o} \frac{g_F(q)}{\|q\|},$$

and for every $q \in J^o$ there is a decomposition

$$(2.10) \qquad q = q_1 - q_2, \qquad \|q_1\| + \|q_2\| \leq \rho_J \|q\|,$$

where $q_1, q_2 \in (J^o)^+$.

Proof. 1.) By definition $\rho < \infty$ implies

$$J = (J^o)^+ - (J^o)^+ = \bigcup_{\lambda > 0} \lambda F - \bigcup_{\lambda > 0} \lambda F = \text{lin } (F).$$

2.) For any $q \in J^o$:

$$g_F(q) = \inf \{\lambda \mid \lambda^{-1} q \in \text{conv } (F \cup - F)$$

$$= \inf \{\lambda \mid \lambda^{-1} q = \lambda_1 p_1 - \lambda_2 p_2 ; p_1, p_2 \in F, \lambda_1 + \lambda_2 = 1\}$$

Writing $q = \lambda \lambda_i p_i$, $i = 1,2$, we obtain

$$\|q_1\| + \|q_2\| = \lambda \lambda_1 + \lambda \lambda_2 = \lambda,$$

and so by substitution :

$$(2.11) \qquad g_F(q) = \inf \{ \|q_1\| + \|q_2\| \mid q = q_1 - q_2 ; q_1, q_2 \in (J^o)^+ \}.$$

This proves (2.9).

3.) Let $q \in J^o$ and write $\lambda = g_F(q)$.

By w^* - compactness, conv $(F \cup - F)$ is radially compact. Hence the gage value is effectively attained, i.e. $q \in \lambda$ conv $(F \cup - F)$.

Now

$$q = \lambda_1 \lambda p_1 - \lambda_2 \lambda p_2 , \qquad \lambda_1 + \lambda_2 = 1 ,$$

where $\lambda_1, \lambda_2 > 0$ and $p_1 , p_2 \in F$. Writing $q_i = \lambda \lambda_i p_i$ for $i = 1,2$, we obtain the desired decomposition (2.10) .

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Proposition 14. If J is an Archimedean ideal of finte characteristic of an order unit space (A,e) and $\varphi : A \to A/J$ is the canonical map, then the order unit norm of $(A/J , \varphi(e))$ is topologically equivalent to the quotient norm induced on A/J from A, and

$$\rho_J = \sup_{a \notin J} \frac{\|\varphi(a)\|_q}{\|\varphi(a)\|_0} \qquad (2.12)$$

Proof. B y Propositions 12, 13, it follows that $(A/J)^*_0 = (A/J)^*_q$, and that

$$\rho_J = \sup_{q \in (A/J)^*} \frac{\|q\|_0}{\|q\|_q}$$

By duality (i.e. by appropriate application of the Hahn - Banach Theorem), one may convert this into the desired formula (2.12).

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It is clear from the definitions that an Archimedean order ideal of finite characteristic must be strongly Archimedean. The reverse implication is in fact also valid , and it is the basis for a series of alternate characterizations of strongly Archimedean ideals. Note that the mutual equivalence of (ii), (iii), (iv) of our next theorem was first proved by D.A. Edwards in a slightly different setting ((6.p.410)).

Theorem 3. If J is an Archimedean ideal of an order unit space (A, e) and $F = J^{\perp}$, then the following statements are equivalent

(i) J is strongly Archimedean.

(ii) The characteristic of J is finite.

(iii) lin (F) is a norm-closed subspace of A^*

(iv) lin (F) is a w^* - closed subspace of A^*

(v) The order unit norm of A/J is topologially equivalent to the quotient norm.

(v1) A/J is complete in order unit norm.

Proof. The proof proceeds in two cycles (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i), (ii) $\Rightarrow$ (v) $\Rightarrow$ (i).

1.) If J is strongly Archimedean, then

$$J^o = (J^o)^+ - (J^o)^+ = \text{lin}\ (F).$$

By w^* - compactness of F and by Proposition S, J^o is complete in the norm g_F. Generally $\|q\| \leq g_F(q)$, and by the Open Mapping Theorem, there must exists a finite number γ such that

$$g_F(q) \leq \gamma \|q\|, \qquad \text{all } q \in J^o.$$

By Proposition 13, J is of finite characteristic $\rho_J < \gamma$.

