Matematisk Seminar
Universitetet i Oslo

Nr. 9
September 1965.

HARMONIC ANALYSIS AND REAL GROUP ALGEBRAS

By

K. E. Aubert

1. <u>Introduction</u>. The literature on real group algebras is very scarce. This is of course due to the fact that Fourier analysis is basically a complex theory, with the complex group algebra $L^1_c(G)$ as the central object of study. Nevertheless, various natural questions pose themselves also in the case of the real group algebra $L^1_R(G)$: What is the structure of this algebra in comparison with well-known results about $L^1_C(G)$? Which relations exist between the convolution product and the pointwise ordering a.e. of the functions in $L^1_R(G)$? To what extent do there exist functions in $L^1_R(G)$ with Fourier-transforms with prescribed properties?

The purpose of the present report is to give some scattered results in connection with the above-mentioned problems. In a previous report [1] in this seminar we showed that there are no other closed convex ideals in $L^1_R(G)$ than those which are contained in the kernel of the Haar-measure. In the proof of this result (Theorem 1) it turns out that a crucial role is played by the positivity of certain convolution products. In [1] we also offered an elementary approach to this question which we here complete on certain points. By means of Theorem 1 the determination of the w^*-closed convex translation invariant subspaces of $L_R(G)$ is quite easy. We also treat the commutativity of a certain diagram which arises in this connection. In a final section we prove a couple of results on Fourier transforms of real-valued functions which parallels similar results in the complex theory.

2. <u>Convex closed ideals in $L^1_R(G)$</u>. By $L^1_R(G)$ - or simply L^1_R - we denote the ordered group algebra of all real-valued integrable functions on a locally compact abelian group G under the ordering $f \geqslant g$ whenever $f(x) \geqslant g(x)$ almost everywhere on G. L^1_C shall denote the usual group algebra of all complex-valued integrable functions on G. We recall that an

ideal $\mathfrak{a}$ [1] $\subset L_R^1$ is said to be convex if $f, g \in \mathfrak{a}$ and $f \geq h \geq g$ implies $h \in \mathfrak{a}$. The following theorem was proved in [1] :

Theorem 1. A proper closed ideal in $L_R^1(G)$ is convex if and only if it is contained in the kernel of the Haar-measure $\mathfrak{M}_0^R$ (consisting of all functions with zero integral).

We give a number of easy corollaries some of which were not mentioned in [1].

Corollary 1. The only regular and convex maximal ideal in L_R^1 is the kernel of the Haar measure. Otherwise expressed: If μ is an order preserving homomorphism of L_R^1 onto a partially ordered field F then F is isomorphic to the field of real numbers and μ is the Haar measure of G.

If $\mathfrak{M}_\alpha$ denotes the maximal ideal in L_C^1 which corresponds to the character $\alpha \in \hat{G}$ we also note the following.

Corollary 2. If G is connected the following statements are equivalent

(1) $\mathfrak{M}_\alpha$ is the kernel of the Haar measure

(2) $\mathfrak{M}_\alpha \cap L_R^1$ is convex

(3) $\mathfrak{M}_\alpha$ does not contain any strictly positive function

(4) $\mathfrak{M}_\alpha \cap L_R^1$ is of real codimension one in L_R^1

(5) $\hat{f}(\alpha)$ is real for all $f \in L_R^1$

(6) $\mathfrak{M}_\alpha \cap L_R^1 = \mathfrak{M}_\beta \cap L_R^1 \Longrightarrow \alpha = \beta$

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1) In the following we shall always assume that $\mathfrak{a}$ is a proper ideal. Hence L_R^1 will not be considered as an ideal in L_R^1.

