Microbundles and Thom classes

by

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In this note we introduce Thom classes of microbundles. We determine the Thom class of the Whitney sum as the cup product of Thom classes and state two applications; one to Gysin sequences of Whitney sums and one to the Atiyah-Bott-Shapiro duality theorem for Thom spaces (cf. Atiyah [2]). Thus our main result states that for microbundles $\mu_1, \mu_2$ over a compact (topological) manifold $X$, if $\pi(X) \oplus \mu_1 \otimes \mu_2$ is stably fibre homotopy trivial, then $\mu_1$ and $\mu_2$ have S-dual Thom spaces. Actually our result is more general since it treats the relative case, i.e. with relative Thom spaces. This makes us able to handle manifolds with boundaries (by passing to the double) among other things. Thus proposition (3.2) in Atiyah [2] has an extended version which just appears as another special case of our duality theorem.

The approach given here to the S-duality theorem shows very clearly that S-duality of Thom spaces is simply Alexander-Spanier duality of compact pairs in the base manifold lifted by Thom isomorphisms. Our approach does not make use of imbeddings of manifolds and seems more relevant to the result. We also think it is conceptually easier than Atiyah's method although there are some technical difficulties due to the fact that we work in a more general setting.

Throughout this paper all base spaces of bundles and microbundles are assumed paracompact unless otherwise stated. Manifolds are manifolds without boundary. For notations and concepts see [4].

Generalizations and details will appear elsewhere.

1) This work was done in Berkeley, California, in spring 1965 while the author was supported from NAVF (Norway) and NSS (contract [contract number]).
1. Throughout this paper \( R \) denotes a fixed principal ideal domain. By a local system on a space \( X \) we understand a local system of \( R \)-modules on \( X \), i.e. a (contravariant) functor from the fundamental groupoid of \( X \) to the category of \( R \)-modules.

If \( \mu : X \rightarrow E \) is an \( \mathbb{R} \)-microbundle with total space \( E \), write \( E^0 = E - sX \). Then there are local systems \( \mathcal{O} = \mathcal{O}(\mu) \), \( \mathcal{O}^\mu = \mathcal{O}^\mu(\mu) \) on \( X \) depending only on the equivalence class of \( \mu \) such that for \( x \in X \), \( \mathcal{O}_x = H^q(E|x, E^0|x) \), \( \mathcal{O}^\mu_x = H^q(E|x, E^0|x) \) and for a path class \( [\omega] \) in \( X \) from \( x_0 \) to \( x_1 \), \( \mathcal{O}_{x_0} : \mathcal{O}_{x_0} \rightarrow \mathcal{O}_{x_1} \) and \( (\mathcal{O}^\mu_{x_0} : \mathcal{O}^\mu_{x_0} \rightarrow \mathcal{O}^\mu_{x_1}) \) are the isomorphisms induced from the (homotopic) map-germs of fibers \( E|x_0 \rightarrow E|x_1 \) determined by \( [\omega] \). (Coefficients in homology or cohomology are taken in \( R \) if not indicated.) \( \mathcal{O} \) and \( \mathcal{O}^\mu \) are constant over each trivializing subset of \( X \) but in general not constant over \( X \).

A Thom class for the microbundle \( \mu \) is a cohomology class \( U \in H^q(E, E^0; \mathcal{O}^\mu) \) such that the restriction to \( H^q(E|x, E^0|x; \mathcal{O}^\mu_x) = Hom(H_q(E|x, E^0|x), H_q(E|x, E^0|x)) \) maps \( U \) to the generator corresponding to the identity automorphism. It is not difficult to show the following

(1) **Theorem.** Any microbundle admits a unique Thom class.

Clearly Thom classes correspond under equivalences and are functorial with respect to pull-backs.