2.) If J is of finite characteristic, then by Proposition 13, the norm g_F on lin (F) is topologically equivalent to the norm induced from A^*. By Proposition 5, lin(F) is complete in g_F, and hence it must be a norm -closed subspace of A^*.

3.) Assume lin (F) to be a norm-closed subspacee of A^*. For every $q \in$ lin (F) , $\| q \| \leq g_F$ (q), and by the Open Mapping Theorem there is a finite number γ such that

$$q_F(q) \leq \gamma \| q \| , \qquad \text{all } q \in \text{lin } (F).$$

Passing to the unit balls in the two norms, we may restate this as follows

$$\text{lin } (F) \cap (A^*)_1 \subseteq \gamma \text{ conv } (F \cup - F).$$

Equivalently

$$\text{lin } (F) \cap (A^*)_1 = \gamma (\text{ conv } (F \cup - F) \cap (A^*)_1) .$$

The right hand term is w^* - compact, hence w^* - closed. By the Banach-Dieudonnè (Krein- Šmulian) Theorem, lin (F) is a w^* - closed subspace of A^*.

4.) Assume lin (F) to be w^* - closed. By Propositions 9, 12, lin (F) is a w^* - dense subspace of the w^* - closed order ideal J^o of A^*. It follows that

$$J^o = \text{lin } (F) = (J^o)^+ - (J^o)^+ .$$

Hence J is a strongly Archimedean ideal.

5.) If J is of finite characteristic, then the two norms of A/J must be topologically equivalent by virtue of Proposition 14.

6.) If the quotient norm of A/J is topologically equivalent to the order unit norm, then the latter must be complete since the former is complete.

7) Assume the order unit norm of A/J to be complete. By Proposition 12,

$\|\varphi(a)\|_o \leq \|\varphi(a)\|_q$ for all $a \in A$. It follows by the Open Mapping Theorem that the two norms are topologically equivalent. Hence $(A/J)_o^* = (A/J)_q$. Now it follows by application of Proposition 12 once more, that $\text{lin } F = J^o$, and J is strongly Archimedean.

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3. Archimedean faces.

In this section we shall study the space $A(K)$ of all continuous affine functions as a compact convex subset K of some locally convex Hausdorff space over the reals. $A(K)$ is seen to be a complete order unit space in the standard ordering of functions, with distinguished order unit equal to 1 and in uniform norm. Applying the results of the preceding sections to this space, we arrive at the following:

Proposition 15. If K is a compact convex subset of a locally convex Hausdorff space, then the map $x \leadsto \hat{x}$ where $\hat{x}(a) = a(x)$ for all $a \in A(K)$, is an affine and topological isomorphism of K onto $S=S(A(K))$. If $\rho: a \leadsto \hat{a}$ is the canonical representation of $A(K)$ over its state space S, then ρ maps $A(K)$ (isomorphically) onto $A(S)$, and

$$(3.1) \qquad \hat{a}(\hat{x}) = a(x),$$

for all $x \in K$, $a \in A(K)$.

If J is an Archimedean ideal of $A(K)$, and if $F = J^{\perp}$ is the corresponding face of S, then for any two elements a, b of A:

$$(3.2) \qquad a \equiv b \pmod{J} \Longleftrightarrow \hat{a}_F = \hat{b}_F$$

(Here $\hat{a}_F$, $\hat{b}_F$ are the restrictions of $\hat{a}, \hat{b}$ to F).

