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# On spectral sequences associated to projective systems.

by

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Introduction. In this paper we shall give some results of a rather general nature on spectral sequences.

These investigations have been undertaken not only for the sake of spectral sequences, but also because, as we shall see, this theory could be based on the general problem of calculating $\varprojlim$ and $\varinjlim$ of projective systems, by "finite means". This again is a question which rises most naturally in the general theory of the functors $\varprojlim$ and $\varinjlim$ with which we have been cencerned for the last 4 years.

A starting point for this work has been the problem posed by Eilenberg and Moore in [1], and the obvious analogy between their conditions and the conditions of our "Mittag-Leffler" theorem, see [3].

The main result of this paper is the theorem (2.2.2) which seems to generalize the known "convergence theorems" in the theory of spectral sequences, the results of Eilenberg and Moore included.

In a forthcoming paper we will consider mapping problems and some applications.

1.1 Let $Z$ be the set of rational numbers ordered in the usual way, let $Z^{\circ}$ be the same set with the oposit ordering and let $\underline{c}_Z$ be the category of all projective systems on $Z^{\circ}$ with values in the category $\underline{c}$. We will suppose that $\underline{c}$ is sufficient good to permit the operations made below, so that in particular we impose the following conditions:

1) $\underline{c}$ is abelian.

2) In $\underline{c}_Z$ the functors $\varprojlim$ and $\varinjlim$ and their derived functors $\varprojlim^{(i)}$ and $\varinjlim_{(i)}$ exists.

3) $\varprojlim^{(i)} = \varinjlim_{(i)} = 0$ for $i \geq 2$.

For information on $\varprojlim$ and $\varinjlim$ the interested reader could consult our paper [5]. For the right conditions to be put on $\underline{c}$, he could look in [1].

Anyhow, the standard model for $\mathfrak{C}$ should be the category of modules on a commutativ ring $L$.

In the sequel we will suppose known the theory of exact couples in the sense of Massey as exposed, for example, in the textbook of Hu, [2].
With his notations the problem in the theory of spectral sequences is, in the most common case of a regular $\partial$-couple, to calculate the group $\mathcal{H}_.$.
This is done by first constructing, for each $n$, a filtration $\mathcal{H}_n = \mathcal{H}_{n,0} \supseteq \mathcal{H}_{n-1,1} \supseteq \cdots \supseteq \mathcal{H}_{-1,n+1} = 0$ of $\mathcal{H}_n$ and proving the "convergence" theorem

$$E^{\infty}_{p,q} \simeq \mathcal{H}_{p,q}/\mathcal{H}_{p-1,q+1}$$

Now we note that the group $\mathcal{H}_.$ is, in fact, an inductive limit of the projective system $D$, more precisely, for each $n \in Z$ we have

$$\mathcal{H}_n = \varinjlim_{p+q=n} D_{p,q}$$

In this case, however, since the conditions imposed on a regular $\partial$-couple are rigorous, the inductive limit is obtained finitely, i.e. we have:

$$\mathcal{H}_n = \varinjlim_{p+q=n} D_{p,q} \simeq D_{n+1,-1} \simeq \cdots \simeq D_{n+r,-r} \simeq \cdots$$

for all $r \geq 1$.

We also note that the elements $\mathcal{H}_{p,q}$ of the filtration of $\mathcal{H}_n$ are of the form:

$$\mathcal{H}_{p,q} = \operatorname{im}\{ D_{p,q} \longrightarrow \varinjlim_{p+q=n} D \}$$

In the case of a regular $\partial$-couple this filtration is, as we have already mentioned, finite. However, in the general case this will obviously not be true.

At last we note that given a completely arbitrary object $D$ of $\underline{c}_Z$, $D : \ldots \longrightarrow D_p \xrightarrow{i_p} D_{p+1} \xrightarrow{i_{p+1}} \ldots$, and given a $p \in Z$ we always could find one, and, in general, lots of objects $E_p$ and morphisms, in $\underline{c}$, such that the diagram

$$\begin{array}{ccc} D_{p-1} & \xrightarrow{i_{p-1}} & D_p \\ {\scriptstyle k}\nwarrow & & \swarrow{\scriptstyle j} \\ & E_p & \end{array}$$

is an exact couple.

