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Notes on the homology of local rings.

by

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Introduction.

Let R be a commutative Noetherian ring with unit, and let $\underline{a}$ and $\underline{b}$ be ideals in 1. In ((7)) Tate has shown that there always exists a fri resolution X of $R/\underline{a}$ such that X is a differential graded algebra over R. It is shown that $\mathrm{Tor}^R(R/\underline{a}, R/\underline{b})$ has a canonical structure as a graded algebra over $R/\underline{a}+\underline{b}$.

Now let R be a local ring. Put $k = R/\underline{m}$. The numbers

$$b_p(R) = \dim_k \mathrm{Tor}_p^R(k,k) \qquad p = 0,1,2,\cdots$$

are called the Betti-numbers of R. We define a formal power series

$$B(R) = \sum_{p=0}^{\infty} b_p(R) Z^p$$

In ((7)) Tate gave a lower bound for $B(R)$ of a non-regular local ring R:

$$R \text{ non-regular} \Rightarrow B(R) \gg \frac{(1+Z)^n}{1-Z^2}, \quad \text{where } n = \dim_k \underline{m}/\underline{m}^2.$$

There also exists upper bounds for $B(R)$ (Serre). There are only a few cases in which one are able to calculate $B(R)$. It is known (Tate, Assmus) that

(1) $B(R) = \frac{(1+Z)^n}{(1-Z^2)^{n-\mathrm{Dim}\,R}}$ if and only if R is a "complete intersection", defined in (2.1). Scheja proved in ((6)) that

(2) $\mathrm{codim}\,R \geq n-2 \Rightarrow B(R)$ is a rational function.

It is still an open question if B(R) is generally rational.

In <u>section 1</u> of the present paper we shall use a technique very similar to one used by Tate in ((7)) to establish some useful algebra isomorphisms concerning $\mathrm{Tor}^R(k,k)$. As a consequence, using a certain characterization of regular rings, we prove that the sequence $\{b_p(R)\}$ of a non-regular ring R is monotonely increasing. We find that $\{b_p(R)\}$ is bounded if and only if $\mathrm{codim}\,R = \dim_k \underline{m}/\underline{m}^2 - 1$.

We also extend a result of Scheja concerning the change of the homology when the ring is divided by a non-zerodivisor t not in $\underline{m}^2$.

In <u>section 2</u> we give a somewhat new proof of the Scheja theorem (2).

In <u>section 3</u> we propose a classification of local rings R based upon considerations on R-algebras which are acyclic after adjunction of a finite set of variables.

In the present paper all rings considered will be local, Noetherian. All modules will be unitary.

Notations.

Rings will be denoted by $R, \tilde{R}, R'$, ideals by $\underline{a}, \underline{b}, \underline{c}, \underline{m}$. The radical of an ideal $\underline{a}$ is denoted by rad $\underline{a}$. If R is a local ring, $\underline{m}$ denotes its maximal ideal, k denotes the field $R/\underline{m}$, and $\hat{R}$ denotes the $\underline{m}$-adic completion of R. Dim R denotes the dimension of R. If M is a R-module, $l(M)$ denotes its length. If M is a vectorspace $\dim(M)$ is its dimension. For convenience we define $n(R) = \dim \underline{m}/\underline{m}^2$. If T is a graded algebra, we let $T[Z_r]$ denote the algebra of polynomials in an indeterminate Z_r of degree r.

1.

The term "R-algebra" will be used in the sense of ((7)) i.e. differential, graded, strict commutative, connected algebras over R, with unit, such that the homogenous components are finitely generated modules over R, which are trivial in negative degrees. The R-algebra obtained by adjoining a variable T of degree w to kill a cycle t of degree $w-1$, we denote by

$$X\langle T\rangle; dT = t.$$

We write $\deg T = w$.

The adjunction of T leads to an exact couple and a long homology sequence, cf. ((7)), the latter being referred to as

<u>the exact homology sequence associated with the adjunction of I.</u>

<u>Lemma 1.1</u>. Let $t_1,\dots,t_n$ be elements in R which build a minimal generating system for $\underline{m}$. Let $\underline{a} = (t_1,\dots,t_{n-1})$. Then if $\underline{a}$ is a non-prime ideal in R, it is $\underline{m}$-primary.