Let \( \mu_1 : X \rightarrow E_1 \overset{p_1}{\rightarrow} X \), \( \mu_2 : X \rightarrow E_2 \overset{p_2}{\rightarrow} X \) be two micro-bundles over \( X \) and let \( \mu : X \rightarrow E \) be their Whitney sum. Let \( \pi_1 : E \rightarrow E_1 \) and \( \pi_2 : E \rightarrow E_2 \) be the canonical projections and \( \sigma_1 : E_1 \rightarrow E \), \( \sigma_2 : E_2 \rightarrow E \) their corresponding sections. The projections \( \pi_1, \pi_2 \) define maps of pairs

\[
\pi_1 : (E, E - \sigma_2 E_2) \rightarrow (E_1, E_1^0), \quad \pi_2 : (E, E - \sigma_1 E_1) \rightarrow (E_2, E_2^0)
\]

which induce isomorphisms in homology. Given a pairing of local systems on \( X \) \( \mathfrak{f} : \mathcal{F}_1 \otimes \mathcal{F}_2 \rightarrow \mathcal{F} \), there is a natural pairing

\[
H^*(E, E_1^0; \mathfrak{f}_1 \otimes \mathfrak{f}_2) \otimes H^*(E_2, E_2^0; \mathfrak{f}_2' \circ \mathfrak{f}_2) \rightarrow H^*(E, E_0^0; \mathfrak{f} \otimes \mathfrak{f})
\]

defined by commutativity of the diagram.
The map $\cup$ is the ordinary cup product, and $\tau_*$ is the homomorphism induced from the composite $\tau: \pi_*^E \otimes \pi_*^E \rightarrow p_*^E \otimes p_*^E = p_*^1 \otimes p_*^2 \approx p_*(\mathcal{J}_1 \otimes \mathcal{J}_2) \otimes p_*^2 p_*^2$. This pairing is invariant under equivalence of microbundles, but in general dependent on the germs of $p_i$ and $s_i$, $i = 1, 2$. The image of an element of $u \otimes v$ will (by a slight abuse of notation) be denoted $\pi_*^X(u) \cup \pi_*^Y(v)$. It follows that there is always a pairing induced from the Künneth formula pairing $\mathcal{E}_1 \otimes \mathcal{E}_2 \rightarrow \mathcal{E}$. The following is true

(2) Theorem. If $\mu, \mu_1, \mu_2$ are microbundles with Thom classes $\mu, \mu_1, \mu_2$ and $\mu = \mu_1 \cap \mu_2$, then $\mathcal{E} = \pi_*^X(U_1) \cup \pi_*^X(U_2)$.

Similarly, for microbundles $\mu, \nu$ whose composite $\mu = \mu_1 \cap \mu_2$ is defined, there is a natural cup product pairing

$H^X(E, E; \mu, \mu_1, \mu_2) \otimes H^X(E, E; \nu, \nu_1, \nu_2) \rightarrow H^X(E, E; \mu, \mu_1, \mu_2) ~ \otimes \rightarrow H^X(E, E; \nu, \nu_1, \nu_2)$

associated to each pairing $p^X \mathcal{S}_1 \otimes \mathcal{S}_2 \rightarrow p^X \mathcal{U}$ (of local systems $(\mathcal{S}, \mathcal{U})$ on the base of $\mu$, $\mathcal{S}$ on the base of $\nu$). The two cup product pairings are actually equivalent; the one is turned into the other by the isomorphism $\nu_2 \approx \mu_2 \otimes \mathcal{s}_2 \mathcal{C}$, $\mu_2 \approx \mu_1 \circ p^X \mathcal{M}_2$, cf. [4], corollary 5. If $\mu: X \rightarrow E$, and $\nu: E' \rightarrow E$, let $\pi: (E', E - \pi^{-1} sX) \rightarrow (E, E')$ be the map given by $\pi$. Then we have

(3) Corollary. If $\nu, \mu, \nu$ are microbundles with Thom classes $\mu, \mu_1, \mu_2$ and $\mu = \mu_1 \cap \mu_2$, then $\mathcal{E} = \pi_*^X U \cup \mu_2$.

By a spectral sequence argument (or by elementary piecing together techniques using a Mayer-Vietoris exact sequence) we can show the existence of a relative Thom isomorphism:
(4) Theorem: Let \( \mu : X \xrightarrow{\rho} E \xrightarrow{p} X \) be an \( \mathbb{R}^q \)-microbundle, \( A \) a subset of \( X \) and \( \mathcal{J} \) a local system on \( X \). Then there are natural isomorphisms of graded modules of degrees \(-q\) and \( q\) respectively

\[
\begin{align*}
\phi : H(E, E^0 \cup E \mid A; p^\mathcal{J}) &\xrightarrow{\sim} H(X; A; \mathcal{O}^\wedge \otimes \mathcal{J}) \\
\phi^* : H^*(X; A; \mathcal{O}^\wedge \otimes \mathcal{J}) &\xrightarrow{\sim} H^*(E, E^0 \cup E \mid A; p^\mathcal{J})
\end{align*}
\]

defined by

\[
\phi(z) = p_{\mathcal{J}}(U_{\mu} \cap z), \quad \phi^*(v) = p^\mathcal{J}(v) \cup U_{\mu}
\]

Note that \( \phi \) and \( \phi^* \) are invariant under equivalence of microbundles.

