Matematisk Seminar Nr. 4.
Universitetet i Oslo May 1966.

EXCISION IN SINGULAR THEORY

by

Egil Heistad

The purpose of this note is to give a proof of the excision theorem in singular homology based on the acyclic model theorem. The acyclic model theorem is the formalization of a simple inductive argument in homological algebra. From this point of view our proof is elementary. This proof does away with most of the calculations connected with the barycentric subdivisions, and seems to be more conceptual than the traditional proofs. The theorem we want to prove is the following.

Excision theorem: Let X be a topological space and $\mathcal{U}$ a covering of X such that $\{\text{int } U\}_{U\in\mathcal{U}}$ also is a covering. Then the inclusion map $\Delta(\mathcal{U}) \subset \Delta(X)$ is a chain equivalence.

For notations and results used in this paper we refer to Spanier: Algebraic Topology, McGraw Hill, 1966, henceforth denoted [S].

Finally I should like to acknowledge that the possibility of proving the theorem by acyclic models and the method of proof was suggested to me by Per Holm.

1. For every natural number $m \geq 0$ and for every topological space X an augmentation-preserving chain map

$$sd^m : \Delta(X) \to \Delta(X)$$

is defined (cf. [S], ch.4,4). sd^m is functorial and is chain homotopic to $1_{\Delta(X)}$ by a functorial chain deformation D^m. We need only the existence and not an explicitly constructed chain deformation $D^m : sd^m \simeq 1_{\Delta(X)}$. The existence of one is given to us by the acyclic model theorem, because the singular chain functor is free and acyclic ([S], ch.4,2,7 and ch. 4,4,1). We now state the following lemma.

Lemma 1: Let $\mathcal{U}$ be a covering of a topological space X such that $\{\mathrm{int}\, U\}_{U\in\mathcal{U}}$ also is a covering. For any singular q-simplex σ of $\Delta(X)$ there is $m \geq 0$ such that $sd^m\sigma \in (\mathcal{U})$.

For the proof we refer to ([S], ch. 4,4,13). Using this lemma we easily prove the following.

Lemma 2: Let $\mathcal{U}$ be a covering of a topological space X such that $\{\mathrm{int}\, U\}_{U\in\mathcal{U}}$ also is a covering, and suppose that the reduced chain complex $\tilde{\Delta}(X)$ is acyclic. Then $\tilde{\Delta}(\mathcal{U})$ is acyclic.

Proof. It is sufficient to show that every singular q-chain of $\Delta(\mathcal{U})$ which is a boundary of $\Delta(X)$, also is a boundary of $\Delta(\mathcal{U})$. Let $c_q \in \Delta(\mathcal{U})$ be a singular q-chain such that $c_q = \partial c'_{q+1}$ for a $q+1$-chain $c'_{q+1} \in \Delta(X)$. c'_{q+1} is a finite linear combination of singular $q+1$-simplexes, and by lemma 1 there is a natural number $m \geq 0$ such that $sd^m c'_{q+1} \in \Delta(\mathcal{U})$. Because the chain deformation $D^m : sd^m \simeq 1_{\Delta(X)}$ is natural $D^m(\Delta(\mathcal{U})) \subset \Delta(\mathcal{U})$ and $D^m c_q \in \Delta(\mathcal{U})$. Put $c_{q+1} = sd^m c'_{q+1} - D^m c_q$, then $c_{q+1} \in \Delta(\mathcal{U})$ and a simple calculation shows that $\partial c_{q+1} = c_q$ and the lemma is proved.

Remark: We only need lemma 2 for the case $X = \Delta^q$, $q \geq 0$.

2. Consider all pairs $(X,\mathcal{U})$ where X is a topological space and $\mathcal{U}$ is a family of subsets of X such that $\mathrm{int}\,\mathcal{U} = \{\mathrm{int}\, U\}_{U\in\mathcal{U}}$ is an open covering of X. A map

$f : (X,\mathcal{U}) \to (Y,\mathcal{V})$ of such pairs is by definition a continuous map $f : X \to Y$ such that $\mathcal{U} = f^{-1}(\mathcal{V})$. Clearly there is a category whose objects are pairs $(X,\mathcal{U})$ and whose morphisms are maps $f : (X,\mathcal{U}) \longrightarrow (Y,\mathcal{V})$. On this category define the two functors

$$(X,\mathcal{U}) \rightsquigarrow \Delta(X)$$

and

$$(X,\mathcal{U}) \rightsquigarrow \Delta(\mathcal{U})$$

and denote these by Δ' and Δ'' respectively. Then Δ' and Δ'' take values in the category of augmented chain complexes. Let $\mathcal{M}$ be the collection of all pairs $(\Delta^q,\mathcal{V})$ for $q \geq 0$ and $\mathcal{V}$ varies over all coverings of Δ^q such that $\mathrm{int}\,\mathcal{V}$ is a covering. If we can show that Δ' and Δ'' are both free and acyclic with respect to the models $\mathcal{M}$, it follows from the theorem of acyclic models that there are natural chain maps from one functor to the other, unique up to natural homotopy, and that any such is a chain equivalence. In particular the inclusion map $\Delta(\mathcal{U}) \subset \Delta(X)$ defines a natural chain map $\Delta'' \longrightarrow \Delta'$, and so is a chain equivalence.

<u>Lemma 3</u>: Δ' and Δ'' are both free and acyclic on the models $\mathcal{M}$.

<u>Proof</u>. For $(X,\mathcal{U})$ an object in our category and $\sigma \in \Delta'_q(X,\mathcal{U})$ an arbitrary singular q-simplex there is <u>one and only one</u> covering, namely $\mathcal{V} = \sigma^{-1}(\mathcal{U})$, of Δ^q such that σ defines a morphism

$$\sigma : (\Delta^q,\mathcal{V}) \longrightarrow (X,\mathcal{U})$$

By this remark we see that the family

$$\{\xi_q \in \Delta_q(\Delta^q;\mathcal{V})\}$$

where ξ_q is the identity map $\Delta^q \subset \Delta^q$ and $\mathcal{V}$ varies over all coverings of Δ^q such that int $\mathcal{V}$ also is a covering, is a basis for Δ'_q. Hence Δ' is free with models $\mathcal{M}$. By the same remark we also see that the family

$$\{\xi_q \in \Delta''_q(\Delta^q;\mathcal{V})\}$$

where ξ_q is the identity map $\Delta^q \subset \Delta^q$ and $\mathcal{V}$ varies over all coverings of Δ^q such that $\{\Delta^q\} \in \mathcal{V}$ is a basis for Δ''_q, and so Δ'' is free with models $\mathcal{M}$.

It is trivial that Δ' is acyclic on the models $\mathcal{M}$, and by lemma 2 it follows that also Δ'' is acyclic. This completes the proof.