Proof: The equivalence of (1), (2) and (3) is valid for arbitrary G and follows from Corollary 1 and the proof of Theorem 1. From the Gelfand-Mazur theorem it follows that $L^1_R / \mathcal{M}_\alpha \cap L^1_R$ is isomorphic to either R or C. This quotient algebra is isomorphic to R if and only if α is a real-valued character which means that α is the identity character if G is connected. From this together with

$$(2.1) \qquad (\mathcal{M}_\alpha \cap L^1_R = \mathcal{M}_\beta \cap L^1_R) \Longleftrightarrow (\alpha = \pm\beta)$$

we easily deduce the latter part of the corollary. The equivalence (2.1) expresses that a maximal ideal in L^1_R can be extended to a maximal ideal in L^1_C in at most two ways and that the extension is unique if and only if the given maximal ideal corresponds to a real-valued character (which means that $\alpha = -\alpha$). For more general information about the relationship between real and complex Benach algebras we can refer the reader to [4] and [5].

For later reference we give the following corollary which is a consequence of the proof of Theorem 1.

Corollary 3. A closed ideal $\mathfrak{A}$ is convex if and only if $\mathfrak{A}$ does not contain a strictly positive function.

Corollary 4. Spectral analysis holds for closed convex ideals in L^1_R while spectral synthesis does not hold.

Corollary 4 is valid since the converse of Theorem 1 is obviously true: Any ideal contained in the kernel of the Haar measure is convex.

If we formulate Theorem 1 in terms of homomorphisms we get the following

Corollary 5. Any order-preserving ring homomorphism of L_R^1 onto a partially ordered ring is a factor in the canonical order-preserving homomorphism of L_R^1 onto R.

3. Convex translation invariant subspaces of L_R^∞. We denote by L_R^∞ the real dual of L_R^1, consisting of all bounded measurable real-valued functions on G. Thus L_R^∞ is nothing else than the family of all the real-valued functions in the usual complex L^∞-space which we shall here denote by L_C^∞. It is a well-known fact that there is a one-to-one correspondence between the closed ideals in L_C^1 and the w^*-closed translation invariant subspaces of L_C^∞ (see [6] p. 184). The same correspondence persists between the real spaces L_R^1 and L_R^∞. By means of this correspondence we shall easily describe the convex w^*-closed translation invariant subspaces of L_R^∞.

If $\mathfrak{a}$ is a closed ideal in L_C^1 we put $\mathfrak{a}^{\perp_C} = \{g \mid g \in L_C^\infty$ and $g * f = 0$ for all $f \in \mathfrak{a}\}$. Similarly if $\mathfrak{a}$ is a closed ideal in L_R^1 we put $\mathfrak{a}^{\perp_R} = \{g \mid g \in L_R$ and $g * f = 0$ for all $f \in \mathfrak{a}\}$. The correspondence between closed ideals in L_R^1 and w^*-closed translation invariant subspaces of L_R is then given by the mapping $\mathfrak{a} \rightarrow \mathfrak{a}^{\perp_R}$

Lemma 1. $\mathfrak{a}$ contains a strictly positive function if and only if $\mathfrak{a}^{\perp_R}$ does not contain a strictly positive functions.

Proof: If $\mathfrak{a}$ contains a strictly positive function it is clear that $\mathfrak{a}^{\perp_R}$ can not contain such a function since $\mathfrak{a} * \mathfrak{a}^{\perp_R} = \{0\}$.

Conversely if $\mathfrak{a}$ does not contain a strictly positive function we know from Corollary 3 of Theorem 1 that $\mathfrak{A} \subset \mathfrak{M}_0^R$ and thus $1 \in \mathfrak{A}^{\perp_R}$ proving that $\mathfrak{A}^{\perp_R}$ contains a strictly positive function.

<u>Proposition 1.</u> $\mathfrak{A}^{\perp_R}$ <u>is convex if and only if</u> $\mathfrak{A}$ <u>is not convex.</u>

<u>Proof</u>: If $\mathfrak{A}$ is convex then $\mathfrak{A} \subset \mathfrak{M}_0^R$ and $1 \in \mathfrak{A}^{\perp_R}$ showing that $\mathfrak{A}^{\perp_R}$ is not convex. Conversely if $\mathfrak{A}$ is not convex then $\mathfrak{A}$ contains a strictly positive function and hence by Lemma 1, $\mathfrak{A}^{\perp_R}$ does not contain a strictly positive function. This means that two functions in $\mathfrak{A}^{\perp_R}$ cannot be comparable without being identical and hence $\mathfrak{A}^{\perp_R}$ is convex.