If ρ' is the canonical representation of $A(K)/J$ over its state space F (cf. Prop. 11), if $\varphi: A(K) \to A(K)/J$ is the canonical map, and if $\pi: A(S) \to A(F)$ is the restriction map, then the diagram

$$\begin{array}{ccc} A(K) & \xrightarrow{\rho} & A(S) \\ {\scriptstyle \varphi}\downarrow & & \downarrow{\scriptstyle \pi} \\ A(K)/J & \xrightarrow{\rho'} & A(F) \end{array}$$

<u>is commutative, and</u>

$$(3.3) \qquad \| \varphi(a) \|_o = \| \hat{a}_F \|$$

$$(3.4) \qquad \| \varphi(a) \|_q = \inf \left\{ \| \hat{b} \| \;\middle|\; b \in A(K),\ \hat{b}_F = \hat{a}_F \right\}$$

<u>The restriction map π is surjective iff J is strongly Archimedean, and in this case</u>

$$(3.5) \qquad \rho_J = \sup_{a \notin J} \inf \left\{ \frac{\| \hat{b} \|}{\| \hat{a}_F \|} \;\middle|\; b \in A(K),\ \hat{b}_F = \hat{a}_F \right\}$$

<u>The proof</u> is straightforward, except perhaps for the verification that $x \rightsquigarrow \hat{x}$ is a surjection of K onto S, and that π is a surjection of A(S) onto A(F) in case J is strongly Archimedean.

1) Let p be an arbitrary state on A(K). By the Hahn-Banach Theorem, there is a norm preserving extension μ of p to a linear functional on $\mathcal{C}(K)$. Now $\|\mu\| = \mu(1) = 1$, and by the existence of barycenters of probability measures (cf. e.g. ((16, Ch. 1)), there exists a point $x \in K$ such that

$$p(a) = a(x) = \hat{x}(a), \qquad \text{all } a \in A(K).$$

2) By a known theorem (cf. e.g. ((16, ch.4))), the F-restrictions of w*-continuous affine functions on A* are uniformly dense in A(F). In particular $\pi(A(S))$ is dense in A(F). By Theorem 3, J is strongly Archimedean iff A(K)/J is complete in order unit norm, and by (3.3) this is

equivalent to $\pi(A(S))$ being uniformly closed, i.e. $\pi(A(S))=A(F)$.

Finally (3.5) follows from Proposition 14 by means of (3.3) and (3.4).

By Proposition 15 one may identify K with $S = S(A(K))$ and $A(K)$ with $A(S)$, by which the canonical homomorphism of $A(K)$ onto the quotient space modulo an Archimedean ideal J turns out to be the restriction map onto the annihilator face F of J, and the quotient norm of an extendable continuous affine function on F becomes the infinum of the norms of all possible extensions to a continuous affine function on the whole convex set.

A closed face F of a compact convex set K in a locally convex Hausdorff space will be said to be (<u>strongly</u>) <u>Archimedean</u> if its annihilator ideal $J = \{ a \in A(K) \mid a_F = 0 \}$ is a (strongly) Archimedean ideal. The characteristic of J will also be said to be the characteristic of F, and it will be denoted by ρ_F as well as ρ_J.

<u>Theorem 4.</u> <u>Let K be a compact convex set in a locally convex Hausdorff space. A closed face F of K is Archimedean iff</u>

$$(3.6) \qquad a \in A(K)\,, \quad a_F \geqq 0$$
$$\Longrightarrow \exists\, c \in A(K)^+ : c \geqq a, \quad c_F = a_F\,.$$

<u>An Archimedean face F of K is strongly Archimedean iff every $a_0 \in A(F)$ admits an extension to a function $a \in A(K)$. In this case it is possible for every $\varepsilon > 0$ to choose $a \in A(K)$ such that</u>

(3.7) $$a_F = a_o \; , \quad \| a \| \leqq (\rho_F + \varepsilon) \| a_o \| \; .$$

Moreover, ρ_F is the smallest number with this property.

Proof. 1) The condition (3.7) is a restatement of condition (2.6) of Theorem 2.

2) By Proposition 15, the restriction map $\pi : A(K) \to A(F)$ is surjective iff F is strongly Archimedean. In this case one may choose $a \in A(K)$ satisfying (3.8) by virtue of the formula (3.5), which also proves ρ_F to be the smallest number for which this is possible.