From this theorem and the homology sequence of \( (E, E^0 \cup E \mid A, E^0) \) and the five lemma follows that \( \{ E^0; E \mid A \} \) is an excisive couple with respect to any local system \( p^\mathcal{J} \) pulled up from \( X \). If \( \mu \) happens to be a bundle, there is an exact Gysin homology sequence

\[
\cdots \to H_*(E, E^0 \mid A; p^\mathcal{J}) \to H_*(X; A; \mathcal{J}) \xrightarrow{\psi^*} H_{*-q}(X, A; \mathcal{O}^\wedge \otimes \mathcal{J}) \to \cdots
\]

obtained from the homology sequence of \( (E, E^0 \cup E \mid A, E \mid A) \), and a similar exact Gysin cohomology sequence. Since by \( \mathcal{L} \) any microbundle has a fundamental system of bundle neighbourhoods of the zero-section, any microbundle has a functorial exact Gysin homology sequence as above, except the \( \mathbb{R} \)-module \( H_*(E, E^0 \mid A; p^\mathcal{J}) \) should be replaced with \( \lim H_*(V^0, V \mid A; p^\mathcal{J}) \), where \( V \) runs through the neighbourhoods of \( sX \) in \( E \).

In the cohomology Gysin sequence of \( \mu \) there is a map

\[
\psi^*_\mu : H^*(X, A; \mathcal{J}) \to H^*(X, A; \mathcal{O}^\wedge \otimes \mathcal{J})
\]

of degree \( q \) which determines an element \( \psi_\mu(1) = \psi^*_\mu(1) \in H^q(X, A; \mathcal{O}^\wedge) \) (for \( A = \emptyset \) and \( \mathcal{J} = \mathcal{O}^\wedge \)) called the characteristic class of \( \mu \). Generally

\[
\psi^*_\mu(z) = \psi_\mu(1) \cup z \quad \text{and} \quad \psi^*_\mu(v) = v \cup \psi_\mu(1).
\]

As an application of (2), we get the following relations between the Gysin sequences of a Whitney sum to those of the components. (For simplicity of notations we consider the bundle case only.)

(5) Theorem. Let \( \mathcal{E}_i, i = 1, 2, \) be \( \mathbb{R}^q \)-bundles over a space \( X \) and \( \mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2 \) their Whitney sum.
The canonical injections \( \phi_i : E_i \to E \) induce maps between the Gysin sequences of \( \mathcal{A}_i \) and \( \mathcal{B} : \)

\[
\cdots \to H_n(E_i^0, E_i^0|A; p_i^0|\cdots) \to H_n(X, A; \varphi_i^0|\cdots) \to H_{n-q}(X, A; \varphi_i^0|\cdots) \to \cdots \\
\downarrow \delta_i^{\varphi_i} \\
\cdots \to H_n(E^0, E^0|A; p^0|\cdots) \to H_n(X, A; \varphi^0|\cdots) \to H_{n-q}(X, A; \varphi^0|\cdots) \to \cdots
\]

where the maps \( \varphi_{i, k} \) are given by

\[
\varphi_{i, 1}(z) = (-1)^{q_1q_2} \cap_2 \cap z, \quad \varphi_{i, 2}(z) = \cap_1 \cap z
\]

Similarly, in the diagrams of the Gysin cohomology sequences the corresponding maps \( \varphi_{1, k}^* \) are given by

\[
\varphi_{1, 1}^*(v) = (-1)^{q_1q_2} v \cup \cap_2, \quad \varphi_{1, 2}^*(v) = v \cup \cap_2
\]

The verification is easy except for determining the \( \varphi_{1, k} \) and \( \varphi_{1, k}^* \) which is done directly by means of theorem 2. Theorem 5 has also been found by Chern and Lashof (unpublished) in the case of orientable orthogonal bundles over a compact manifold.