In fact, it suffices to take an object $E_p$ and morphisms of $\underline{c}$ such that the sequence

$$0 \longrightarrow \operatorname{coker} i_{p-1} \longrightarrow E_p \longrightarrow \ker i_{p-1} \longrightarrow 0$$

is exact, and this is obviously the same as picking an element from

$$\operatorname{Ext}^1(\ker i_{p-1}, \operatorname{coker} i_{p-1}).$$

Thus the set

$$S(D) = \prod_{p \in Z} \operatorname{Ext}^1(\ker i_{p-1}, \operatorname{coker} i_{p-1})$$

is in one-to-one correspondence with the set of all (up to isomorphisms), graded exact couples

$$\begin{array}{ccc} D & \xrightarrow{i} & D \\ {\scriptstyle k}\nwarrow & & \swarrow{\scriptstyle j} \\ & E & \end{array}$$

with $D = \coprod_{p \in Z} D_p$, $E = \coprod_{p \in Z} E_p$, $i = \coprod_{p \in Z} i_p$, and where $j$

and k should have degrees respectively 0 and -1.

From these observations we are led to consider the following general problem: Given an object D in $\underline{c}_Z$, calculate $\varprojlim_{Z^\circ} D$ and $\varinjlim_{Z^\circ} D$ using the spectral sequence associated to one of the exact couples in S(D). We start the work by "filtering" $\varprojlim_{Z^\circ} D$ and $\varinjlim_{Z^\circ} D$.

1.2. So let D be an object of $\underline{c}_Z$

$$D : \dots \longrightarrow D_{p-1} \xrightarrow{i_{p-1}} D_p \xrightarrow{i_p} D_{p+1} \longrightarrow \dots$$

and let for $p \geq p'$ $\eta_p^{p'} : D_{p'} \longrightarrow D_p$ be the obvious composition of the i's.
Now look at the objects in $\underline{c}$

$$H^\bullet = \varprojlim D \qquad {}^1H^\bullet = \varprojlim{}^{(1)} D$$
$$H_\bullet = \varinjlim D \qquad {}_1H_\bullet = \varinjlim{}_{(1)} D$$

We define a canonical filtration $\{H^p\}_{p \in Z}$ of $H^\bullet$ and a canonical cofiltration $\{H_p\}_{p \in Z}$ of $H_\bullet$, by puting:

$$H^p = \ker \pi^p$$
$$H_p = \operatorname{coker} \amalg_p$$

where $\pi^p : \varprojlim D \longrightarrow D_p$ and $\amalg_p : D_p \longrightarrow \varinjlim D$ are the canonical morphisms.

Now consider the diagram of exact sequences

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \leftarrow & \operatorname{im}\eta^{p'}_{p} & \leftarrow & D_{p'} & \leftarrow & \ker\eta^{p'}_{p} & \leftarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & D_p & & & & & & \\
 & & \downarrow & & & & & & \\
 & & \operatorname{coker}\eta^{p'}_{p} & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

with $p' \leqq p$. Using first $\varprojlim$ and then $\varinjlim$ we get the following commutative diagrams

$$\begin{array}{l}
 \qquad\qquad\qquad\qquad\qquad\quad 0 \\
 \qquad\qquad\qquad\qquad\qquad\quad \downarrow \\
0 \to \varprojlim_{p'\in Z_0} \ker\eta^{p'}_{p} \to H^{\cdot} \to \varprojlim_{p'\in Z_0} \operatorname{im}\eta^{p'}_{p} \to \varprojlim^{(1)}_{p'\in Z_0} \ker\eta^{p'}_{p} \to {}^{1}H^{\cdot} \to \varprojlim^{(1)}_{p'\in Z_0} \operatorname{im}\eta^{p'}_{p} \to \\
 \qquad\qquad\qquad \pi^{p} \searrow \qquad\quad \downarrow \\
 \qquad\qquad\qquad\qquad\qquad\quad D_p \\
 \qquad\qquad\qquad\qquad\qquad\quad \downarrow \\
 \qquad\qquad\qquad\qquad\qquad\quad \varprojlim_{p'\in Z_0} \operatorname{coker}\eta^{p'}_{p}
\end{array}$$

