Two-sided ideals in C*-algebras.

by

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If $\mathcal{A}$ is a $C^*$-algebra and $\mathcal{I}$ and $\mathcal{J}$ are uniformly closed two-sided ideals in $\mathcal{A}$ then so is $\mathcal{I} + \mathcal{J}$. The following problem has been proposed by J. Dixmier [1, Problem 1.9.12]: is $(\mathcal{I} + \mathcal{J})^+ = \mathcal{I}^+ + \mathcal{J}^+$, where $\mathcal{L}^+$ denotes the set of positive operators in a family $\mathcal{L}$ of operators? He suggested to the author that techniques using the duality between invariant faces of the state space $\mathcal{S}(\mathcal{A})$ of $\mathcal{A}$ and two-sided ideals in $\mathcal{A}$, as shown by E. Effros, might be helpful in studying it. In this note we shall use such arguments to solve the problem to the affirmative.

By a face of $\mathcal{S}(\mathcal{A})$ we shall mean a convex subset $\mathcal{F}$ such that if $\rho \in \mathcal{F}$, $\omega \in \mathcal{S}(\mathcal{A})$ and $a \omega \leq \rho$ for some $a > 0$, then $\omega \in \mathcal{F}$. $\mathcal{F}$ is an invariant face if $\rho \in \mathcal{F}$ implies the state $B \rightarrow \rho(A^*BA)\rho(A^*A)^{-1}$ belongs to $\mathcal{F}$ whenever $\rho(A^*A) \neq 0$ and $A \in \mathcal{A}$. We denote by $\mathcal{F}^\perp$ the set of operators $A \in \mathcal{A}$ such that $\rho(A) = 0$ for all $\rho \in \mathcal{F}$. If $\mathcal{I} \subset \mathcal{A}, \mathcal{J}^\perp$ shall denote the set of states $\rho$ such that $\rho(A) = 0$ for all $A \in \mathcal{J}$. E. Effros [2] has shown that the map $\mathcal{I} \rightarrow \mathcal{I}^\perp$ is an order inverting bijection between uniformly closed two-sided ideals of $\mathcal{A}$ and $w^*$-closed invariant faces of $\mathcal{S}(\mathcal{A})$. Moreover, $(\mathcal{I}^\perp)^\perp = \mathcal{I}$, and $(\mathcal{F}^\perp)^\perp = \mathcal{F}$ when $\mathcal{F}$ is a $w^*$-closed invariant face. If $\mathcal{I}$ and $\mathcal{J}$ are uniformly closed two-sided ideals in $\mathcal{A}$ then $(\mathcal{I} \cap \mathcal{J})^\perp = \text{conv}(\mathcal{I}^\perp, \mathcal{J}^\perp)$ - the convex
hull of $\mathcal{J}^+$ and $\mathcal{J}^+$, and $(\mathcal{J} + \mathcal{J})^+ = \mathcal{J}^+ \cap \mathcal{J}^+$. If $A$ is a self-adjoint operator in $\mathfrak{A}$ let $\hat{A}$ denote the $w^*$-continuous affine function on $S(\mathfrak{A})$ defined by $\hat{A}(\rho) = \rho(A)$. It has been shown by R. Kadison, [3] and [4], that the map $A \to \hat{A}$ is an isometric order-isomorphism of the self-adjoint part of $\mathfrak{A}$ onto all $w^*$-continuous real affine functions on $S(\mathfrak{A})$. Moreover, if $\mathfrak{J}$ is a uniformly closed two-sided ideal in $\mathfrak{A}$, and $\psi$ is the canonical homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{J}$, then the map $\rho \to \rho \circ \psi$ is an affine isomorphism of $S(\mathfrak{A}/\mathfrak{J})$ onto $\mathcal{J}^+$. Thus the map $\psi(A) \to \hat{A}/\mathcal{J}^+$ is an order-isomorphic isometry on the self-adjoint operators in $\mathfrak{A}/\mathfrak{J}$.

We shall below make extensive use of these facts. For other references see [1, §.1].