2. This section considers some relations between \( \mathbb{R}^q \)-bundles and \( \mathbb{D}^q \)-bundles. In the following \( \mathbb{D}^q \) denotes the closed unit ball centered at the origin in \( \mathbb{R}^q \). A \( \mathbb{D}^q \)-bundle is a fibre bundle with fibre \( \mathbb{D}^q \) and a zero-section. Thus the structure group of a \( \mathbb{D}^q \)-bundle is the group of all homeomorphisms of \( \mathbb{D}^q \) leaving the origin invariant. Denote this by \( G(D^q, 0) \). Any \( \mathbb{D}^q \)-bundle \( \eta \) determines an \( S^{q-1} \)-bundle \( \eta \) , its boundary bundle.

Conversely, \( \eta \) is determined by \( \eta \) up to equivalence in the following way: Any homeomorphism of \( S^{q-1} \) can be radially extended to a homeomorphism of \( \mathbb{D}^q \) leaving the origin invariant. If \( G(S^{q-1}) \) is the group of homeomorphisms of \( S^{q-1} \), this gives a canonical imbedding \( i : G(S^{q-1}) \to G(D^q, 0) \). By the invariance of domain theorem restricting maps from \( \mathbb{D}^q \) to \( S^{q-1} \) determines a
retraction \( r : G(D^q, 0) \to G(S^{q-1}) \). The Alexander radialization process \( \mathcal{R} \) shows immediately that it is homotopic rel \( G(S^{q-1}) \) to the identity on \( G(D^q, 0) \), i.e., \( G(S^{q-1}) \) is a strong deformation retract of \( G(D^q, 0) \). It follows that any \( D^q \)-bundle \( \eta \) is isomorphic to \( M_\eta \), the mapping cylinder of its boundary bundle, by an isomorphism which is the identity on \( \eta \). In particular, if \( \eta_1, \eta_2 \) are \( D^q \)-bundles over the same base, then any isomorphism \( \eta_1 \simeq \eta_2 \) extends to an isomorphism \( \eta_1 \simeq \eta_2 \). This is also true in the category of PL-bundles, where the argument is more delicate, cf. Hirsch [3].

For any integer \( q \geq 0 \) let \( R^q \subset S^q \) be an embedding of \( R^q \) into its one-point compactification, fixed throughout this paper. Let \( \infty \) denote the complementary point of \( R^q \) in \( S^q \). Any homeomorphism of \( D^q \) leaving the origin fixed extends to a homeomorphism of \( S^q \) also leaving \( \infty \) fixed. Therefore the embedding \( R^q \subset S^q \) induces an embedding \( G(R^q, 0) \subset G(S^q, 0, \infty) \). It follows that any \( R^q \)-bundle \( \xi \) has a functorially associated \( S^q \)-bundle \( \xi_\infty \) (over the same base) with two sections \( s_0, s_\infty \), and that \( \xi \) is naturally imbedded in \( \xi_\infty \) with zero-section corresponding to \( s_0 \) and total space \( E \) corresponding to \( E_\infty = \text{im} \ s_\infty \) (cf. remark at the end in ).

For any \( D^q \)-bundle \( \eta \) let \( \xi \) denote the interior \( R^q \)-bundle of \( \eta \), \( \eta < \xi \). We state and indicate a short proof of a result of Hirsch and Mazur (PL-category) in the general topological case.

1. **Theorem.** Let \( \xi \) be an \( R^q \)-bundle over \( X \). Then \( \xi \oplus E \) is isomorphic to the interior bundle of the \( D^q+1 \)-bundle \( M_\xi_\infty \).

Let \( \eta_1, \eta_2 \) be \( D^q \)-bundles over \( X \). Then any isomorphism \( \eta_1 \simeq \eta_2 \) extends to an isomorphism \( \eta_1 \oplus \xi^1 \simeq \eta_2 \oplus \xi^1 \).

Here \( \xi^1 = \xi^1(X) \) means the trivial \( R^1 \)-bundle over \( X \) and \( \xi^1 = \xi^1(X) \) means the trivial \( D^1 \)-bundle over \( X \). According to Hirsch [3], W. Browder has shown the existence of a submanifold of Euclidean space having a normal \( R^q \)-bundle which is not the interior of a \( D^q \)-bundle. The Hirsch-Mazur theorem shows that such pathology cannot occur for split bundles. We indicate a proof of theorem 6: The mapping cylinder bundle \( M_\xi_\infty \) has a section \( \tilde{s}_0 \) defined by \( \tilde{s}_0(x) = [s_0(x), \frac{1}{2}] \). Form the "dented" mapping