<u>Proof</u>. Assume that $\underline{a}$ is non-prime, then $^R/\underline{a}$ is a non-regular ring. Hence

$$\mathrm{Dim}\,^R/\underline{a} < n(^R/\underline{a}) = 1 \quad \text{so} \quad \mathrm{Dim}\,^R/\underline{a} = 0.$$

Q.E.D.

<u>Lemma 1.2</u>. Let codim R = c. Then there exists a minimal generating system $t_1,\dots,t_n$ of $\underline{m}$ such that $t_1,\dots,t_c$ is a maximal R-sequence.

<u>Proof</u>. If $\underline{m}$ contains a non-zerodivisor, then a lemma of Murthy ((3)) shows that there exists a non-zerodivisor $t \in \underline{m}$, $t \notin \underline{m}^2$. Using this, lemma (1.2) is easily proved by induction on codim R.

Q.E.D.

<u>Proposition 1.3</u>. R is a non-regular local ring if and only if there exists a minimal generating system $t_1,\dots,t_n$ for $\underline{m}$, and a positive integer m such that

(1) $$t_n{}^m \in (t_1,\dots,t_n).$$

Proof. If R is regular, obviously no minimal generating system of $\underline{m}$ can satisfy (1). Now assume that R is non-regular. Since

$$R \cap (\underline{b}\hat{R}) = \underline{b}$$

for every ideal $\underline{b}$ in R, it suffices to show the proposition for $\hat{R}$. We can therefore assume that R is complete. By a wellknown theorem of I.S. Cohen there exists an epimorphism

(2) $f : \tilde{R} \to R$ where $\tilde{R}$ is regular, with

maximal ideal $\tilde{\underline{m}}$. We can assume that $\ker f \subset \tilde{\underline{m}}^2$, see ((1, § 2)), so that $n(\tilde{R}) = n(R)$. Since R is non-regular, we have $\ker f \neq 0$. Consequently $\ker f$ contains a non-zerodivisor a. Put $R' = \tilde{R}/(a)$. f induces an epimorphism

$$f' : R' \to R$$

Since codim $R' = n-1$ there exists by (1.2) a minimal generating system $t_1', \ldots, t_n'$ for the maximal ideal $\underline{m}'$ in R' such that $t_1', \ldots, t_{n-1}'$ is a maximal R'-sequence in $\underline{m}'$. From the maximality of this sequence it follows that the ideal

$$\underline{a} = (t_1', \ldots, t_{n-1}')$$

is non-prime. By (1.1) we have that $t_n' \in \operatorname{rad} \underline{a}$. Consider the elements

$$t_i = f'(t_i'),\ i = 1, \ldots, n$$

They are easily seen to build a minimal generating system for $\underline{m}$ which satisfy (1).

Q.E.D.

Definition 1.4. Let $K(R)$ denote the set of all R-algebras

$$R\langle T_1,\dots,T_n\rangle\ ; dT_i = t_i$$

where $t_1,\dots,t_n$ runs through the set of all minimal generating systems of $\underline{m}$.

From (1.3) follows the well known

Corollary 1.5. Let R be a non-regular ring and let $E \in K(R)$. Then

$$\sum_{i=0}^{n(R)} (-1)^i \dim H_i(E) = 0.$$

Proof. Since R is non-regular, let $t_1,\dots,t_n$ be as in (1.3) where $n = n(R)$. Construct

$$E' = R\langle T_1,\dots,T_{n-1}\rangle\ ; dT_i = t_i$$

Since the elements of $K(R)$ are all isomorphic as R-algebras, cf. ((1, § 6)), it is no loss of generality assuming

$$E = E'\langle T_n\rangle\ ; dT_n = t_n.$$

From (1.3(1)) it follows that the homology modules $H_i(E')$

have finite length for all i, so the Euler-Poincare characteristic

$$\chi(H(E')) = \sum_i (-1)^i l(H_i(R'))$$

is defined. We have an exact sequence of complexes

$$(1) \qquad 0 \longrightarrow E' \xrightarrow{i} E \xrightarrow{j} E' \longrightarrow 0$$

where i is the inclusion map and j is given by $j(e+e'T) = e'$. From (1) it follows that

$$\chi(H(E)) = \chi(H(E')) - \chi(H(E')) = 0.$$

Q.E.D.