Our next theorem shows how the concepts of Archimedicity and strong Archimedicity can be characterized by the existence of continuous affine extensions with prescribed lower bounds from the cone:

(3.8) $$Q(K) = \{ a_1 \vee \dots \vee a_n \mid a_i \in A(K), \quad i=1,\dots,n \} .$$

Theorem 5. Let F be a closed face of a convex set K in a locally convex Hausdorff space. F is Archimedean iff

(3.9) $$a \in A(K) \; , \quad g \in Q(K), \quad a_F \geqq g_F$$
$$\Longrightarrow \exists \, c \in A(K) \; : \; c \geqq g \; , \quad c_F = a_F \; ,$$

and F is strongly Archimedean iff

(3.10) $$a \in A(F) \; , \quad g \in Q(K) \; , \quad a \geqq g_F$$
$$\Longrightarrow \exists \, c \in A(K) \; : \; c \geqq g \; , \quad c_F = a \; .$$

Proof 1) Writing $g = a \vee o$, we obtain (3.6) from (3.9). To prove the reverse implication, we assume (3.6) and procede by induction.

Assume first that $a, a_1 \in A(K)$ and that $a \geqq a_1$ on F. By application of (3.6) with $a-a_1$ in the place of a, we determine $b_1 \in A(K)^+$ with $b_1 = a-a_1$, and $b=a-a_1$ on F. Writing $c_1 = b_1+a_1$, we obtain $c_1 = a_1$, and $c_1 = a$ on F. This proves (3.9) in case $g = a_1 \in A(K)$.

Next we assume (3.9) valid whenever $g = a_1 \vee \dots \vee a_{n-1}$ where $a_1, \dots, a_{n-1} \in A(K)$. Let $a_1, \dots, a_n \in A(K)$, and assume that

$$a = a_1 \vee \dots \vee a_n \qquad \text{on } F.$$

By the induction hypothesis, there is a $c_{n-1} \in A(K)$ such that

$$c_{n-1} \geqq a_1 \vee \dots \vee a_{n-1} , \qquad c_{n-1} = a \quad \text{on } F.$$

By application of (3.6) with $c_{n-1} - a_n$ in the place of a, we determine $b_n \in A(K)^+$ such that $b_n = c_{n-1} - a_n$, and $b_n = c_{n-1} - a_n$ on F. Writing $c_n = b_n+a_n$, we obtain

$$c_n = a_1 \vee \dots \vee a_n , \quad c_n = a \quad \text{on } F.$$

This completes the induction.

2) Clearly (3.10) implies (3.9) and also the extendability of every $a \in A(F)$ to a function in $A(K)$. Hence (3.10) implies strong Archimedity of F by virtue of Theorem 4.

Conversely, if F is strongly Archimedean, then every $a_o \in A(F)$ is extendable to a function $a \in A(K)$, and so we may apply (3.9) to yield (3.10).

Remark. There is an essential difference between the two conditions (3.9) and (3.10). The former states that if a function in $A(F)$ is extendable to a function in $A(K)$, then it admits an extension above the prescribed bound g. The latter states that every function in $A(F)$ admits an $A(K)$-extension above g.

Note also that by Theorem 4 one may conclude that if an Archimedean face F has the "extension property", i.e. if every $a_o \in A(F)$ is extendable to an $a \in A(K)$, then F has the "bounded extension property, i.e. there exists a $\gamma < \infty$ such that every $a_o \in A(F)$ can be extended to an $a \in A(K)$ with $\|a\| \leqq \gamma \|a_o\|$.

Finally we shall give a measure theoretic characterization of strong Archimedicity and a formula for the characteristic of an Archimedean face in terms of representing boundary measures. We shall use the standard notation $\mathcal{M}(K)$ to denote the space of (Raden-) measures on the compact convex set K and the symbol $\mathcal{M}_1^+(K)$ to denote the convex subset of positive normalized measures on K. Also we shall use the symbol $\mathcal{M}(\partial_e K)$ (and $\mathcal{M}_1^+(\partial_e K)$) to denote the space of (positive normalized) boundary measures on K. (Cf. e.g. ((2,p.98)).).

Two measures $\mu, \nu \in \mathcal{M}(K)$ are said to be equivalent if $\mu(K) = \nu(K)$ and if μ and ν have common resultant. Thus $\mu \sim \nu$ iff

$$(3.11) \qquad \int a \, d\mu = \int a \, d\nu , \qquad \text{all } a \in A(K).$$

If F is a closed face of K, if $\mu \in \mathcal{M}_1^+(F)$, and if ν is any positive measure on K such that $\mu \sim \nu$, then $\nu \in \mathcal{M}(F)$. (Cf. ((2,p.98)). Note that we identify $\mathcal{M}(F)$ and $\{\mu \mid \mu \in \mathcal{M}(K), \operatorname{Spt}(\mu) \subset F\}$). If ν is allowed to be a signed measure, then the situation is different; now ν may have resultant in F without being supported by F.