...
cylinder bundle $\tilde{M}_{s_{\infty}}$ from $M_{s_{\infty}}$ by collapsing the radius to the $\infty$-point in each fiber. Then $\tilde{M}_{s_{\infty}}$ defines a section of $\tilde{M}_{s_{\infty}}$. With $\xi_{s_{\infty}}$ as zero-section $\tilde{M}_{s_{\infty}}$ is a $D^{q+1}$-bundle whose interior is easily seen to be isomorphic to $\xi_{s_{\infty}} \oplus E_{1}$. On the other hand the boundary bundle of $\tilde{M}_{s_{\infty}}$ is clearly isomorphic to the boundary bundle of $M_{s_{\infty}}$ and this isomorphism extends to an isomorphism $\tilde{M}_{s_{\infty}} \approx M_{s_{\infty}}$ by the remarks in the first part of this section. Finally, if $\eta$ is any $D^{q}$-bundle one establishes immediately natural isomorphisms between $\eta_{s_{\infty}}$ and the boundary bundle of $\eta \oplus S^{1}$, which again extends to isomorphisms $\tilde{M}_{s_{\infty}} \eta \oplus \eta_{s_{\infty}} \oplus S^{1}$. This gives the last part of theorem 6.

We make a remark about Thom spaces. If $\xi: X \to E \to X$ is an $\mathbb{R}^{q}$-bundle with associated $S^{q}$-bundle $\xi_{s_{\infty}}: X \to E \to X$, and $(A, B)$ is a pair of subsets of $X$, the Thom space $T_{\xi}(A, B)$ is defined to be the pointed space

$$T_{\xi}(A, B) = p_{-1}^{-1}A/(s_{\infty} \cup p_{-1}^{-1}B)$$

By the representation theorem in Holm [4], this definition carries over to microbundles, cf. last part of [4]. Suppose $\xi$ contains or is the interior of a $D^{q}$-bundle $\eta$. Then, again by the microbundle representation theorem, $\xi \approx \eta$. Collapsing the boundary in each fiber of $\eta$ gives rise to an $S^{q}$-bundle $\eta/\eta$ and a canonical bundle map $\eta/\eta \to \eta/\eta$, which is an imbedding on $\eta$. Moreover, there is a natural isomorphism of bundles $\xi_{s_{\infty}} \approx \eta/\eta$ which is the identity on $\eta$. It follows that we have homeomorphisms

$$T_{\xi}(X, B) \approx T_{\xi}(X, B) \approx E_{1}/(E_{1} \cup p_{-1}^{-1}B)$$

When $B = \emptyset$, this gives the classical definition of the Thom space of a vector bundle.

We next show that $T_{\xi}(X, B)$ reflects the homology properties of the bundle pair $(\xi, E_{1}|B)$, at least when $B$ is nicely imbedded in $X$. A radial neighbourhood of a closed set $B$ in $X$ is an open set $U \supset B$ such that $U$ is homeomorphic to the mapping cylinder of a map $f: \hat{U} \to B$ ($\hat{U}$ = boundary of $U$) sending $\hat{U}$ onto $\hat{B}$, by a homeomorphism which is the identity on $\hat{U} \cup B$. Call $f$ the defining map of the radial neighbourhood. In this case $B$ is radially imbedded or r-imbedded in $X$. If $A$ is r-imbedded in $X$ and $U \subset X$ is a radial neighbourhood of $A$,
then again X-U is r-imbedded in X and H(U) \cong H(A), H(X-U) \cong H(X-A). We have the following

\begin{enumerate}
\item \textbf{Lemma.} Let \( \xi: X \stackrel{S}{\to} E \stackrel{p}{\to} X \) be an \( \mathbb{R}^q \)-bundle and let \( B \subset X \) be a closed set, r-imbedded with radial neighbourhood \( U \) and defining map \( f \). Then \( E|B, E|_{\infty}|B \) are r-imbedded in \( E,E_{\infty} \) respectively, with radial neighbourhoods \( E|U, E_{\infty}|U \) and defining maps \( f',f_{\infty} \) such that \( p \circ f' = f \circ p, p_{\infty} \circ f_{\infty}' = f \circ p_{\infty} \).