$$\begin{array}{l}
 \qquad\qquad\qquad\qquad\qquad\quad 0 \\
 \qquad\qquad\qquad\qquad\qquad\quad \uparrow \\
0 \leftarrow \varinjlim_{p'\in Z_0} \operatorname{coker}\eta^{p}_{p'} \leftarrow H_{\cdot} \leftarrow \varinjlim_{p'\in Z_0} \operatorname{im}\eta^{p}_{p'} \leftarrow \varinjlim_{p'\in Z_0 (1)} \operatorname{coker}\eta^{p}_{p'} \leftarrow {}_{1}H_{\cdot} \leftarrow \\
 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \varinjlim_{p'\in Z_0 (1)} \operatorname{im}\eta^{p}_{p'} \leftarrow 0 \\
 \qquad\qquad\qquad \pi_{p} \nwarrow \qquad\quad \uparrow \\
 \qquad\qquad\qquad\qquad\qquad\quad D_p \\
 \qquad\qquad\qquad\qquad\qquad\quad \uparrow \\
 \qquad\qquad\qquad\qquad\qquad\quad \varinjlim_{p'\in Z_0} \ker\eta^{p}_{p'}
\end{array}$$

From these diagrams we read out:

$$H^{p} \simeq \varprojlim_{p'\, Z} \ker\eta^{p'}_{p} \qquad\qquad H_{p} \simeq \varinjlim_{p'\, Z} \operatorname{coker}\eta^{p}_{p'}$$

and, naturally, we put:

$$ {}_1H^p \simeq \varprojlim_{p'\in Z^\circ}{}^{(1)} \ker \eta^{p'}_{p} \qquad {}_1H_p \simeq \varinjlim_{p'\in Z^\circ}{}^{(1)} \operatorname{coker} \eta^{p}_{p'} $$

<u>2.1</u>. Let $D$ be an object of $\underline{c}_Z$ and let $E \in S(D)$. Consider the associated exact couple

$$\begin{array}{ccc} D & \xrightarrow{\;i\;} & D \\ & {}_{k}\nwarrow \quad \swarrow_{j} & \\ & E & \end{array}$$

The objects $D$ and $E$ are graded, so when we form the derived exact couples $(D^r, E^r, i^r, j^r, k^r)$ we are free to give these objects proper gradings. We put:

$$(1)\qquad \begin{aligned} 'D^r_p &= \operatorname{im}(i_{p-1}\circ \cdots \circ i_{p-r+1}) \quad \text{for } r \geqq 2. \\ 'E^2_p &= E^r_p \\ ''D^r_p &= \operatorname{im}(i_{p+r-2}\circ \cdots \circ i_p) \quad \text{for } r \geqq 2 \\ ''E^r_p &= E^r_p \end{aligned}$$

Thus we have:

$$(2)\qquad \begin{aligned} &'D^r_p \subseteq D_p, \quad ''D^r_p \subseteq D_{p+r-1}, \quad 'E^r_p = ''E^r_p = E^r_p \\ &'D^r_p = ''D^r_{p-r+1}, \quad 'i^r_p = ''i^r_{p-r+1}, \quad 'j^r_p = ''j^r_{p-r+1}, \quad 'k^r_p = ''k^r_p \end{aligned}$$

and exact sequences:

$$(3)\qquad \begin{aligned} &\cdots \to {}'D^r_{p+r-1} \xrightarrow{'j^r_{p+r-1}} E^r_p \xrightarrow{'k^r_p} {}'D^r_{p-1} \xrightarrow{'i^r_{p-1}} D^r_p \xrightarrow{'j^r_p} \\ &\qquad\qquad \longrightarrow E^r_{p-r+1} \xrightarrow{'k^r_{p-r+1}} {}'D^r_{p-r} \longrightarrow \cdots \\ &\cdots \to {}''D^r_{p'r-1} \xrightarrow{''j^r_{p+r-1}} E^r_{p+r-1} \xrightarrow{''k^r_{p+r-1}} {}''D^r_{p-1} \xrightarrow{''i^r_{p-1}} {}''D^r_p \xrightarrow{''j^r_p} \\ &\qquad\qquad \longrightarrow E^r_p \xrightarrow{''k^r_p} {}''D^r_{p-r} \longrightarrow \cdots \end{aligned}$$

Now $'D^r : \dots \to {}'D^r_{p-1} \xrightarrow{'i^r_{p-1}} {}'D^r_p \xrightarrow{'i^r_p} {}'D^r_{p+1} \to \dots$

and $''D^r : \dots \to {}''D^r_{p-1} \xrightarrow{''i^r_{p-1}} {}''D^r_p \xrightarrow{''i^r_p} {}''D^r_{p+1} \to \dots$

are objects in $\underline{c}_Z$ and by the general method of [4], see also [1], we find that

$$\varprojlim{}^{(i)}\,'D^r \simeq \varprojlim{}^{(i)} D \qquad \text{for all } i \geq 0.$$
$$\varinjlim{}_{(i)}\,''D^r \simeq \varinjlim{}_{(i)} D$$

Consequently, by the formulas (1) we have canonical isomorphisms

$$(4) \qquad \begin{aligned} \varprojlim{}^{(i)}\,'D^r &\simeq \varprojlim{}^{(i)}\,''D^r \simeq \varprojlim{}^{(i)} D \\ \varinjlim{}_{(i)}\,''D^r &\simeq \varinjlim{}_{(i)}\,'D^r \simeq \varinjlim{}_{(i)} D \end{aligned}$$

Moreover we find easily that if:

$$'\pi^p_r : \varprojlim\,'D^r \to {}'D^r_p \qquad '\amalg^r_{p+r-1} : {}'D^r_{p+r-1} \to \varinjlim\,'D^r$$

$$''\pi^{p-r+1}_r : \varprojlim\,''D^r \to {}''D^r_{p-r+1} \qquad ''\amalg^r_p : {}''D^r_p \to \varinjlim\,''D^r$$

then:

$$\ker\,'\pi^p_r = \ker\,''\pi^{p-r+1}_r \simeq \ker \pi^p$$

$$\operatorname{coker}\,''\amalg^r_p = \operatorname{coker}\,'\amalg^r_{p+r-1} \simeq \operatorname{coker} \amalg_p$$

so that the filtration of $\varprojlim D$ and the cofiltration of $\varinjlim D$ defined in (1.2), by the isomorphisms (4), coincides with the filtration of $\varprojlim\,'D^r$ given by $'D^r$, respectively with the cofiltration of $\varinjlim\,''D^r$ given by $''D^r$.

As it will be obvious from the next paragraph the derived filtration $\{{}^1H^p\}$ and the derived cofiltration $\{{}_1H_p\}$ will also be invariant under the process of taking derived exact couple $('D^r, E^r)$ respectively $(''D^r, E^r)$.

2.2. Now we fix our attention to the exact couples $('D^r, E^r)$.

Consider the associated exact sequence

$$\ldots \longrightarrow {}'D^r_{p+r-2} \xrightarrow{{}'i^r_{p+r-2}} {}'D^r_{p+r-1} \xrightarrow{{}'j^r_{p+2-1}} E^r_p \xrightarrow{{}'k^r_p} {}'D^r_{p-1} \xrightarrow{{}'i^r_{p-1}} {}'D^r_p \longrightarrow \ldots$$

which induces the short exact sequence

$$(5) \qquad 0 \longrightarrow \operatorname{coker} {}'i^r_{p+r-2} \longrightarrow E^r_p \longrightarrow \ker {}'i^r_{p-1} \longrightarrow 0$$

Using the notations in Hu : [2] we would like to prove:

Lemma.(2.2.1) For every $k \geq 0$ (5) induces an exact sequence

$$0 \longrightarrow \operatorname{coker} {}'i^r_{p+r-2} \longrightarrow Z^r_{p,k} \longrightarrow \ker {}'i^{r+k}_{p-1} \longrightarrow 0$$

Proof. As $\operatorname{coker} {}'i^r_{p+r-2} = \ker {}'k^r_p$ the inclusion

$$\operatorname{coker} {}'i^r_{p+r-2} \subseteq Z^r_{p,k} \qquad \text{for all} \quad k \geq 0$$

is evident.