Theorem 6. If F is an Archimedean face of a compact convex subset K of a locally convex Hausdorff space, then

$$(3.12) \qquad \rho_F = \sup_{\mu \in \mathcal{M}(F)} \frac{\inf\{\|\lambda\| \mid \lambda \in \mathcal{M}(\partial_e F), \lambda \sim \mu\}}{\inf\{\|\nu\| \mid \nu \in \mathcal{M}(\partial_e K), \nu \sim \mu\}}$$

Proof. We denote the annihilator ideal of F in A(K) by J, and we first establish the following two auxilliary formulas where $p \in (A(K)/J)_o^*$ and $q \in (A(K)/J_q^*$:

$$(3.13) \qquad \|p\|_o = \inf\{\|\lambda\| \mid \lambda \in \mathcal{M}(\partial_e F),\ \lambda(a) = q(a_F) \text{ all } a \in A(K)\}$$

$$(3.14) \qquad \|q\|_q = \inf\{\|\nu\| \mid \nu \in \mathcal{M}(\partial_e K),\ \nu(a) = q(a_F) \text{ all } a \in A(K)\}$$

(Recall that A(K)/J is identified with the space of all F-restrictions of functions in A(K), and so the right-hand terms of (3.13) and (3.14) are well-defined).

Since $(A(K)/J)_o^*$ is a base norm space, we may decompose p into positive components $p = p_1 - p_2$, where $\|p\|_o = \|p_1\|_o + \|p_2\|_o$. By Proposition 15, the space of F-restrictions of functions in A(K) is dense in A(F). Hence p_1, p_2 may be extended by continuity to positive linear functionals $\tilde{p}_1, \tilde{p}_2$ on A(F) such that $\|p\|_o = \|\tilde{p}_1\| + \|\tilde{p}_2\|$. Applying the Choquet Integral Theorem, we obtain two positive boundary measures μ_1, μ_2 on F which represent the positive linear functionals $\tilde{p}_1, \tilde{p}_2$ on A(F). (Cf. e.g. ((5))((16))). Writing $\mu = \mu_1 - \mu_2$, we shall have

$$(3.15) \qquad \|\mu\| = \|p\|_o, \quad \mu(a_F) = p(a_F) \text{ for all } a \in A(K).$$

If μ' is any (Radon-) measure on F such that $\mu(a_F) = p(a_F)$ for all $a \in A(K)$, then μ' is an extension of the linear functional p, and so $\|\mu'\| \geqq \|p\|_0$. By (3.15) this completes the proof of (3.13).

Every $q \in (A(K)/J)_q^*$ corresponds to a (unique) linear functional $\tilde{q}$ over A(K) vanishing on J, and $\|q\|_q = \|\tilde{q}\|$ (cf. e.g. ((4,ch.4,§5,no4))). Applying the Choquet Theorem in the same way as above for the linear functional $\tilde{q}$ on A(K), we arrive at the formula (3.14).

To prove (3.12) we first assume F to be strongly Archimedean, i.e. $\varrho_F < \infty$. By Propositions 11,12 $(A(K)/J)_0^* = (A(K)/J)_q^*$, and

$$(3.16) \qquad \varrho_F = \sup_{q \in (A(K)/J)^*} \frac{\|q\|_0}{\|q\|_q} .$$

Clearly every linear functional q admits a $\mu \in \mathfrak{M}(F)$ such that $\mu(a_F) = q(a_F)$ for all $a \in A(K)$. Hence we may apply the formulas (3.13), (3.14) to rewrite (3.15) in the form (3.12).