Observe that $\mathrm{Tor}^R(R/\underline{a}, R/\underline{b})$ equipped with a trivial differential, can be regarded as an R-algebra, so that the algebra

$$\mathrm{Tor}^R(R/\underline{a}, R/\underline{b})\langle T\rangle; dT = 0$$

is defined.

<u>Theorem 1.6</u>. Let R be a local ring. Let $t_1,\dots,t_{n-1},t$ be a minimal generating system for $\underline{m}$. Let $\underline{a} = (t_1,\dots,t_{n-1})$. Then we have one of the following canonical isomorphisms of graded algebras

(i) If $\underline{a}$ is a prime ideal then

$$(1) \qquad \mathrm{Tor}^R(k,k) \approx \mathrm{Tor}^R(R/\underline{a}, k)[Z_1]/(Z_1^2)$$

where Z_1 is an indeterminate of degree 1.

(ii) If $\underline{a}$ is non-prime then

(2) $$\mathrm{Tor}^R(k,k) \approx \mathrm{Tor}^R(R/\underline{a},k)\langle T,S\rangle ; dT = 0,\ dS = 0$$

where $\deg T = 1$ and $\deg S = 2$. In particular if the characteristic of k is zero, the right hand side of (2) is canonically isomorphic (as a graded algebra) to

(3) $$\mathrm{Tor}^R(R/\underline{a},k)[Z_1,Z_2]/(Z_1^2)$$

where $\deg Z_1 = 1$, $\deg Z_2 = 2$.

Proof. Construct the R-algebra

$$E' = R\langle T_1,\ldots,T_{n-1}\rangle ; dT_i = t_i$$

Adjoin sufficiently many variables of degree ≥ 2 to kill the cycles of positive degrees. In that way one obtains an acyclic R-algebra X' such that $H_o(X') = R/\underline{a}$. Now put

$$Y = X'\langle T\rangle ; dT = t$$

and consider the exact homology sequence associated to the adjunction of T

(4) $$\begin{array}{ccccccc} \ldots \rightarrow & H_{i+1}(X') & \rightarrow H_{i+1}(Y) \rightarrow & H_i(X') & \rightarrow \ldots & & (i = 1) \\ & \| & & \| & & & \\ & 0 & & 0 & & & \end{array}$$

(5) $$\cdots \to \underset{\substack{\| \\ 0}}{H_1(X')} \to H_1(Y) \xrightarrow{j_{*1}} \underset{\substack{\| \\ R/\underline{a}}}{H_0(X')} \xrightarrow{d_{*0}} \underset{\substack{\| \\ R/\underline{a}}}{H_0(X')} \to \underset{\substack{\| \\ k}}{H_0(Y)} \to 0$$

From (4) we get

(6) $$H_i(Y) = 0 \quad \text{for} \quad i = 2.$$

We observe that d_{*0} is multiplication by t. If therefore $\underline{a}$ <u>is a prime ideal</u>, then d_{*0} is injective. From (5) then follows $H_1(Y) = 0$, so Y is acyclic. Tensoring with k commutes with the adjunction of T. Therefore we have canonical isomorphisms of algebras

$$\mathrm{Tor}^R(k,k) \approx H(Y \underset{R}{\otimes} k)$$

$$\approx H(X' \underset{R}{\otimes} k\langle T\rangle; dT = t \otimes 1).$$

Since $t \otimes 1 = 0$ this is further isomorphic to

$$H(X' \underset{R}{\otimes} k)\langle T\rangle; dT = 0$$

$\approx \mathrm{Tor}^R(R/\underline{a},k)\ T\ ; dT = 0$. Since $\deg T = 1$ we easily conclude (i).

Now suppose that $\underline{a}$ is <u>non-prime.</u> By (1.1) $\underline{a}$ is $\underline{m}$-primary. Let m be the least integer such that $t^m \in \underline{a}$, and let

(7) $$t^m + r_1 t_1 + \ldots + r_{n-1} t_{n-1} = 0, \quad r_i \in R$$

Since $R/\underline{a}$ is artinian, it follows that $l(H_i(X')) < \infty$ for all i. Therefore (5) gives

$$(8) \qquad l(H_1(Y)) = l(H_o(Y)) = 1$$

Let σ be the homology class of the cycle

$$s = t^{m-1}T+r_1T_1+\ldots+r_{n-1}T_{n-1} \quad Z_1(Y)$$

We will show that σ generate $H_1(Y)$. By (8) it suffices to show that $\sigma \neq 0$. Looking at the definition of j_{*1} in (5) see ((7, § 3)), one finds that $j_{*1}(\sigma)$ equals the homology class of t^{m-1} in $H_o(X')$. As m is minimally chosen, we have $t^{m-1} \notin \underline{a}$. Therefore $j_{*1}(\sigma) \neq 0$, so $\sigma \neq 0$.