**Theorem.** Let $\mathfrak{A}$ be a $C^*$-algebra. If $\mathfrak{J}$ and $\mathfrak{J}'$ are uniformly closed two-sided ideals in $\mathfrak{A}$ then

$$(\mathfrak{J} + \mathfrak{J}')^+ = \mathfrak{J}^+ + \mathfrak{J}'^+.$$  

In order to prove the theorem we may assume $\mathfrak{A}$ has an identity, denoted by $I$. We first prove a

**Lemma.** With the assumptions as in Theorem let $A$ belong to $(\mathfrak{J} + \mathfrak{J})^+$, and let $\varepsilon > 0$ be given, $\varepsilon < 1$. Then there exist $B$ in $\mathfrak{J}^+$ and $C$ in $\mathfrak{J}'^+$ such that $0 \leq A - B - C \leq \varepsilon I$.

**Proof.** We may assume $\|A\| \leq 1$. Let $\psi$ denote the canonical homomorphism of $\mathfrak{A}$ onto $\mathfrak{A}/\mathfrak{J}$. Then $\psi(\mathfrak{J} + \mathfrak{J}') = \psi(\mathfrak{J}')$. Now
\[ \psi(A) \geq 0. \] Therefore there exists \( B_1 \in \mathcal{J}^+ \) such that \( \psi(B_1) = \psi(A) \). Then \( \hat{B}_1 \psi = 0 \) and \( \hat{B}_1 \gamma = \hat{A} \gamma \). Since \( (\gamma \cap \gamma^\perp) = \text{conv}(\gamma^\perp, \gamma^\perp), \hat{B}_1 \gamma \cap \gamma^\perp \leq \hat{A} \gamma \cap \gamma^\perp \). Let \( \phi \) denote the canonical homomorphism of \( \mathcal{A} \) onto \( \mathcal{A}/\gamma \cap \gamma^\perp \). Then \( 0 \leq \phi(B_1) \leq \phi(A) \).

Let \( f \) be the real continuous function \( f(x) = (\varepsilon/3)^2 \) for \( x \leq (\varepsilon/3)^2 \), \( f(x) = x \) for \( x > (\varepsilon/3)^2 \). Let

\[
S = f(A) - B_1 f(A) - A.
\]

Then \( S \in \mathcal{J}^+ \), and

\[
(1) \quad 0 \leq \phi(S) = f(\phi(A)) - \phi(B_1) f(\phi(A)) - \phi(A) - \phi(A) f(\phi(A)) - \phi(I).
\]

Let \( g \) be the real continuous function \( g(x) = x \) for \( x \leq 1 \), \( g(x) = 1 \) for \( x > 1 \). Since \( g(0) = 0 \), \( g(S) \) is by the Stone-Weierstrass Theorem a uniform limit of polynomials in \( S \) without constant terms. Since \( S \in \mathcal{J}^+ \), and \( \mathcal{J} \) is uniformly closed, \( g(S) \in \mathcal{J}^+ \). By (1)

\[
(2) \quad \phi(g(S)) = g(\phi(S)) = \phi(S).
\]

Let

\[
B = (f(A)^{1/2} - \varepsilon/3 I) g(S) (f(A)^{1/2} - \varepsilon/3 I).
\]

Since \( g(S) \in \mathcal{J}^+ \) so is \( B \). Now \( (f(x)^{1/2} - \varepsilon/3)^2 \leq x \) for \( x \geq 0 \), and \( g(S) \leq I \). Hence \( 0 \leq B \leq A \). By (2)
\[ \varrho(B) = (f(\varrho(A))^{\frac{1}{2}} \varepsilon/3 \varrho(I) - \varrho(g(S)) (f(\varrho(A))^{\frac{1}{2}} \varepsilon/3 \varrho(I)) \]
\[ = \varrho(B_1) - \varepsilon/3 \left[ (f(\varrho(A))^{\frac{1}{2}} \varrho(S) + \varrho(S) f(\varrho(A))^{\frac{1}{2}} - \varepsilon/3 \varrho(S) \right]. \]

Since \( \| f(\varrho(A))^{\frac{1}{2}} \| \leq 1, \| \varrho(S) \| \leq 1, \) and \( \varepsilon \leq 1 \)
\[ \| B \| \left( \mathcal{J} \cap \mathbb{J} \right) \supset B \supset (\mathcal{J} \cap \mathbb{J})^\perp \leq \| \varrho(B) - \varrho(B_1) \| \leq \varepsilon. \]

In particular,
\[ \| B \| \left( \mathcal{J} \cap \mathbb{J} \right) \supset \mathcal{B} \supset (\mathcal{J} \cap \mathbb{J})^\perp \leq \varepsilon. \]

