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by

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If $\mathfrak{A}$ is a C^*-algebra and $\mathfrak{I}$ and $\mathfrak{J}$ are uniformly closed two-sided ideals in $\mathfrak{A}$ then so is $\mathfrak{I}+\mathfrak{J}$. The following problem has been proposed by J. Dixmier [1, Problem 1.9.12] : is $(\mathfrak{I}+\mathfrak{J})^+ = \mathfrak{I}^+ + \mathfrak{J}^+$, where $\mathfrak{L}^+$ denotes the set of positive operators in a family $\mathfrak{L}$ of operators ? He suggested to the author that techniques using the duality between invariant faces of the state space $S(\mathfrak{A})$ of $\mathfrak{A}$ and two-sided ideals in $\mathfrak{A}$, as shown by E. Effros, might be helpful in studying it. In this note we shall use such arguments to solve the problem to the affirmative.

By a <u>face</u> of $S(\mathfrak{A})$ we shall mean a convex subset F such that if $\rho \in F$, $\omega \in S(\mathfrak{A})$ and $a\omega \leq \rho$ for some $a > 0$, then $\omega \in F$. F is an <u>invariant face</u> if $\rho \in F$ implies the state $B \to \rho(A^*BA)\cdot\rho(A^*A)^{-1}$ belongs to F whenever $\rho(A^*A) \neq 0$ and $A \in \mathfrak{A}$. We denote by $F^\perp$ the set of operators $A \in \mathfrak{A}$ such that $\rho(A) = 0$ for all $\rho \in F$. If $\mathfrak{I} \subset \mathfrak{A}$, $\mathfrak{I}^\perp$ shall denote the set of states ρ such that $\rho(A) = 0$ for all $A \in \mathfrak{I}$. E. Effros [2] has shown that the map $\mathfrak{I} \to \mathfrak{I}^\perp$ is an order inverting bijection between uniformly closed two-sided ideals of $\mathfrak{A}$ and w^*-closed invariant faces of $S(\mathfrak{A})$. Moreover, $(\mathfrak{I}^\perp)^\perp = \mathfrak{I}$, and $(F^\perp)^\perp = F$ when F is a w^*-closed invariant face. If $\mathfrak{I}$ and $\mathfrak{J}$ are uniformly closed two-sided ideals in $\mathfrak{A}$ then $(\mathfrak{I} \cap \mathfrak{J})^\perp = \mathrm{conv}(\mathfrak{I}^\perp, \mathfrak{J}^\perp)$ - the convex hull of $\mathfrak{J}^{\perp}$ and $\mathfrak{F}^{\perp}$, and $(\mathfrak{J}+\mathfrak{F})^{\perp} = \mathfrak{J}^{\perp} \cap \mathfrak{F}^{\perp}$. If A is a self-adjoint operator in $\mathfrak{A}$ let $\hat{A}$ denote the w^*-continuous affine function on $S(\mathfrak{A})$ defined by $\hat{A}(\rho) = \rho(A)$. It has been shown by R. Kadison, [3] and [4], that the map $A \to \hat{A}$ is an isometric order-isomorphism of the self-adjoint part of $\mathfrak{A}$ onto all w^*-continuous real affine functions on $S(\mathfrak{A})$. Moreover, if $\mathfrak{J}$ is a uniformly closed two-sided ideal in $\mathfrak{A}$, and ψ is the canonical homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{J}$, then the map $\rho \to \rho \circ \psi$ is an affine isomorphism of $S(\mathfrak{A}/\mathfrak{J})$ onto $\mathfrak{J}^{\perp}$. Thus the map $\psi(A) \to \hat{A}|\mathfrak{J}^{\perp}$ is an order-isomorphic isometry on the self-adjoint operators in $\mathfrak{A}/\mathfrak{J}$. We shall below make extensive use of these facts. For other references see [1, § .1].

<u>Theorem</u>. Let $\mathfrak{A}$ be a C^*-algebra. If $\mathfrak{J}$ and $\mathfrak{F}$ are uniformly closed two-sided ideals in $\mathfrak{A}$ then

$$(\mathfrak{J}+\mathfrak{F})^{+} = \mathfrak{J}^{+} + \mathfrak{F}^{+}.$$

In order to prove the theorem we may assume $\mathfrak{A}$ has an identity, denoted by I. We first prove a

<u>Lemma</u>. With the assumptions as in Theorem let A belong to $(\mathfrak{J}+\mathfrak{F})^{+}$, and let $\varepsilon > 0$ be given, $\varepsilon < 1$. Then there exist B in $\mathfrak{J}^{+}$ and C in $\mathfrak{F}^{+}$ such that $0 \leq A-B-C \leq \varepsilon I$.