We will show that the R-algebra

$$L = Y\langle S\rangle; dS = s \text{ is acyclic.}$$

Consider the exact homology sequence associated with the adjunction of S:

$$(9) \qquad \ldots \to \underset{0}{\underset{\parallel}{H_4(Y)}} \to H_4(L) \to H_2(L) \to \underset{0}{\underset{\parallel}{H_3(Y)}} \to H_3(Y) \to H_1(L) \to \underset{0}{\underset{\parallel}{H_2(Y)}}$$

$$(10) \qquad \underset{0}{\underset{\parallel}{H_2(Y)}} \to H_2(L) \to H_o(L) \to H_1(Y) \to \underset{0}{\underset{\parallel}{H_1(L)}} \to \ldots$$

By construction $H_1(L) = 0$. From (9) and (6) it follows that it suffices to show that $H_2(L) = 0$. However, this follows from (8) and (10), remembering that $H_o(L) = H_o(Y)$. Therefore we have

$$(11) \qquad \mathrm{Tor}^R(k,k) \approx H(L \underset{R}{\otimes} k)$$

$$\approx H(X' \underset{R}{\otimes} k\langle T,S\rangle; dT = t \otimes 1, dS = s \otimes 1)$$

Again $t \otimes 1 = 0$. Since $t_1,\ldots,t_{n-1},t$ is lineary independant modulo $\underline{m}^2$, it follows from (7) that

$$s \in \underline{m}Y_1 \quad \text{so} \quad s \otimes 1 = 0$$

Therefore the adjunction of T and S kommutes with H. (11) gives

$$\mathrm{Tor}^R(k,k) \approx H(X' \underset{R}{\otimes} k)\langle T,S\rangle; dT = 0, dS = 0$$

$$\approx \mathrm{Tor}^R({}^R/\underline{a},k)\langle T,S\rangle; dT = 0, dS = 0.$$

So (ii) is proved.

In the case of characteristic zero one easily finds the isomorphism between (3) and the right hand side of (2).

Q.E.D.

<u>Corollary 1.7</u>. Let $\underline{a}$ be as in (1.5).

(i) If $\underline{a}$ is a prime ideal then $dh_R k = dh_R {}^R/\underline{a} + 1$.

(ii) If $\underline{a}$ is non-prime then

$$b_p(R) = \sum_{i=0}^{p} c_i, \quad \text{where} \quad c_i = \dim \mathrm{Tor}_i^R({}^R/\underline{a},k).$$

Corollary 1.8. Let R be a non-regular local ring. Then

(i) $B(R) \gg \frac{(1+Z)^n}{1-Z^2}$ (Tate)

(ii) The sequence $\{b_p(R)\}$ is monotonely increasing.

Proof. Since R is non-regular, there is by (1.3) a minimal generating system $t_1,\dots,t_n$ for $\underline{m}$ such that $t_n \in \text{rad}\ \underline{a}$, where $\underline{a} = (t_1,\dots,t_n)$. Therefore (ii) follows from (1.7 (ii)).

The R-algebra

$$E' = R\langle T_1,\dots,T_{n-1}\rangle; dT_i = t_i$$

can be regarded as an exterior algebra with differential. It follows from ((5, appendice I)) that E' can be imbedded (as a graded R-module) as a direct factor of a minimal resolution of $R/\underline{a}$. It follows that

$$(1) \qquad \dim \text{Tor}_i(R/\underline{a},k) \geq \dim E'_i \otimes_R k = \binom{n-1}{i}$$

Now (i) follows from (1) and (1.6 (ii)).

Q.E.D.