Given an \( \mathbb{R}^q \)-bundle \( \xi: X \stackrel{S}{\to} E \stackrel{p}{\to} X \) and a space \( Y \), a fibrewise proper map \( f: Y \to E \) is a map such that \( f|f^{-1}p^{-1}x \) is a proper map for any fiber \( p^{-1}x \). A fibrewise proper fibre homotopy \( H: Y \times I \to E \) is a fibre homotopy which is a fibrewise proper map. Then we have

\item \textbf{Lemma.} Let \( \xi, \xi' \) be two bundles and \( f: \xi \to \xi' \) a bundle map. Then \( f \) extends uniquely to a bundle map \( f_{\infty}: \xi_{\infty} \to \xi'_{\infty} \) which maps \( \text{im } s_{\infty} \) to \( \text{im } s'_{\infty} \) if and only if \( f \) is fibrewise proper. Similarly, if \( H: f \sim g \) is a fibre homotopy between bundle maps \( f,g: \xi \to \xi' \), then \( H \) extends uniquely to a fibre homotopy \( H_{\infty}: f_{\infty} \sim g_{\infty} \) mapping \( \text{im } s_{\infty} \times I \) to \( \text{im } s'_{\infty} \) if and only if \( H \) is fibrewise proper.

Using lemma 2 and ordinary homotopy-type reasoning we get

\item \textbf{Corollary.} Let \( \xi: X \stackrel{S}{\to} E \stackrel{p}{\to} X \) be an \( \mathbb{R}^q \)-bundle and \( B \subset X \) an r-imbedded closed set. Then there is an isomorphism

\[ H(T_{\xi}(X,B)) \cong H(E,E_0 \cup E|B) \]

natural with respect to fibrewise proper bundle maps \((\xi,\xi|B) \to (\xi',\xi'|B)\).

Let \( \xi: X \stackrel{S}{\to} E \stackrel{p}{\to} X \) and \( \xi': X' \stackrel{S'}{\to} E' \stackrel{p'}{\to} X' \) two nonlinear bundles (or microbundles) and let \((X,B),(X',B')\) be pairs of subsets of \( X,X' \) resp. Then \((X,B) \times (X',B')\) is a pair of subsets of \( X \times X' \) and we may consider the Thom space \( T_{\xi \times \xi'}((X,B) \times (X',B')) \). One establishes a natural continuous bijection (of pointed spaces)
which always induces isomorphisms in homology. This map is a relative homeomorphism, and if one of \((X,B), (X',B')\) is a compact pair or both of them and their product are compactly generated, then the map is a homeomorphism. In this case the two pointed spaces will be identified.

For any space \(X\) and any \(\mathbb{R}^q\)-bundle \(\xi\) over \(X\) there is a natural isomorphism \(\xi \otimes \mathbb{C}^n(X) \cong \xi \times \mathbb{C}^n(pt)\); thus if \(B \subset X\), then \((\xi \otimes \mathbb{C}^n(X))[B] \cong \xi[B] \times \mathbb{C}^n(pt)\). It follows that

\[
T \xi \otimes \mathbb{C}^n(X, B) = T\xi(X, B) \wedge T\mathbb{C}^n(pt).
\]

(We write \(T\mathbb{C}^n(pt)\) for \(T\mathbb{C}^n(pt, \xi)\).) But \(T\mathbb{C}^n(pt) \cong S^n\), therefore we have

\[
T \xi \otimes \mathbb{C}^n(X, B) = S^n T\xi(X, B).
\]

Next, define two non-linear bundles \(\xi, \xi'\) to be \(j\)-equivalent or to have the same stable fibre homotopy type if for some integers \(m, n\) there exists fibrewise proper bundle maps

\[
g : \xi \otimes \mathbb{C}^m \rightarrow \xi' \otimes \mathbb{C}^n, \quad h : \xi' \otimes \mathbb{C}^n \rightarrow \xi \otimes \mathbb{C}^m \quad \text{and fibrewise proper fibre homotopies} \quad G : 1 \rightarrow \xi \otimes \mathbb{C}^m \quad \text{and} \quad H : 1 \rightarrow \xi' \otimes \mathbb{C}^n
\]

\(g \circ h\). We then have the following result similar to (2.6) in Atiyah [1.]

\[\text{(5) Lemma. If } \xi, \xi' \text{ are } j\text{-equivalent non-linear bundles over } X, \text{ then for any } B \subset X, \quad T\xi(X, B) \text{ and } T\xi'(X, B) \text{ are of the same } S\text{-type.}\]

It is not hard to see (using the Hirsch-Mazur theorem and lemma 3) that the above definition of \(j\)-equivalence is equivalent to the classical one in the case of vector bundles. We skip the details.