<u>Proof</u>. We may assume $\|A\| \leq 1$. Let ψ denote the canonical homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{F}$. Then $\psi(\mathfrak{J}+\mathfrak{F}) = \psi(\mathfrak{J})$. Now

$\psi(A) \geq 0$. Therefore there exists $B_1 \in \mathfrak{J}^+$ such that $\psi(B_1) = \psi(A)$. Then $\hat{B}_1 | \mathfrak{J}^\perp = 0$ and $\hat{B}_1 | \mathfrak{Y}^\perp = \hat{A} | \mathfrak{Y}^\perp$. Since $(\mathfrak{J} \cap \mathfrak{Y})^\perp = \mathrm{conv}(\mathfrak{J}^\perp, \mathfrak{Y}^\perp)$, $\hat{B}_1 | (\mathfrak{J} \cap \mathfrak{Y})^\perp \leq \hat{A} | (\mathfrak{J} \cap \mathfrak{Y})^\perp$. Let ϕ denote the canonical homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{J} \cap \mathfrak{Y}$. Then $0 \leq \phi(B_1) \leq \phi(A)$. Let f be the real continuous function $f(x) = (\varepsilon/3)^2$ for $x \leq (\varepsilon/3)^2$, $f(x) = x$ for $x > (\varepsilon/3)^2$. Let

$$S = f(A)^{-\frac{1}{2}} B_1 f(A)^{-\frac{1}{2}} .$$

Then $S \in \mathfrak{J}^+$, and

(1)
$$\begin{aligned} 0 \leq \phi(S) &= f(\phi(A))^{-\frac{1}{2}} \phi(B_1) f(\phi(A))^{-\frac{1}{2}} \\ &\leq f(\phi(A))^{-\frac{1}{2}} \phi(A) f(\phi(A))^{-\frac{1}{2}} \\ &\leq \phi(I). \end{aligned}$$

Let g be the real continuous function $g(x) = x$ for $x \leq 1$, $g(x) = 1$ for $x > 1$. Since $g(0) = 0$, $g(S)$ is by the Stone-Weierstrass Theorem a uniform limit of polynomials in S without constant terms. Since $S \in \mathfrak{J}^+$, and $\mathfrak{J}$ is uniformly closed, $g(S) \in \mathfrak{J}^+$. By (1)

(2)
$$\phi(g(S)) = g(\phi(S)) = \phi(S).$$

Let

$$B = (f(A)^{\frac{1}{2}} - \varepsilon/3\, I)\, g(S)\, (f(A)^{\frac{1}{2}} - \varepsilon/3\, I).$$

Since $g(S) \in \mathfrak{J}^+$ so is B. Now $(f(x)^{\frac{1}{2}} - \varepsilon/3)^2 \leq x$ for $x \geq 0$, and $g(S) \leq I$. Hence $0 \leq B \leq A$. By (2)

$$\phi(B) = (f(\phi(A))^{1/2} - \varepsilon/3\,\phi(I))\,\phi(g(S))\,(f(\phi(A))^{1/2} - \varepsilon/3\,\phi(I))$$
$$= \phi(B_1) - \varepsilon/3\left[f(\phi(A))^{1/2}\phi(S) + \phi(S)f(\phi(A))^{1/2} - \varepsilon/3\,\phi(S)\right].$$

Since $\| f(\phi(A))^{1/2} \| \leq 1$, $\| \phi(S) \| \leq 1$, and $\varepsilon < 1$

$$\| \hat{B}|(\mathcal{J} \cap \mathcal{Y})^{\perp} - \hat{B}_1|(\mathcal{J} \cap \mathcal{Y})^{\perp} \| = \| \phi(B) - \phi(B_1) \| \leq \varepsilon .$$

In particular,

(3) $$\| \hat{B}|\mathcal{Y}^{\perp} - \hat{A}|\mathcal{Y}^{\perp} \| = \| \hat{B}|\mathcal{Y}^{\perp} - \hat{B}_1|\mathcal{Y}^{\perp} \| \leq \varepsilon .$$