Lemma 1.9. Let $t_1,\dots,t_{n-1},t$ be a minimal generating system for $\underline{m}$. Assume that t is a non-zerodivisor. Put

$\underline{a} = (t_1,\dots,t_{n-1})$ and $\overline{R} = R/(t)$. Then we have canonical isomorphisms of graded algebras:

(i) If $\underline{a}$ is a prime ideal

$$\mathrm{Tor}^{\overline{R}}(k,k) \approx \mathrm{Tor}^{R}(R/\underline{a},k).$$

(ii) If $\underline{a}$ is non-prime

$$\mathrm{Tor}^{\overline{R}}(k,k) \approx \mathrm{Tor}^{R}(R/\underline{a},k)\langle S\rangle ; dS = 0$$

where $\deg S = 2$.

Proof. Let X be a free R-algebra such that $H_o(X) = R/\underline{a}$. We have an exact sequence of complexes

(1) $0 \to X \xrightarrow{t\cdot} X \to \overline{X} \to 0$, where $\overline{X} = X/t\cdot X$

has a canonical structure as an R-algebra. Consider the exact homology sequence associated with (1)

(2) $$\ldots \quad \underset{\overset{\|}{0}}{H_{i+1}(X)} \to H_{i+1}(\overline{X}) \to \underset{\overset{\|}{0}}{H_i(X)} \to \ldots \quad (i \geq 1)$$

(3) $$\ldots \to \underset{\overset{\|}{0}}{H_1(X)} \to H_1(\overline{X}) \to \underset{\overset{\|}{R/\underline{a}}}{H_o(X)} \xrightarrow{t\cdot} \underset{\overset{\|}{R/\underline{a}}}{H_o(X)} \to \underset{\overset{\|}{k}}{H_o(\overline{X})} \to 0$$

We observe that (2) and (3) are analogous to (1.6 (4) and (5)). If $\underline{a}$ is prime, then $\overline{X}$ is acyclic and (i) follows. If $\underline{a}$ is non-prime, $H_i(X)$ has finite length for all i. (3) gives $l(H_1(\overline{X})) = 1$, and one finds that $H_1(\overline{X})$ is generated by a single homology class $\bar{\sigma}$, represented by a cycle $\bar{s} \in \underline{m}\bar{X}_1$. Similar arguments as used in (1.6) shows that

$$L = \bar{X}\langle S\rangle ; dS = \bar{s} \quad \text{is acyclic, so}$$

$$\mathrm{Tor}^{\bar{R}}(k,k) \approx H(L \otimes_{\bar{R}} k) \approx H(X \otimes_{R} k)\langle S\rangle ; dS = 0.$$

Q.E.D.

From (1.6) and (1.9) now follows

Theorem 1.10. Let t be a non-zerodivisor in a local ring R. Assume that $t \in \underline{m}$, $t \notin \underline{m}^2$. Then we have a canonical isomorphism of graded algebras

$$\mathrm{Tor}^{R}(k,k) \approx \mathrm{Tor}^{\bar{R}}(k,k)[Z]/(Z^2)$$

where Z is an indeterminate of degree 1. In particular

$$B(R) = (1+Z)B(\bar{R}) \quad \text{(Scheja).}$$

Proposition 1.11. The following conditions are equivalent

(i) The sequence $\{b_p(R)\}$ is bounded

(ii) $\operatorname{codim} R \geq n(R)-1$.

Proof. We observe that if t is a non-zerodivisor in R such that $t \in \underline{m}, t \notin \underline{m}^2$, and $\bar{R} = R/(t)$, then by (1.10) R satisfy (i) (and (ii)) if and only if $\bar{R}$ does. Therefore there is no loss of generality to assume that $\operatorname{codim} R = 0$.

That (ii) implies (i) is nearly obvious, see ((6)), so we show that (i) implies (ii). Assume that

$$\operatorname{codim} R < n(R)-1, \quad \text{that is} \quad n(R) \geq 2.$$

Let $\underline{a}$ be as in (1.6). Since $\underline{a} \neq 0$, it follows from ((4,9, Theorem 23)) that

$$dh_R \, R/\underline{a} = \infty$$

By (1.3) we can assume that $\underline{a}$ is non-prime. From (1.7 (ii)) it now follows that $\{b_p(R)\}$ is non-bounded.

Q.E.D.

2.

In this section we make use of the invariants $\mathcal{E}_i = \mathcal{E}_i(R)$, $i = 1,2,3$ of a given local ring R. $\mathcal{E}_i$ are defined in ((6)). We recall that $\mathcal{E}_i$ are non-negative integers defined by means of $b_p(R)$, $p = 1,2,3,4$.