3. Let \(X\) be a \(q\)-manifold and let \(\mu_1 : X \overset{\iota_1}{\rightarrow} E \overset{p_1}{\rightarrow} X, \quad \mu_2 : X \overset{\iota_2}{\rightarrow} E \overset{p_2}{\rightarrow} X\) be microbundles over \(X\) of dimension \(q_1, q_2\) respectively. Assume the Whitney sum \(I \oplus \mu_1 \oplus \mu_2\) is trivial, where \(\iota : X \rightarrow X \times X \rightarrow X\) is the tangent microbundle of \(X\). Consider the composite microbundle \(\mu = \iota \circ (\mu_1, \mu_2)\). We have
\[ \mu : X \rightarrow \mathbb{E}_1 \times \mathbb{E}_2 \rightarrow X \text{ with } s = (s_1 \times s_2) \circ \Delta \text{ and } \\
p = \text{pr}_1 \circ (p_1 \times p_2). \text{ After Milnor } \mu \text{ is equivalent to } \\
\tau \circ \Delta^X (\mu_1 \times \mu_2) = \tau \circ \mu_1 \oplus \mu_2 \text{ and thus trivial (cf. (5) in } \text{[5]} \text{).} \]

Therefore there exists a neighbourhood \( W \) of \( sX \) in \( \mathbb{E}_1 \times \mathbb{E}_2 \) and a homeomorphism

\[ W \cong X \times D^Q \quad Q = q + q_1 + q_2 \]

which fits into a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\phi} & X \\
\downarrow s & & \downarrow \phi \\
W & \cong & X \times D^Q \\
\downarrow \text{pr} & & \downarrow \text{pr} \\
X & \xrightarrow{\psi} & X \\
\end{array}
\]

There is a map

\[ (W, W - sX) \cong X \times (D^Q, D^Q - 0) \xrightarrow{\text{pr}_2} (D^Q, D^Q - 0) \rightarrow (D^Q / D^Q, D^Q / D^Q - 0) \cong (S^Q, S^Q - 0) \]

with extensions

\[
(E_1 \times E_2, E_1 \times E_2 - sX) \rightarrow (S^Q, S^Q - 0) \\
((E_1 \times E_2)_\infty, \text{im}(s_1 \times s_2)_\infty) \rightarrow (S^Q, \infty)
\]

where \( \infty \) is the antipodal point of 0 (= the collapsed boundary \( D^Q \)). Passing to quotients give a map

\[ (T\mu_1 \times \mu_2, \ast) \rightarrow (S^Q, \infty) \]

where \( \ast \) is the basepoint of the Thom space \( T\mu_1 \times \mu_2 = T\mu_1 \times T\mu_2 \). If \( X \) is compact, \( T\mu_1 \times \mu_2 = T\mu_1 \land T\mu_2 \). In this case we therefore get a map

\[ (T\mu_1 \land T\mu_2, \ast) \rightarrow (S^Q, \infty) \]

\( \ast \) this is an \( S \)-duality map in the sense of Spanier-Whitehead.

More generally, let \( (A, B) \) be a compact \( r \)-pair in the (possibly non-compact) manifold \( X \) with a radial neighbourhood \( (U, V) \), say, and defining map \( (f, g) \) (i.e. \( U \) and \( V \) are
radial neighbourhoods of the compact subsets $A$ and $B$ in $X$ with defining maps $f$ and $g$ respectively, and $V \cap A$ is a radial neighbourhood of $B$ in $A$ with defining map $g|_{A}$. Then there is a map

$$f: \mu_{1}^{(A,B)} \cap \mu_{2}^{(X-V,X-U)} \rightarrow (S^{Q}, \infty)$$

To show that this is an $S$-duality map we must show that slanting with the spherical cohomology class in

$$H^{Q}(\mu_{1}^{(A,B)} \cap \mu_{2}^{(X-V,X-U)})$$

gives a duality isomorphism

$$H_{n}(\mu_{1}^{(X-V,X-U)}) \cong H^{Q-n}(\mu_{1}^{(A,B)}).$$

According to corollary 4 in section 2 there are natural isomorphisms in homology and cohomology of the pair $(\mu_{1}^{(A,B)} \cap \mu_{2}^{(X-V,X-U)}$, $x)$ and the pair $(E_{1}|A, E_{1}^{0}|A \cup E_{1}|B) \times (E_{2}|(X-V), E_{2}^{0}|(X-V) \cup E_{2}|(X-U))$. Notice also that $(E_{1}|A, E_{1}^{0}|A \cup E_{1}|B) \times (E_{2}|(X-V), E_{2}^{0}|(X-V) \cup E_{2}|(X-U))$ is contained in the pair $(E_{1} \times E_{2}, E_{1} \times E_{2}-sX)$. Denote the inclusion map by