Now look at the commutative diagram:

$$\begin{array}{ccccccc}
Z^r_{p,k-1} \subseteq E^r_p & \xrightarrow{{}'k^r_p} & {}'D^r_{p-1} & & \\
\varphi \big\downarrow \text{surj.} & & \cup\!\!| & & \\
Z^{r+k-1}_{p,1} = E^{r+k-1}_p & \xrightarrow{{}'k^{r+k-1}_p} & {}'D^{r+k-1}_{p-1} & & \\
\big\downarrow \text{surj.} & & \cup\!\!| & & \\
E^{r+k}_p & \xrightarrow{{}'k^{r+k}_p} & {}'D^{r+k}_{p-1} & \xrightarrow{{}'i^{r+k}_{p-1}} & {}'D^{r+k}_p
\end{array}$$

Taking into account the definition of $Z^r_{p,k} = \varphi^{-1}(Z^{r+k-1}_{p,1})$ it becomes fairly evident that ${}'k^r_p$ maps $Z^r_{p,k}$ onto $\ker {}'i^{r+k}_{p-1}$. QED.

Now use the functor $\varprojlim_{k\in Z}$ on the exact sequence of objects in $\underline{c}_Z$ of (2.2.1).
Since the projective system coker $'i^r_{p+r-2}$ is constant with respect to $k\in Z$ we obtain the exact sequence:

(6) $$0 \to \text{coker } 'i^r_{p+r-2} \to \varprojlim_{k\in Z} Z^r_{p,k} \to \varprojlim_{k\in Z} \ker\, 'i^{r+k}_{p-1} \to 0$$

and the isomorphisme

(7) $$\varprojlim_{k\in Z}{}^{(1)} Z^r_{p,k} \simeq \varprojlim_{k\in Z}{}^{(1)} \ker\, 'i^{r+k}_{p-1}$$

$$\simeq \varprojlim_{k\in Z}{}^{(1)} \ker\, 'i^{k}_{p-1}$$

In particular we find that the projective systems on $r\in Z$, $\varprojlim_{k\in Z} \ker\, 'i^{r+k}_{p-1} = \varprojlim_{k\in Z} \ker\, 'i^{k}_{p-1}$ and $\varprojlim_{k\in Z}{}^{(1)} Z^r_{p,k}$ are constant.

Remembering that in the notations of Hu [2]:

$$\varprojlim_{k\in Z} Z^r_{p,k} = \bigcap_{k\in Z} Z^r_{p,k} = \overline{E}^r_p$$

we know that, by definition

$$E^\infty_p = \varinjlim_{r\in Z^o} \overline{E}^r_p = \varinjlim_{r\in Z^o} \varprojlim_{k\in Z} Z^r_{p,k} \,.$$

Since $\varprojlim_{k\in Z}{}^{(1)} Z^r_{p,k}$ is constant with respect to $r$ we may define:

$${}^1E_p = \varprojlim_{k\in Z}{}^{(1)} Z^r_{p,k}$$

Thus using the functor $\varinjlim_{r\in Z^o}$ on the sequence (6) we get an exact sequence:

(8) $$0 \longrightarrow \varinjlim_{r\in Z^0} \operatorname{coker}\, {}'i^r_{p+r-2} \longrightarrow E^\infty_p \longrightarrow \varprojlim_{k\in Z} \ker\, {}'i^k_{p-1} \longrightarrow 0$$

and isomorphisms:

(9) $${}^1E^\infty_p \simeq \varprojlim_{k\in Z}{}^{(1)} \ker\, {}'i^k_{p-1} \quad \text{and} \quad {}_1E^\infty_p \simeq \varinjlim_{r\in Z^0}{}_{(1)} \operatorname{coker}\, {}'i^r_{p+r-2}$$

where in analogy with the definition above we have put:

$${}_1E^\infty_p = \varinjlim_{r\in Z^0}{}_{(1)} \bar{E}^r_p$$

Now, look at the commutative diagrams of exact sequences

$$\begin{array}{ccccccc}
0 \to \ker\eta^{p-k}_{p-1} & \longrightarrow & D_{p-k} & \longrightarrow & {}'D^k_{p-1} & \to & 0 \\
\text{inj.}\downarrow & & SI & & \downarrow {}'i^k_{p-1} & & \\
0 \to \ker\eta^{p-k}_{p} & \longrightarrow & D_{p-k} & \longrightarrow & {}'D^k_{p} & & \\
 & & \searrow & & \nearrow & & \\
 & & & {}'D^{k+1}_p & & &
\end{array}$$

$$\begin{array}{ccccccc}
 & {}'D^{r+1}_{p+r-1} & & & & & \\
\nearrow & & \searrow & & & & \\
{}'D^r_{p+r-2} & \longrightarrow & D_{p+r-1} & \longrightarrow & \operatorname{coker}\eta^{p-1}_{p+r-1} & \to & 0 \\
{}'i^r_{p+r-2}\downarrow & & SI & & \downarrow \text{surj.} & & \\
0 \to {}'D^r_{p+r-1} & \longrightarrow & D_{p+r-1} & \longrightarrow & \operatorname{coker}\eta^{p}_{p+r-1} & \to & 0\,.
\end{array}$$

Using the snakes lemma we get exact sequences:

$$0 \to \ker\eta^{p-k}_{p-1} \to \ker\eta^{p-k}_{p} \to \ker\, {}'i^k_{p-1} \to 0$$

$$0 \to \operatorname{coker}\, {}'i^r_{p+r-2} \to \operatorname{coker}\eta^{p-1}_{p+2-1} \to \operatorname{coker}\eta^{p}_{p+r-1} \to 0$$

Taking into account the results of (1.2) and the isomorphisms (9), then applying the functors $\varprojlim_{k\in Z}$ respectively $\varinjlim_{r\in Z^{o}}$ on these sequences we are left with the exact sequences:

$$0 \to H^{p-1} \to H^{p} \to \varprojlim_{k} \ker {}'i^{k}_{p-1} \to {}^{1}H^{p-1} \to {}^{1}H^{p} \to {}^{1}E_{p} \to 0$$

(10)

$$0 \to {}_{1}E_{p} \to {}_{1}H_{p-1} \to {}_{1}H_{p} \to \varinjlim_{r} \operatorname{coker} {}'i^{r}_{p+r-2} \to H_{p-1} \to H_{p} \to 0$$

Together (8) and (10) give us the very general result:

<u>Theorem (2.2.2)</u>. For any $E \in S(D)$ we have the following diagram of exact sequences:

$$\begin{array}{ccccccccccccccc}
 & & & & & & & & 0 & & & & & & \\
 & & & & & & & & \downarrow & & & & & & \\
0 & \to & {}_{1}E_{p} & \to & {}_{1}H_{p-1} & \to & {}_{1}H_{p} & \to & * & \to & H_{p-1} & \to & H_{p} & \to & 0 \\
 & & & & & & & & \downarrow & & & & & & \\
 & & & & & & & & E_{p} & & & & & & \\
 & & & & & & & & \downarrow & & & & & & \\
 & & & & 0 & \to & H^{p-1} \to H^{p} & \to & * & \to & {}^{1}H^{p-1} & \to & {}^{1}H^{p} & \to & {}^{1}E_{p} \to 0 \\
 & & & & & & & & \downarrow & & & & & & \\
 & & & & & & & & 0 & & & & & &
\end{array}$$

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