<u>Corollary.</u> <u>A w^*-closed translation invariant subspace of L_R^∞ is convex if and only if it annihilates a strictly positive function.</u>

Let $\mathfrak{J}_C$ and $\mathfrak{J}_R$ denote the families of closed ideals in L_C^1 and L_R^1 respectively and let $\mathcal{S}_C$ and $\mathcal{S}_R$ denote the families of w^*-closed translation invariant subspaces of L_C^∞ and L_R^∞ respectively. We define the mappings $\varphi : \mathfrak{J}_C \to \mathfrak{J}_R$ and $\psi : \mathcal{S}_C \longrightarrow \mathcal{S}_R$ by

$$\varphi(\mathfrak{A}) = \mathfrak{A} \cap L_R^1 \quad \text{and} \quad \psi(\mathfrak{A}^{\perp_C}) = \mathfrak{A}^{\perp_C} \cap L_R^\infty .$$

The question arises whether the following diagram is commutative or not:

$$D: \begin{tikzcd} \mathfrak{J}_C \arrow[r, "\perp_C"] \arrow[d, "\varphi"'] & \mathcal{S}_C \arrow[d, "\psi"] \\ \mathfrak{J}_R \arrow[r, "\perp_R"] & \mathcal{S}_R \end{tikzcd}$$

If $\psi(\mathfrak{a}^{\perp_C}) = (\varphi(\mathfrak{a}))^{\perp_R}$ we shall say that <u>D is commutative for $\mathfrak{a}$</u>. It is clear that φ is a surjection since $\varphi(\mathfrak{b}) = \mathfrak{a}$ when $\mathfrak{a} \subset L^1_R$ and $\mathfrak{b} = \mathfrak{a} + i\mathfrak{a} = \{f_1 + if_2 \mid f_1, f_2 \in \mathfrak{a}\}$. This ideal $\mathfrak{b}$ is the unique minimal closed ideal in L^1_C such that $\varphi(\mathfrak{b}) = \mathfrak{a}$. These ideals $\mathfrak{a} + i\mathfrak{a}$ which are in one-to-one correspondence with the closed ideals in L^1_R will be called <u>symmetric ideals</u> in L^1_C.

<u>Proposition 2.</u> <u>The diagram</u> D <u>is commutative for</u> $\mathfrak{a}$ <u>if and only if</u> $\mathfrak{a}$ <u>is a symmetric ideal.</u>

<u>Proof</u>: If $\mathfrak{b} = \mathfrak{a} + i\mathfrak{a}$ with $\mathfrak{a} \in \mathcal{J}_R$ then $(\varphi(\mathfrak{b}))^{\perp_R} = \{g \mid g \in L^\infty_R$ and $g * f = 0$ for all $f \in \mathfrak{a}\}$ and this set is evidently the same as $\psi(\mathfrak{b}^{\perp_C})$. Assume conversely that $\mathfrak{b}$ is not symmetric, i.e., that $\mathfrak{b}$ <u>properly</u> contains $\varphi(\mathfrak{b}) + i\varphi(\mathfrak{b})$. Due to the fact that $\perp_C$ is one-to-one there exists a $g \in L^\infty_C$ such that $g * f = 0$ for all $f \in \varphi(\mathfrak{b}) + i\varphi(\mathfrak{b})$ but not for all $f \in \mathfrak{b}$. Writing $g = g_1 + ig_2$ we get

$$(g_1 + ig_2) * f = g_1 * f + i(g_2 * f) = 0 \text{ for all } f \in \varphi(\mathfrak{b}).$$

Thus $g_1 * f = g_2 * f = 0$ for all $f \in \varphi(\mathfrak{b})$. On the other hand both g_1 and g_2 cannot annihilate $\mathfrak{b}$ since $g = g_1 + ig_2$ would then do the same. This proves that either g_1 or g_2 will annihilate $\varphi(\mathfrak{b})$ without annihilating $\mathfrak{b}$. This implies that the diagram D is not commutative for $\mathfrak{b}$.