Next we assume that F is not strongly Archimedean, i.e. we assume $\varrho_F = \infty$. In this case we must have

$$(3.17) \qquad \sup_{p \in (A(K)/J)_0^*} \frac{\|p\|_0}{\|p\|_q} = \infty ,$$

for otherwise the two norms on $(A(K)/J)_0^*$ would be topologically equivalent, and then also the two norms on A(K)/J would be equivalent, in contradiction with Theorem 3. By virtue of (3.13), (3.14), the right hand term of (3.12) must be infinite, and hence the equality is established also in this case.

Remark. Without lack of generality one may choose the measure μ of (3.12) to be of the form $\mu = \alpha_1 \varepsilon_{x_1} - \alpha_2 \varepsilon_{x_2}$ where $\alpha_1, \alpha_2 \in \mathbb{R}$ and $x_1, x_2 \in F$. In fact this measure is used only to specify an equivalence class of measures, i.e. a certain moment and net charge of a "charge-distribution" on F. The nominator of (3.12) expresses the least total charge of a charge-distribution on $\partial_e K$ with the prescribed moment and net charge. The denominator, however, expresses the least total charge of a charge-distribution on $\partial_e F$ with the prescribed moment and net charge.

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4. Examples and special properties of Archimedean faces.

We shall first state an application to C*-algebras which is essentially due to E. Størmer ((17,ch.5,2)). Recall that if $\mathfrak{A}$ is a C*-algebra with identity I, then the self-adjoint part $\mathfrak{A}_{sa}$ is a complete order unit space with distinguished order unit I, whose state space is denoted by $S(\mathfrak{A})$. A face F of $S(\mathfrak{A})$ is said to be invariant if $p \in F$ implies $p_A \in F$, where p_A is defined by

$$(4.1) \qquad p_A(B) = p(A^*A)^{-1}\, p(A^*BA)$$

whenever $p(A^*A) \neq 0$, and $A \in \mathfrak{A}$.

Proposition 16. A w*-closed face F of the state space $S(\mathfrak{A})$ of a C*-algebra $\mathfrak{A}$ is Archimedean iff it is invariant, in which case it is strongly Archimedean with characteristic 1.

Proof. By a theorem of E. Effros ((9,Th.28)), F is invariant iff $F_\perp$ is the self-adjoint part of a norm-closed two-sided ideal J of $\mathfrak{A}$. By Størmer's theorem ((17,Th.5.2)), this in turn is equivalent to $F_\perp$ being an Archimedean (order) ideal of $\mathfrak{A}_{sa}$.

2) If F is invariant, then by the first part of the proof we shall have $F_\perp = J_{sa}$, where J is a norm-closed two-sided ideal. The quotient $\mathfrak{A}/J$ is itself a C*-algebra (in quotient norm), and its self adjoint part (considered as ordered vectorspace over $\mathbb{R}$) is equal to the order unit space $\mathfrak{A}_{sa}/F_\perp$. Let $\varphi : \mathfrak{A}_{sa} \to \mathfrak{A}_{sa}/F$ be the canonical map, and recall that the state space of $\mathfrak{A}_{sa}/F$ is equal to F (Prop. 11). Hence the order unit norm can be expressed as follows:

$$\| \varphi(a) \|_o = \sup \{ |p(a)| \mid p \in F \}, \quad a \in \mathfrak{A}_{sa}. \tag{4.2}$$

The right hand term of (4.2) is equal to the norm of the self-adjoint element $\varphi(a)$ in the C*-algebra $\mathfrak{A}/J$. Hence we shall have

$$\| \varphi(a) \|_o = \| \varphi(a) \|_q , \tag{4.3}$$

for all $a \in \mathfrak{A}_{sa}$.

Now it follows from Proposition 14 that $\rho_F = 1$, and so F is strongly Archimedean with characteristic 1.

<u>Proposition 17.</u> <u>Every closed face F of a Choquet simplex K is strongly Archimedean with characteristic 1</u>.

<u>Proof</u>. It follows from known properties of simplexes, that

(3.6) is valid, and that every $a \in A(F)$ admists a norm-preserving extension to a continuous affine function on K. (These results are based on D.A. Edwards' theorem ((7)), and are stated more explicitely in ((8)), ((10)) and ((15))). By Theorem 4, the proof is complete.