Definition 2.1. An R-algebra X is said to be ordinary or an ordinary extension of E if

(i) There is an E $K(R)$ such that X has the form

$$X = E\langle U_1, U_2, \ldots \rangle ; dU_i = u_i$$

(ii) $2 \leq \deg U_i \leq \deg U_{i+1}$, $1 = 1,2,\ldots$

(iii) dU_1 is a non-boundary in E. dU_i is a non-boundary in

$$E \langle U_1, U_2, \ldots, U_{i-1} \rangle ; dU_i = u_i \quad \text{for} \quad i \geq 2.$$

X is of <u>degree</u> g if $\sup_i \deg U_i = g$.

<u>Lemma 2.2</u>. (Assmus) An ordinary acyclic R-algebra X has the properties

(i) $dX_p \subset \underline{m}X$ for $1 \leq p \leq 5$

(ii) ε_i equals the number of adjoined variables of degree $i+1$, $i = 1,2,3$.

<u>Proof</u>. (i) is proved in ((2, § 2)). Calculating $\mathrm{Tor}(k,k)$ by means of X, one easily proves (ii).

<u>Lemma 2.3</u>. Let $E \in K(R)$. Then there exists an acyclic R-algebra X which is an ordinary extension of E, such that

$$Z_3(X) \subset \underline{m}E_3 \oplus \coprod_{i=1}^{\varepsilon_1} Z_1(E)S_i \oplus \coprod_{j=1}^{\varepsilon_2} \underline{m}V_j$$

where $\{S_i\}$ and $\{V_j\}$ is the sets of adjoined variables of degree 2 and 3.

<u>Proof</u>. In constructing an ordinary extension of E, it is by ((1,2.5)) possible to represent the elements of

$$H_2(E\langle S_1,\dots,S_{\varepsilon_1}\rangle; dS_i = s_i) \text{ by cycles in } Z_2(E).$$

We may therefore assume that $dV_j \in Z_2(E)$. Examining a general element in X_3, using (2.2) one finds

$$Z_3(X) \subset (E_3 \oplus \coprod_{i=1}^{\varepsilon_1} Z_1(E)S_i \oplus \coprod_{j=1}^{\varepsilon_2} R\cdot V_j) \cap \underline{m}X_3$$

Q.E.D.

<u>Lemma 2.4</u>. If R is complete in its $\underline{m}$-topology. Then for every pair (E,M) where $E \in K(R)$ and M is a submodule of $H_1(E)$, there exists

(i) an epimorphism of rings $f' : R' \to R$ such that $n(R') = n(R)$.

(ii) a morphism of R'-algebras

$$\Phi : E' \quad E, \text{ where } E' \in K(R') \text{ and } \Phi_0 = f'$$

(iii) Φ induces a morphism of graded algebras

$$\Phi_* : H(E') \to H(E) \text{ such that } \Phi_{*1} : H_1(E') \to H_1(E)$$

is monomorphic with image M.

<u>Proof</u>. Since R is complete, let $\tilde{R}$, $\tilde{\underline{m}}$, f be as in (1.3 (2)). Let E be defined by $E = R\langle T_1,\dots,T_n\rangle; dT_i = t_i$. Let $\sigma_1,\dots,\sigma_m$ be a k-base of M, represented by cycles

$$s_j = \sum_{i=1}^{n} r_i^j T_i \quad j = 1,\dots,m, \quad \text{where } r_i^j \in R.$$

Choose elements $\tilde{r}_i^j$ and $\tilde{t}_i$ in $\tilde{R}$ which are mapped by f to r_i^j and t_i. Let $\tilde{\underline{a}}$ be the ideal in $\tilde{R}$ generated by

$$\sum_i \tilde{r}_i^j \tilde{t}_i, \quad j = 1,\ldots,m.$$

Put $R' = \tilde{R}/\tilde{\underline{a}}$. Let $g : R \to R'$ be the canonical map, and let

(1) $f' : R' \to R$ be the map induced by f.

Put

$$E' = R'\langle T_1', \ldots, T_n' \rangle ; dT_i' = g(\tilde{t}_i)$$

By means of f, E is given R'-algebra structure, and can be identified, as an R'-algebra, with $E' \otimes_{R'} R$. Now let

(2) $\Phi : E' \to E$ be defined by $\Phi(e') = e' \otimes 1$, for $e' \in E'$.