In order to determine more specifically the maximal ideals for which D is commutative it is convenient to have the following

Lemma 2. *The ideal* $\mathfrak{A}_\alpha = \mathfrak{M}_\alpha \cap \mathfrak{M}_{-\alpha}$ *consists of all functions* $f = f_1 + if_2$ *such that* $\hat{f}_1(\alpha) = \hat{f}_2(\alpha) = 0$. *Otherwise expressed:* $\mathfrak{A}_\alpha = \varphi(\mathfrak{M}_\alpha) + i\varphi(\mathfrak{M}_\alpha)$ $(= \varphi(\mathfrak{M}_{-\alpha}) + i\varphi(\mathfrak{M}_{-\alpha}))$.

Proof: If $\hat{f}_1(\alpha) = \hat{f}_2(\alpha) = 0$ then also $\hat{f}(\alpha) = 0$ and $f \in \mathfrak{M}_\alpha$. Since $f_1, f_2 \in \varphi(\mathfrak{M}_\alpha) = \varphi(\mathfrak{M}_{-\alpha})$ we also have $\hat{f}_1(-\alpha) = \hat{f}_2(-\alpha) = 0$ and hence $\hat{f}(-\alpha) = 0$. This implies $f \in \mathfrak{M}_\alpha \cap \mathfrak{M}_{-\alpha}$. Assume conversely that $h = h_1 + ih_2 \in \mathfrak{A}_\alpha$, i.e.

$$(3.1) \qquad \int h(x)\overline{(x,\alpha)}dx = \int h(x)(x,\alpha)dx = 0$$

By adding and subtracting the two left hand terms of (3.1) we get

$$(3.2) \qquad \int h(x)(\overline{(x,\alpha)} + (x,\alpha))dx = \int h(x)(\overline{(x,\alpha)} - (x,\alpha))dx = 0.$$

Since $\overline{(x,\alpha)} + (x,\ )$ and $\overline{(x,\alpha)} - (x,\alpha)$ are real and purely imaginary respectively, (3.2) is also valid when substituting h_1 or h_2 for h. Adding up the two expressions on the left hand side of (3.2) with h_1 instead of h we get $\hat{h}_1(\alpha) = 0$. Similarly $\hat{h}_2(\alpha) = 0$.

Proposition 3. *The diagram* D *is commutative for a maximal ideal* $\mathfrak{M}_\alpha$ *if and only if* α *is a real-valued character.*

Proof: According to Proposition 2 and Lemma 2, D is commutative for $\mathfrak{M}_\alpha$ if and only if $\mathfrak{A}_\alpha = \mathfrak{M}_\alpha$ or equivalently if and only if $\mathfrak{M}_\alpha = \mathfrak{M}_{-\alpha}$. But $\alpha = -\alpha$ means that α is real-valued (i.e., α assumes only the values ± 1).

Since the identity character is the only continuous real character in case G is connected we get the following

Corollary. *If* G *is connected the diagram* D *is commutative for* $\mathfrak{M}_\alpha$ *if and only if* $\mathfrak{M}_\alpha$ *is the kernel of the Haar measure.*

4. <u>The positivity of certain convolution products on compactly generated abelian groups</u>. This section contains some revisions and supplements to section 4 in [1] .

The scarcity of closed convex ideals in L_R^1 was shown in [1] to be mainly due to the existence of certain positive convolution products on G. Though the proof of Lemma A in [1] was not difficult, it used a couple of fairly deep-lying results of harmonic analysis. We shall in this section show that in certain cases we can establish the existence of the pertinent positive convolution products in a quite elementary way. In fact if we restrict the given function f in Lemma A to have compact support we can obtain an everywhere strictly positive integrable function by convoluting f with a function which is "almost constant" - in a sense which will be made precise below. This, however, raises the question as to which groups G possess such almost constant integrable functions as well as which closed convex ideals possess functions with compact support and non-vanishing integral.