Apply the preceding to \( A-B \) instead of \( A \) and to \( \mathcal{J} \) instead of \( \mathbb{J} \). Choose \( C_1 \in \mathcal{J}^+ \) such that \( C_1 \leq A-B \), and
\[ \| C_1 \| \mathcal{J} \supset (A-B) \mathcal{J} \supset \| \leq \varepsilon. \]

Since \( C_1 \| \mathcal{J} \| = 0 \)
\[ \| C_1 \| \mathcal{J} \supset (A-B) \mathcal{J} \supset \| \leq \varepsilon. \]

By (4) and (5)
\[ \| \varrho(C_1) - \varrho(A-B) \| = \| C_1 \| \mathcal{J} \supset (A-B) \mathcal{J} \supset \| \leq \varepsilon. \]

Let \( D = A-(B+C_1) \). Then \( D \geq 0 \), and \( \| \varrho(D) \| \leq \varepsilon \). Let \( h \) be the real continuous function \( h(x) = 0 \) for \( x \leq \varepsilon \), \( h(x) = x-\varepsilon \) for \( x > \varepsilon \). Then \( \varrho(h(D)) = h(\varrho(D)) = 0 \), and \( h(D) \in (\mathcal{J} \cap \mathbb{J})^+ \subset \mathcal{J}^+ \). Furthermore
Let \( C = C_1 + h(D) \). Then \( C \in \mathcal{Y}^+ \), and by (6)

\[
0 \leq B + C \leq B + C_1 + D = A \leq B + C_1 + h(D) + \varepsilon I = B + C + \varepsilon I.
\]

The proof is complete.

Proof of Theorem. Let \( A \in (\mathcal{Y} + \mathcal{Y})^+ \). Multiplying \( A \) by a scalar we may assume \( 0 \leq A \leq I \). By Lemma choose \( B_0 \in \mathcal{Y}^+ \), \( C_0 \in \mathcal{Y}^+ \) such that

\[
0 \leq A - B_0 - C_0 \leq 2^{-1}I.
\]

Then \( \|B_0\| = \|A\| \leq 1 \), \( \|C_0\| = \|A\| \leq 1 \). Suppose inductively

\[
B_0, B_1, \ldots, B_{n-1}, C_0, C_1, \ldots, C_{n-1}
\]

are chosen in \( \mathcal{Y}^+ \) and \( C_0, C_1, \ldots, C_{n-1} \) are chosen in \( \mathcal{Y}^+ \) such that \( \|B_j\| \leq 2^{-j}, \|C_j\| \leq 2^{-j} \), and

\[
0 \leq A - \sum_{j=0}^{n-1} B_j - \sum_{j=0}^{n-1} C_j \leq 2^{-n}I.
\]

Apply Lemma to \( A - \sum_{j=0}^{n-1} B_j - \sum_{j=0}^{n-1} C_j \) and to \( \varepsilon = 2^{-n-1} \).

Then there exist \( B_n \in \mathcal{Y}^+ \), \( C_n \in \mathcal{Y}^+ \) such that

\[
0 \leq A - \sum_{j=0}^{n-1} B_j - \sum_{j=0}^{n-1} C_j - B_n - C_n \leq 2^{-n-1}I,
\]

or

\[
0 \leq A - \sum_{j=0}^{n-1} B_j - \sum_{j=0}^{n-1} C_j \leq 2^{-n-1}I.
\]
Moreover, by (7) \( \| B_n \| \leq 2^{-n}, \| C_n \| \leq 2^{-n} \); the induction argument is complete. Let

\[
B = \sum_{j=0}^{\infty} B_j, \quad C = \sum_{j=0}^{\infty} C_j.
\]

Then \( B \in \mathcal{J}^+, C \in \mathcal{J}^+ \), and

\[
\| A - B - C \| = \lim_{n \to \infty} \| A - \sum_{j=0}^{n} B_j - \sum_{j=0}^{n} C_j \| \leq \lim_{n \to \infty} 2^{-n-1} = 0.
\]

Thus \( A = B+C \in \mathcal{J}^+ + \mathcal{J}^+ \), and \((\mathcal{J} + \mathcal{J})^+ \subseteq \mathcal{J}^+ + \mathcal{J}^+ \). Since the converse inclusion is trivial, the proof is complete.

References.

1) J. Dixmier, Les \( C^\star \)-algèbres et leurs representations, Gauthier-Villars, Paris (1964).


University of Oslo.