Φ is easily seen to satisfy (i). Since

$$\dim H_1(E') = \dim \underline{a}/\underline{m}\cdot\underline{a} \quad ((1, \S\ 2))$$

we have $\dim H_1(E') \leq \dim M$. On the other hand $M \subset \Phi_*(H_1(E'))$, so $M = \Phi_*(H(E'))$, and (iii) follows.

Q.E.D.

Definition 2.5. R is a complete intersection if

$$n(R) - \operatorname{Dim} R = \varepsilon_1(R).$$

It should be observed that R is a complete intersection if and only if the completion $\hat{R}$ is. This is by ((6))

equivalent to saying that $\hat{R}$ can be obtained by dividing a regular local ring $\tilde{R}$ by a suitable $\tilde{R}$-sequence.

<u>Proposition 2.6</u>. Assume that R is a non-regular ring, such that $n(R) \leq 2$. Let $E \in K(R)$. Then

(i) if $H_1(E)^2 = 0$ we have

(i') $B(R) = \frac{(1+Z)^2}{1-\varepsilon_1 Z^2-\varepsilon_2 Z^3}$ and $\varepsilon_3 = \binom{\varepsilon_1}{2}$

(i") $\varepsilon_2 = \varepsilon_1 - 1$.

(ii) If $H_1(E)^2 \neq 0$, then R is a complete intersection such that $\mathrm{Dim}\, R = 0$.

<u>Proof</u>. First assume $H_1(E)^2 = 0$. Consider the R-algebra of (2.3). Since $E_q = 0$ for $q \geq 3$, we have

$$\underline{m}V_j,\ Z_1(E)S_i \subset Z_3(X) \quad \text{for all } i,j.$$

Therefore the inclusion of (2.3) becomes equality. Applying the functor $\mathrm{Tor}_p^R(-,k)$ to this equation, one gets

(1) $$b_{p+4} = \varepsilon_1 b_{p+2} + \varepsilon_2 b_{p+1} \quad \text{for } p \geq 1.$$

Since $Z_3(X) \subset \underline{m}X$, (1) is also correct for $p = 0$. Hence (i'). From (2.2) and ((1,2.5)) one obtains

$$\varepsilon_2 = \dim {}^{H_2(E)}/_{H_1(E)^2}$$

Therefore we have $\mathcal{E}_2 = \dim H_2(E)$ and (i") follows by (1.5). Now assume that $H_1(E)^2 \neq 0$. Since $B(R) = B(\hat{R})$, $\mathcal{E}_i(R) = \mathcal{E}_i(\hat{R})$ cf. ((6)), and since we have an algebra-isomorphism $H(E) \approx H(E \otimes_R \hat{R})$, we may assume that R is complete.

Let $\sigma_1, \sigma_2 \in H_1(E)$ such that $\sigma_1 \cdot \sigma_2 \neq 0$. Let M be the submodule of $H_1(E)$ generated by $\{\sigma_1, \sigma_2\}$.

Apply (2.4) to the pair (E,M), and let R', E', f', Φ, Φ_* be as in (2.4). Since $\dim H_1(E') = \dim M = 2$, (1.5) gives

(2) $\dim H_2(E') = 1$ i.e. $\dim 0 : \underline{m}' = 1$, where $\underline{m}'$ is the maximal ideal in R'.

Since

(3) $$\Phi_*(H_1(E')^2) = \Phi_*(H_1(E'))^2 = M^2 \neq 0$$

it follows from (2) that

(4) $H_2(E') = H_1(E')^2$ i.e. R' is a complete intersection cf. ((1, § 2)). Further

(5) $\operatorname{Dim} R' = n(R') - \mathcal{E}_1(R') = 0$. We are going to show that $R = R'$. Assume that $\ker f' \neq 0$. From (2) and (5) it follows that $0 : \underline{m}'$ is contained in every non-zero ideal in R' so $0 : \underline{m}' \subset \ker f'$. Therefore

$$\Phi_*(H_2(E')) = \Phi((0 : \underline{m}')E_2') = f(0 : \underline{m}')\Phi(E_2') = 0$$

which is absurd by (3).

Q.E.D.