<u>Definition</u>. When ε is a strictly positive real number we shall say that a nowhere vanishing function f on G is <u>almost constant of type</u> (ε,K) if

$$1-\varepsilon < \frac{f(x_1)}{f(x_2)} < 1+\varepsilon$$

whenever $x_1 - x_2 \in K$, where K is a compact subset of G. We shall say that G <u>possesses integrable almost constant functions</u> if for any given ε and K there exists a (positive) integrable almost constant function of type (ε,K) on G.

<u>Lemma 3. Any compactly generated abelian group</u> G <u>possesses integrable almost constant functions.</u>

<u>Proof</u>: By the structure theorem for compactly generated abelian groups (see [3], p. 90) any such group may be written as $G = R^n \times Z^m \times F$ where R denotes the reals, Z the integers, F a compact group, and m and n are non-negative integers. This essentially reduces the question to proving the lemma for the three groups R, Z and F, which indeed is quite easy. For a compact F we can just take any strictly positive constant function. If K is a compact subset of R and $\varepsilon > 0$, the function f defined on R by

$$f(x) = e^{-\frac{|x|}{a}\delta} \tag{4.1}$$

is an integrable almost constant function of type (ε, K) if the positive real numbers a and δ are chosen such that $K \subset [-a,a]$ and $e^{\delta} < 1 + \varepsilon$. In fact if $|x_2 - x_1| \leqslant a$ then also $\big| |x_2| - |x_1| \big| \leqslant a$ and

$$1 - \varepsilon < \frac{1}{1+\varepsilon} < e^{-\delta} \leqslant \frac{f(x_1)}{f(x_2)} \leqslant e^{\delta} < 1 + \varepsilon .$$

The restriction of the function (4.1) to Z will give an integrable almost constant function on Z which is of type (ε, K) if a is again chosen such that $K \subset [-a,a]$. Such a choice is obviously possible since K is finite, being a compact subset of a discrete group.

Roughly speaking we therefore only have to show how two integrable almost constant functions on the groups G and H, respectively, can be used to define an integrable almost constant function on $G \times H$. Assume therefore that g and h are two integrable almost constant functions on G and H and are of type (ε_1, K_1) and (ε_2, K_2) respectively. Without loss of generality we can assume that $\varepsilon_1, \varepsilon_2 < 1$. Then define f as a function on $G \times H$ by putting

$$f(x,y) = g(x) \cdot h(y).$$

It is clear that f is an integrable, everywhere positive function on $G \times H$ such that

$$(1-\varepsilon_1)(1-\varepsilon_2) < \frac{f(x_1,y_1)}{f(x_2,y_2)} < (1+\varepsilon_1)(1+\varepsilon_2)$$

whenever $(x_1-x_2, y_1-y_2) \in K_1 \times K_2$. In order to produce an integrable almost constant function f on $G \times H$ of type (ε, K) it is therefore enough to choose ε_1, ε_2, K_1 and K_2 such that $K \subset K_1 \times K_2$ and $(1+\varepsilon_1)(1+\varepsilon_2) < 1 + \varepsilon$. This completes the proof of Lemma 2.

Since an almost constant function is nowhere equal to zero it is clear that a group must in any case be σ-compact in order to possess integrable almost constant functions. We do not know, however, whether the existence of integrable almost constant functions characterizes the class of compactly, generated groups or may be the class of σ-compact groups. In any case we have the following

<u>Lemma 4.</u> <u>Let G be any locally compact abelian group which possesses integrable almost constant functions.</u> <u>If f is a function in</u> $L_R^1(G)$ <u>with compact support and non-vanishing integral, then there exists a function</u> $g \in L_R^1(G)$ <u>such that</u> $f * g > 0$.

For the proof we refer the reader to [1].