$$(1) \quad \text{Lemma. Under the natural isomorphism of 2.4 the spherical class in } H^{Q}(\mu_{1}^{(A,B)} \cap \mu_{2}^{(X-V,X-U)}) \text{ corresponds to}$$

$\gamma_{\mu} = \gamma \times \mu'$, the Thom class of $\mu$ restricted to

$$(E_{1}|A, E_{1}^{0}|A \cup E_{1}|B) \times (E_{2}|(X-V), E_{2}^{0}|(X-V) \cup E_{2}|(X-U)).$$

Moreover $\gamma \times \mu'$ has coefficients $\gamma \times \mu = \gamma \times \nu_{1} \times \mu$ where

$\gamma_{1} : (E_{1}|A, E_{1}^{0}|A \cup E_{1}|B) \subset (E_{1}, E_{1}^{0}).$ There is a map (for each $n$)

$$\gamma_{\mu} : H_{n}(E_{1}|(X-V), E_{1}^{0}|(X-V) \cup E_{2}|(X-U)) \rightarrow H^{Q-n}(E_{1}|A, E_{1}^{0}|A \cup E_{1}|B;$$

$$\gamma_{\mu}^{*} \times \nu_{1} \times \mu$$

defined by $\gamma_{\mu}(z) = \gamma_{\mu}/z,$ corresponding to the map

$$H_{n}(\mu_{2}^{(X-V,X-U)}) \rightarrow H^{Q-n}(\mu_{1}^{(A,B)})$$

defined by slanting with the spherical class in

$$H^{Q}(\mu_{1}^{(A,B)} \cap \mu_{2}^{(X-V,X-U)}).$$

(Notice that $\nu_{1} \times \mu$ hence $\gamma \times \mu$ is constant since $\mu$ is trivial). Finally notice that the $r$-structure on $(A,B)$ induces an isomorphism in homology

$$H(E_{2}|(X-V), E_{2}^{0}|(X-V) \cup E_{2}|(X-U)) \cong H(E_{2}|(X-B), E_{2}^{0}|(X-B) \cup E_{2}|(X-A))$$
according to 2.2. Therefore there is a map
\[ \gamma(A,B) : H_n(E_2|(X-B), E_2^{0}|(X-B) \cup E_2|(X-A)) \to H^{n-q_2}(A; \mathcal{G}_2) \]
(for each \( n \)) corresponding to \( \gamma \). The following result is the one we have been heading for:

(2) **Theorem.** The map \( \gamma = \gamma(A,B) \) is an isomorphism. More precisely we have the commutative diagram
\[
\begin{array}{ccc}
H_n(E_2|(X-B), E_2^{0}|(X-B) \cup E_2|(X-A)) & \cong & H^{n-q_2}(A; \mathcal{G}_2) \\
\downarrow \phi_{2*} & & \phi_{1*} \uparrow \cong \\
H_{n-q_2}(X-B, X-A; \mathcal{G}_2) & \cong & H^{q+q_2-n}(A; \mathcal{G}_2(\tau) \otimes \mathcal{G}_2)
\end{array}
\]

where \( \phi_{1*}, \phi_{2*} \) are the relative Thom isomorphisms and \( \gamma \) the Alexander-Spanier duality map for the manifold \( X \).

Using 2.5 we get

(3) **Corollary.** Let \( X \) be a manifold which is a CW-complex and let \( \mu_1, \mu_2 \) be two microbundles over \( X \) such that \( \tau \oplus \mu_1 \oplus \mu_2 \) has trivial stable fibre homotopy type. Let \( (A,B) \) be a compact relative CW-complex in \( X \) such that \( A \) is a subcomplex of \( X \). Then \( (A,B) \) is an r-pair in \( X \) with radial neighbourhoods \( (U,V) \), say, and \( T_{\mu_1}(A,B) \) and \( T_{\mu_2}(X-V,X-U) \) are S-duals.
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