Theorem 2.7 (Scheja). Let R be a non-regular local ring. Assume that codim $R \geq n(R)-2$. Then

$$B(R) = \frac{(1+Z)^n}{1-\varepsilon_1 Z^2-\varepsilon_2 Z^3-(\varepsilon_3-\binom{\varepsilon_1}{2})Z^4}, \quad \text{where} \quad n = n(R)$$

If R is not a complete intersection then

$$\varepsilon_2 = \varepsilon_1 - 1 \quad \text{and} \quad \varepsilon_3 = \binom{\varepsilon_1}{2}.$$

Proof. Dividing R by the maximal R-sequence in (1.2) Scheja shows that it suffices to show the Theorem for rings R such that $n(R) \leq 2$, in which case (2.6) and the formula (1) in the introduction give the sufficient information.

Q.E.D.

3.

We now restrict ourselves to those local rings R whose residue class fields k are of characteristic $p \neq 0$. We give a few results, without proofs, on the following classification:

Definition 3.1. Let $\underline{C}_1$ be the class of regular local rings. For each $i \geq 2$ define

$\underline{C}_i$ is the class of rings R which satisfy

(i) There is an acyclic R-algebra X which is ordinary of degree i.

(ii) $R \notin \underline{C}_q$ for $q < i$.

The elements of $\underline{C}_1 \cup \underline{C}_2$ are exactly the complete intersections.

Observe that if $R \in \underline{C}_i$ for some i, then B(R) has a radius of convergence ≥ 1. By (2.7) rings R such that codim $R \geq n(R)-2$, escape the above classification unless they are complete intersections.

Remembering characteristic $k = p \neq 0$, one can prove

<u>Proposition 3.2</u>. Let $E \in K(R)$, and let X be an extension of the form

$$X = E\langle U_1, U_2, \ldots\rangle; dU_i = u_i$$

deg U_i being arbitrary. Then the homology algebra H(X) is nilpotent, that is every homology class of positive degree is nilpotent.

Using (3.2) one can show

<u>Proposition 3.3</u>.

$$\underline{C}_{2i+1} \text{ is empty for } i = 1,2,3,\ldots$$

To show that $\underline{C}_3$ is empty, we need no restriction on the characteristic.

Proposition 3.4. If $R \in \underline{C}_i$ for some $i \geq 3$, then

(i) codim $R \leq n - \varepsilon_1 - 2$

(ii) $\dim \underline{c}:\underline{m}/\underline{c} = 1$ each time $\underline{c}$ is generated by a maximal R-sequence in $\underline{m}$.

The requirements (i) and (ii) seems to be very restrictive. It seems doubtful that the two conditions can be satisfied simultaneously. As an example the Macaulay rings do not even satisfy (i).

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Errata

p.1, l.6 for "ideals in 1" read "ideals in R"

p.2, l.2 for "(2.1)" read "(2.5)"

p.2, l.13 read "Codim $R \geqslant \dim_k \underline{m}/\underline{m}^2 - 1$

p.4 bottom read "$t_n{}^m \in (t_1, \ldots, t_{n-1})$"

p.6, l.16 for "((1,§6))" read "((7,§6))"

p.7, l.3 for "$H_i(R')$" read "$H_i(E')$"

p.8 bottom for "$(i = 1)$" read "$(i \geqslant 1)$"

p.9, l.3 for "$i=2$" read "$i \geqslant 2$"

p.9, l.13 read "$\mathrm{Tor}^R(R/\underline{a}, k) \langle T \rangle$; $dT = 0$

p.10, l.4 read "$s = t^{m-1}T + r_1T_1 + \ldots + r_{n-1}T_{n-1} \in Z_1(Y)$"

p.11, l.15 for "(1.5)" read "(1.6)"

p.12, l.6 for "$(t_1, \ldots, t_n)$" read "$(t_1, \ldots, t_{n-1})$"

p.12, l.15 for "(1.6(ii))" read "(1.7(ii))"

p.15, bottom for "E K(R)" read $E \in K(R)$"

p.16, l.2 for "dU_i" read "dU_j"

p.16, l.4 read "$E \langle U_1, U_2, \ldots, U_{j-1} \rangle$; $dU_i = u_i$ for $j \geqslant 2$"

p.17, l.11 for "Φ : E' E "read "$\Phi : E' \longrightarrow E$"