The above lemma is more restrictive than Lemma A in [1] in two ways: We have imposed conditions both on G and on f. The condition on f is not inessential since we can easily show that there exist closed ideals which are not contained in the kernel of the Haar measure and which do not contain functions with non-vanishing integral and compact support: Take $G = R$ and put

$$\mathfrak{a} = \bigcap_{\alpha \in [1,2]} \mathfrak{m}_\alpha$$

This ideal $\mathfrak{a}$ is not contained in the kernel of the Haar measure and consists of all functions in $L^1_C(R)$ which have a Fourier transform vanishing on the interval $[1,2]$. If such a function f has compact support its Fourier transform will be the restriction of an analytic function and hence vanish identically on R. Hence f itself is identically zero and $\mathfrak{a}$ contains no function with non-vanishing integral and compact support.

5. <u>Fourier transforms of real-valued functions</u>. An ever returning fundamental question in Fourier analysis is the following: To what extent do there exist functions in $L^1_C(G)$ with Fourier transforms with prescribed properties? The literature contains a variety of important contributions to this general problem. Let us only mention various types of separation theorems (like the well-known complex analogue of Theorem 2 below) as well as the crucial lemma in the proof

of Wiener's Tauberian theorem saying that the family of functions having a Fourier transform with compact support is dense in $L^1_C(G)$.

It is natural to ask whether the various theorems of this kind still hold if we restrict ourselves to real-valued functions, i.e. if we pass from $L^1_C(G)$ to $L^1_R(G)$. Such a study will be helpful for the investigation of the structure of $L^1_R(G)$. We shall here content ourselves by proving two separation theorems. We first establish an easy real-valued analogue of a well-known separation theorem of Godement [2]

<u>Definition</u>. If $K \subset 0 \subset \hat{G}$ where K is compact and 0 is open we shall say that the Fourier transforms of $L^1_R(G)$ separate K and 0 if there exists a function $f \in L^1_R$ such that $\hat{f} = 1$ on K and $\hat{f} = 0$ on $\complement 0$

<u>Theorem 2. The Fourier transforms of $L^1_R(G)$ separate K and 0 if and only if $-K \subset 0$.</u> ($-K = \{-k \mid k \in K\}$)

<u>Proof</u>: It is clear that $-K \subset 0$ is a necessary condition for separation since $\hat{f}(\alpha) = \hat{f}(-\alpha)$ whenever f is real-valued. If on the other hand $-K \subset 0$ then also $K \cup -K \subset 0$ and it is sufficient to show that the Fourier-transforms of L^1_R separate $K_1 = K \cup -K$ and 0. We choose the open neighbourhood U of the zero element in $\hat{G}$ so small that $K_1 + U \subset 0$. Let further g and h denote the characteristic functions of K_1 and U respectively. Since K_1 is symmetric $\hat{g}^* = \hat{g}$ and $\frac{1}{2}(\hat{g} + \hat{g}^*)$ is also the characteristic function of K_1. We now consider the function

$$(5.1) \qquad f = \frac{1}{m(U)} \left(\tfrac{1}{2}(\hat{g} + \hat{g}^{*})\right) * \hat{h}$$

We have here exactly the same situation as in the usual complex proof (see [2]) but in the present case we can write the right hand side of (5.1) as a linear combination of positive definite functions with real coefficients:

$$\tfrac{1}{2}(\hat{g} + \hat{g}^{*}) * \hat{h} = \tfrac{1}{4}(\hat{g} + \hat{h}) * (\hat{g} + \hat{h})^{*} - \tfrac{1}{4}(\hat{g} - \hat{h}) * (\hat{g} - \hat{h})^{*}$$

By the Fourier inversion theorem there exists a function $m(U)f \in L^1_C$ having this function as its Fourier transform and by the very form of f it is obvious that f is real-valued and that f separates K and 0.