By Theorem 2, every Archimedean face F of a compact convex set K satisfies $F = (F_\perp)^\perp$, i.e.

$$(4.4) \qquad F = \bigcap \left\{ a^{-1}(0) \mid a \in A(K),\ \ a_F = 0 \right\}$$

In the terminology of ((2)) this means that F is its own "set of determinacy" with respect to A(K). This was proved for closed faces of a simplex K with closed extreme boundary $\partial_e K$ (an "r-simplex") in ((2,Prop.1)), and the condition on $\partial_e K$ was avoided by A. Lazar ((15,Th.1,Cor.1)).

A face F of a compact convex set is said to be exposed relatively to A(K) if there exists an $a \in A(K)$ which "peaks" exactly at F, or what is equivalent (since F is a face), if

$$(4.5) \qquad F = a^{-1}(0).$$

It was proved independently by D.A. Edwards and A. Lazar that every closed face of a metrizable simplex K is A(K)-exposed ((15,Th.1,Cor.2)),((8,Th.3,Cor.)). The proof is based on the property (3.6), and so it applies to Archimedean faces of any metrizable compact convex set.

Proposition 18. An Archimedean face F of a metrizable compact convex set K is $A(K)$-exposed.

Proof. By metrizability and compactness, there is a covering of $K \setminus F$ by compact sets $C_n \subset K \setminus F$, $n=1,2,\dots$. By Hahn-Banach separation, there exists for every natural number n, a continuous affine function a_n on K such that $a_n \geqq 0$ on F and $a_n < 0$ on C_n. By (3.6) there are functions $c_n = a_n \vee 0$ $c_n - a_n = 0$ on F, $n=1,2,\dots$. Define:

$$b_n = \| c_n - a_n \|^{-1} (c_n - a_n), \qquad n=1,2,\dots .$$

Now $b_n = 0$ on F, $b_n > 0$ on C_n, and $\| b_n \| = 1$ for $n=1,2,\dots$. Define next:

$$a = \sum_{n=1} 2^{-n} b_n$$

Then $a \in A(K)$, $a=0$ on F, and $a > 0$ on $K \setminus F = \bigcup_{n=1}^{\infty} C_n$. This completes the proof.

If K is a compact convex set in $\mathbb{R}^n$, $n < \infty$, then $A(K)$ is a vectorspace of dimension at most $n+1$. If F is an Archimedean face of K and J is the corresponding ideal of $A(K)$, then $A(K)/J$ is finite dimensional. By Theorem 3, J (and F) must be strongly Archimedean in this case.

Proposition 19. There exists a compact convex set K in an (infinite dimensional) locally convex Hausdorff space possessing a face F which is Archimedean, but not strongly Archimedean.

Proof. Let E be a countable product of Euclidean planes, i.e. $E=(\mathbb{R}^2)^N$, and define a convex compact subset $K = \prod_{n=1}^{\infty} K_n$ and a closed face $F = \prod_{n=1}^{\infty} F_n$ of K by

$$(4.6) \qquad K_n = \left\{ (\xi,\eta) \mid 0 \leqq \xi \leqq 1, \quad 0 \leqq \eta \leqq \frac{1}{n}((n-1)\xi+1) \right\},$$

$$(4.7) \qquad F_n = \left\{ (0,\eta) \mid 0 = \eta = \frac{1}{n} \right\}.$$

We claim that $A(K)$ consists of all functions of the form

$$(4.8) \qquad a(x) = \alpha_0 + \sum_{i=1}^{\infty} (\alpha_i \xi_i + \beta_i \eta_i)$$

where $x = \{(\xi_1,\eta_1), (\xi_2,\eta_2), \dots \} \in K$, and where

$$(4.9) \qquad \sum_{i=1}^{\infty} (|\alpha_i| + |\beta_i|) < \infty \ .$$

To prove this claim, we consider an $a \in A(K)$. Without lack of generality we assume $a(0) = 0$. Clearly $x = \lim_n x_n$, where

$$x = \{(\xi_1,\eta_1),(\xi_2,\eta_2),\dots\}; x_n = \{(\xi_1,\eta_1),\dots,(\xi_n,\eta_n),(0,0)\dots\}.$$

It follows by the continuity of a, that it can be expressed in the form (4.8) for some sequence $\{(\alpha_i,\beta_i)\}_{i=1,2,\dots}$ of coefficients.