Apply the preceding to $A-B$ instead of A and to $\mathcal{Y}$ instead of $\mathcal{J}$. Choose $C_1 \in \mathcal{Y}^+$ such that $C_1 \leq A-B$, and

(4) $$\| \hat{C}_1|\mathcal{J}^{\perp} - (\hat{A}-\hat{B})|\mathcal{J}^{\perp} \| \leq \varepsilon .$$

Since $\hat{C}_1|\mathcal{Y}^{\perp} = 0$ 3) implies

(5) $$\| \hat{C}_1|\mathcal{Y}^{\perp} - (\hat{A}-\hat{B})|\mathcal{Y}^{\perp} \| \leq \varepsilon .$$

By (4) and (5)

$$\| \phi(C_1) - \phi(A-B) \| =$$
$$= \| \hat{C}_1|\operatorname{conv}(\mathcal{J}^{\perp},\mathcal{Y}^{\perp}) - (\hat{A}-\hat{B})|\operatorname{conv}(\mathcal{J}^{\perp},\mathcal{Y}^{\perp}) \| \leq \varepsilon .$$

Let $D = A-(B+C_1)$. Then $D \geq 0$, and $\| \phi(D) \| \leq \varepsilon$. Let h be the real continuous function $h(x) = 0$ for $x \leq \varepsilon$, $h(x) = x-\varepsilon$ for $x > \varepsilon$. Then $\phi(h(D)) = h(\phi(D)) = 0$, and $h(D) \in (\mathcal{J} \cap \mathcal{Y})^+ \subset \mathcal{Y}^+$. Furthermore

$$D-\varepsilon I \leq h(D) \leq D. \tag{6}$$

Let $C = C_1+h(D)$. Then $C \in \mathfrak{J}^+$, and by (6)

$$0 \leq B+C \leq B+C_1+D = A \leq B+C_1+h(D)+\varepsilon I = B+C+\varepsilon I.$$

The proof is complete.

<u>Proof of Theorem</u>. Let $A \in (\mathcal{J}+\mathfrak{J})^+$. Multiplying A by a scalar we may assume $0 \leq A \leq I$. By Lemma choose $B_o \in \mathcal{J}^+$, $C_o \in \mathfrak{J}^+$ such that

$$0 \leq A-B_o-C_o \leq 2^{-1}I.$$

Then $\|B_o\| \leq \|A\| \leq 1$, $\|C_o\| \leq \|A\| \leq 1$. Suppose inductively $B_o, B_1, \ldots, B_{n-1}$ are chosen in $\mathcal{J}^+$ and $C_o, C_1, \ldots, C_{n-1}$ are chosen in $\mathfrak{J}^+$ such that $\|B_j\| \leq 2^{-j}$, $\|C_j\| \leq 2^{-j}$, and

$$0 \leq A - \sum_{j=0}^{n-1} B_j - \sum_{j=0}^{n-1} C_j \leq 2^{-n}I.$$

Apply Lemma to $A - \sum_{j=0}^{n-1} B_j - \sum_{j=0}^{n-1} C_j$ and to $\varepsilon = 2^{-n-1}$.

Then there exist $B_n \in \mathcal{J}^+$, $C_n \in \mathfrak{J}^+$ such that

$$0 \leq A - \sum_{j=0}^{n-1} B_j - \sum_{j=0}^{n-1} C_j - B_n - C_n \leq 2^{-n-1}I, \tag{7}$$

or

$$0 \leq A - \sum_{j=0}^{n} B_j - \sum_{j=0}^{n} C_j \leq 2^{-n-1}I.$$

Moreover, by (7) $\|B_n\| \leq 2^{-n}$, $\|C_n\| \leq 2^{-n}$; the induction argument is complete. Let

$$B = \sum_{j=0}^{\infty} B_j \quad , \qquad C = \sum_{j=0}^{\infty} C_j .$$

Then $B \in \mathfrak{J}^+$, $C \in \mathfrak{Y}^+$, and

$$\|A-B-C\| = \lim_{n\to\infty} \|A - \sum_{j=0}^{n} B_j - \sum_{j=0}^{n} C_j\| \leq \lim_{n\to\infty} 2^{-n-1} = 0.$$

Thus $A = B+C \in \mathfrak{J}^+ + \mathfrak{Y}^+$, and $(\mathfrak{J}+\mathfrak{Y})^+ \subset \mathfrak{J}^+ + \mathfrak{Y}^+$. Since the converse inclusion is trivial, the proof is complete.

References.

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