Let us give another separation theorem where the proof is slightly more technical

Theorem 3. Given two characters $\alpha_1, \alpha_2 \in \hat{G}$ we can find a function $f \in L^1_C$ such that the following requirements are simultanously fulfilled

(1) f is real-valued

(2) f has compact support

(3) $\hat{f}$ is real-valued

(4) $\hat{f}(\alpha_1) \neq 0$ and $\hat{f}(\alpha_2) = 0$

if and only if $\alpha_1 \neq \alpha_2$ and $\alpha_1 \neq -\alpha_2$

Proof. The necessity of $\alpha_1 \neq \alpha_2$ is obvious. The necessity of $\alpha_1 \neq -\alpha_2$ stems from the fact that when f is real-valued then $\hat{f}(\alpha_1) = 0$ implies that

$$\hat{f}(-\alpha_1) = \int f(x)\overline{(x, -\alpha_1)}\,dx = \overline{\int f(x)\overline{(x, \alpha_1)}\,dx} = \overline{\hat{f}(\alpha_1)} = 0$$

If on the other hand the two conditions $\alpha_1 \neq \alpha_2$ and $\alpha_1 \neq -\alpha_2$

are fulfilled then the difference set $L^1_R \cap \mathfrak{M}_{\alpha_2} - L^1_R \cap \mathfrak{M}_{\alpha_1}$ is non-void and we can pick a function g in this set. This function g will then have the properties (1) and (4). Replacing g by $h = g * g^*$ we obtain a function which satisfies (1), (3) and (4). By considering a suitable real multiple rh instead of h we can assume that h satisfies (1), (3) and (4) and that $\hat{h}(\alpha_1)$ is an arbitrarily large positive real number. Since the set of real-valued continuous functions with compact support (denoted by C^R_{oo}) is dense in L^1_R and the Fourier transformation is non-decreasing we can find a function $k \in C^R_{oo}$ such that

$$|\hat{k}(\alpha_2)| < \varepsilon \quad \text{and} \quad \mathrm{Re}(\hat{k}(\alpha_1)) > \frac{1}{\varepsilon}$$

for any given $\varepsilon > 0$. By this transition from h to k we have gained the property (2) at the expense of losing the two properties (3) and (4). Some further adjustments are therefore necessary. The function $l = 2(k + k^*)$ is a real-valued function with compact support such that

$$|\hat{l}(\alpha_2)| < 4\varepsilon \quad \text{and} \quad \mathrm{Re}(\hat{l}(\alpha_1)) > \frac{4}{\varepsilon} \tag{5.2}$$

We can choose an approximate identity $\{u\}$ for L^1_R consisting of continuous real-valued functions with compact support such that $u = u^*$. Because of (5.2) there exists a suitable u such that

$$|\widehat{l * u}(\alpha_2)| < 4\varepsilon \quad \text{and} \quad \mathrm{Re}(\widehat{l * u}(\alpha_1)) > \frac{4}{\varepsilon}$$

But since

$$1 * u = 2(k + k^*) * u = (k + u) * (k + u)^* - (k - u) * (k - u)^*$$

it is clear that $m = 1 * u$ will satisfy (1), (2) and (3). If therefore $\hat{m}(\alpha_2) = 0$ we already have a function of the required type. If $\hat{m}(\alpha_2) \neq 0$ we put

$$f = m - \frac{1}{\hat{m}(\alpha_2)}(m * m)$$

and get

$$\hat{f}(\alpha_2) = 0 \quad \text{and} \quad \hat{f}(\alpha_1) = \hat{m}(\alpha_1) - \frac{(\hat{m}(\alpha_1))^2}{\hat{m}(\alpha_2)}$$

By choosing ε small enough we get $\hat{f}(\alpha_1) \neq 0$ and this completes the proof of the theorem.

Sometimes we can trivially deduce a theorem about L^1_R from a corresponding theorem about L^1_C. This is for instance the case with Wiener's tauberian theorem: If $\mathfrak{a}$ is a proper closed ideal in L^1_R then $\mathfrak{a} + i\mathfrak{a}$ is a proper closed ideal in L^1_C. Hence there exists by Wiener's theorem a maximal ideal $\mathfrak{M}_\alpha \subset L^1_C$ such that $\mathfrak{a} + i\mathfrak{a} \subset \mathfrak{M}_\alpha$. From this it follows that $\mathfrak{a}$ is contained in the maximal ideal $L^1_R \cap \mathfrak{M}_\alpha$ in L^1_R. So for this purpose we do not need to consider any real-valued analogue of the lemma mentioned at the beginning of this section.

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