To verify (4.9), we first evaluate a at the point $x_1 = \{(1,1),(1,1),\dots\}$, obtaining

$$(4.10) \qquad \sum_{i=1}^{\infty} (\alpha_i + \beta_i) = a(x_1) < \infty \ .$$

Next we evaluate a at the point $x_2 = \{(\xi_i,\eta_i)\}_{i=1,2,\dots}$ defined by

$$\xi_i = \begin{cases} 1 & \text{if } \alpha_i \geqq 0 \\ 0 & \text{if } \alpha_i < 0 , \end{cases} \qquad \eta_i = \begin{cases} 1 & \text{if } \beta_i \geqq 0 \\ 0 & \text{if } \beta_i < 0 , \end{cases}$$

obtaining

$$(4.11) \qquad \sum_{i=1}^{\infty} (\alpha_i^+ + \beta_i^+) = a(x_2) < \infty$$

It follows by (4.10) and (4.11) that

$$\sum_{i=1}^{\infty} (|\alpha_i| + |\beta_i|) = \lim_{n \to \infty} \left[2 \sum_{i=1}^{n} (\alpha_i^+ + \beta_i^+) - \sum_{i=1}^{n} (\alpha_i + \beta_i) \right] < \infty .$$

Conversely we assume that a is defined by (4.8), and that (4.9) holds. To prove continuity we consider an element $x = \{(\xi_i, \eta_i)\}_{i=1,2,\dots}$ of K and an $\varepsilon > 0$. Choose a natural number n such that

$$(4.12) \qquad \sum_{i=n+1}^{\infty} (|\alpha_i| + |\beta_i|) < \frac{\varepsilon}{2} ,$$

and let V be the neighbourhood of x consisting of all $x' = \{(\xi_i', \eta_i')\}_{i=1,2,\dots}$ such that

$$(4.13) \qquad |\xi_i - \xi_i'| < \frac{\varepsilon}{2M} , \quad |\eta_i - \eta_i'| < \frac{\varepsilon}{2M} , \quad i=1,2,\dots n,$$

where $M = \sum_{i=1}^{\infty} (|\alpha_i| + |\beta_i|)$.

It follows by (4.12) and (4.13) that for every $x' \in V$

$$|a(x)-a(x')| \leq \sum_{i=1}^{\infty} |\alpha_i(\xi_i - \xi_i') + \beta_i(\eta_i - \eta_i')|$$

$$\leq \frac{\varepsilon}{2M} \sum_{i=1}^{n} (|\alpha_i| + |\beta_i|) + \sum_{i=n+1}^{\infty} (|\alpha_i| + |\beta_i|) < \varepsilon .$$

This proves the continuity of a.

Now we shall apply Theorem 4 to show that F is Archimedean, but not strongly Archimedean.

In fact, let $a \in A(K)$ and assume $a_F \geqq 0$. Let a be represented in the form (4.8), (4.9), and define

$$c(x) = \alpha_0 + \sum_{i=1} (\alpha_i \xi_i + |\beta_i| \eta_i)$$

It is seen that c is of the same form (4.8), (4.9); hence $c \in A(K)$. Moreover $c \geqq a$, $c \geqq 0$, and $c_F = a_F$. This proves F to be Archimedean.

For every natural number n, define $a_n \in A(F)$ by

$$a_n(x) = n \eta_n ,$$

where $x = \{(0, \eta_1), (0, \eta_2), \ldots\}$.

Clearly every extension of the function $\eta \leadsto n\eta$ defined on the edge $[(0,0), (0, \frac{1}{n})]$ of the trapezoid F_n to an affine function on all of F_n must assume an absolute value exceding $\frac{n}{2}$ at either of the two vertices (1,0) or (1.1). Hence

$$\inf \{ \|\tilde{a}\| \mid \tilde{a} \in A(K) , \tilde{a}_F = a_n \} \geqq \frac{n}{2} .$$

This proves $\rho_F \geqq \frac{n}{2}$, and since n was arbitrary, this implies $\rho_F = \infty$. Hence F can not be strongly Archimedean.

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