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ON JORDAN ALGEBRAS OF SELF-ADJOINT OPERATORS

By

Erling Störmer

The present seminar report represents a continuation of the work reported in ((7)) . Consequently a number of mistakes will be patched up, all of which were due to ((7, Lemma 3.3)) , in which we, as pointed out by L. Ingelstam, forgot about the quaternions. Terminology and notation will be as in ((7)) and will not be repeated here. Just recall that a JC-algebra (resp. JW-algebra) is a uniformly (resp. weakly) closed Jordan algebra of self-adjoint operators on a Hilbert space. A JW-factor (or Jordan factor) is a JW-algebra whose center relative to ordinary operator multiplication is the real scalars. Such a Jordan algebra is of type I if it has minimal projections, and more specifically, of type I_n if there are n orthogonal minimal projections in it with sum the identity.

Jordan, von Neumann, and Wigner ((2)) have classified all ("abstractly defined") finite dimensional irreducible Jordan algebras over the reals. In the present note we shall classify all irreducible JW-algebras, and all JW-factors of type I . The results are then quite analogous to that in ((2)) , except we do not get hold of the Jordan algebra $\mathfrak{M}_3^8$ of that paper, hence we obtain as a corollary a well known result of Albert ((1)) that $\mathfrak{M}_3^8$ has no representation as a JW-algebra. The results can be summarized as follows: every irreducible JW-algebra is of type I ; those of type I_n , $n \geq 3$, are roughly all self-adjoint operators on Hilbert spaces over either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$, where these letters denote the reals, complexes or the quaternions respectively. The JW-factors of type I_2 are quite different; they are the spin factors (see ((9))), and except when the dimensions are small, are exactly those JW-factors which are not reversible (i.e. which are not the self-adjoint parts of real operator algebras).

This note is void of proofs, only rough indications will be given.

Recall from ((7, Lemma 2.2)) that if $\mathfrak{A}$ is a JC-algebra and $\mathfrak{J}$ denotes the set of A in $\mathfrak{A}$ such that $BAC + C^*AB^* \in \mathfrak{A}$ for all

B,C in $\mathcal{R}(\mathfrak{A})$, $\mathcal{R}(\mathfrak{A})$ = the uniformly closed real algebra generated by $\mathfrak{A}$, then $\mathfrak{J}$ is a Jordan ideal in $\mathfrak{A}$, and is a reversible JC-algebra. We defined $\mathfrak{A}$ to be <u>totally non reversible</u> if $\mathfrak{J}$ is zero. We recall ((7, Theorem 2.4)) .

T h e o r e m 1 . Let $\mathfrak{A}$ be a JW-algebra. Then there exist three central projections E , F , and G in $\mathfrak{A}$ with $E + F + G = I$ such that

i) $E\mathfrak{A}$ is the self-adjoint part of a von Neumann algebra,
ii) $F\mathfrak{A}$ is reversible and $\mathcal{R}(F\mathfrak{A}) \cap i\mathcal{R}(F\mathfrak{A}) = \{0\}$,
iii) $G\mathfrak{A}$ is totally non reversible.

C o r o l l a r y 2 . A JW-factor is either reversible or totally non reversible.

These results divide the study of JW-algebras into three groups, one of which is at once taken care of by the theory of von Neumann algebras. If $\mathfrak{A}$ is a JW-algebra we say two projections E and F in $\mathfrak{A}$ are <u>equivalent</u> if there exists a self-adjoint unitary operator S in $\mathfrak{A}$ such that $SES = F$. If there are at least three orthogonal equivalent non zero projections in $\mathfrak{A}$ then there is a great deal of freedom with respect to multiplication in $\mathfrak{A}$. As a result of this we obtain the key to the whole theory.

L e m m a 3 . Let $\mathfrak{A}$ be a JW-algebra such that there exists a family $\{E_\sigma\}_{\sigma \in J}$ of orthogonal, non zero, equivalent projections in $\mathfrak{A}$ with $\sum_{\sigma \in J} E_\sigma = I$, and card $J \geq 3$. For $\sigma, \rho \in J$ let $\mathfrak{S}_{\sigma\rho} = E_\sigma \mathfrak{A} E_\rho$. Then the following relations hold:

i)
$$\mathfrak{S}_{\sigma\tau}\mathfrak{S}_{\eta\rho} = \begin{cases} 0 \text{ if } \tau \neq \eta \\ \mathfrak{S}_{\sigma\rho} \quad \text{if } \tau = \eta \ , \ \sigma \neq \rho \\ \mathfrak{S}_{\sigma\varphi}\mathfrak{S}_{\varphi\sigma} \text{ for all } \varphi \text{ if } \sigma = \rho \neq \tau = \eta \end{cases}$$

ii) $\mathfrak{S}_{\sigma\rho} = \mathfrak{S}_{\sigma\rho}\mathfrak{S}_{\rho\sigma}\mathfrak{S}_{\sigma\rho}$ if $\sigma \neq \rho$

iii) If $\sigma \neq \rho$, let $\mathfrak{A}_\sigma$ be the uniformly closed real linear space generated by $\mathfrak{S}_{\sigma\rho}\mathfrak{S}_{\rho\sigma}$. Then $\mathfrak{A}_\sigma$ is a real self-adjoint algebra with identity E_σ, $\mathfrak{A}_\sigma$ is independent of ρ and $\mathfrak{A}_{\sigma\,SA} = \mathfrak{S}_{\sigma\sigma}$.

iv) $$E_\sigma \mathcal{R}(\mathfrak{A}) E_\rho = \begin{cases} \mathfrak{S}_{\sigma\rho} & \text{if } \sigma \neq \rho \\ \mathfrak{A}_\sigma & \text{if } \sigma = \rho \end{cases}$$

v) $\mathfrak{A}$ is reversible.

As a first consequence of this we have

L e m m a 4 . A totally non reversible JW-algebra is of type I_2.

We have thus reduced our problem to either reversible JW-algebras or those of type I_2. In order to classify the latter we follow Topping ((9)) and define a <u>spin system</u> to be a set $\mathcal{P}$ of self-adjoint unitaries $\neq \pm I$ such that $ST + TS = 0$ for S, T in $\mathcal{P}$ and $S \neq T$. If $\mathcal{P}$ is a spin system let $\mathcal{K}$ denote the weak closure of the real linear space spanned by $\mathcal{P}$. If a JW-factor can be written in the form $\mathbb{R}I \oplus \mathcal{K}$ with $\mathcal{K}$ as above, it is said to be a <u>spin factor</u>.

T h e o r e m 5 . Let $\mathfrak{A}$ be a JW-factor. Then the following are equivalent:

i) $\mathfrak{A}$ is of type I_2,

ii) $\mathfrak{A}$ is a spin factor.

If dim $\mathfrak{A}$ as a vector space over $\mathbb{R}$ is greater than 6 then the above conditions are equivalent to

iii) $\mathfrak{A}$ is totally non reversible.

It is clear from the above that from some points of view the interesting JW-algebras are the reversible ones $\mathfrak{A}$ such that $\mathcal{R}(\mathfrak{A}) \cap i\mathcal{R}(\mathfrak{A}) = \{0\}$.

In order to prove that every irreducible JW-algebra is of type I the following lemma is essential.

L e m m a 6 . Let $\mathcal{R}$ be a uniformly closed real self-adjoint operator algebra with identity acting on a Hilbert space. If $\mathcal{R} \cap i\mathcal{R} = \{0\}$ then for all $A, B \in \mathcal{R}$, $\| A + iB \| \geq \max\{ \|A\|, \|B\| \}$. Moreover, $\mathcal{R} + i\mathcal{R}$ is a C^*-algebra.

From this and the Kaplansky density theorem we have

L e m m a 7 . If $\mathfrak{A}$ is a reversible JW-algebra then the von Neumann algebra generated by $\mathfrak{A}$ equals $\mathcal{R}(\mathfrak{A})^- + i\,\mathcal{R}(\mathfrak{A})^-$.

It is now a easy matter to prove

T h e o r e m 8 . Every irreducible JW-algebra is of type I .

From now on we consider JW-factors of type I_n with $n \geq 3$ and make use of Lemma 3 applied to orthogonal minimal projections. Such projections are all equivalent ((9)) . The first main result towards the complete characterization of such JW-factors is the following modification of a result of Kaplansky ((5)) .

L e m m a 9 . Let $\mathcal{R}$ be a real self-adjoint algebra of operators on a Hilbert space such that every self-adjoint operator in $\mathcal{R}$ is a scalar multiple of the identity I . Then $\mathcal{R}$ is characterized as follows:

i) $\mathcal{R} = \mathbb{R}\, I$.

ii) $\mathcal{R} = \mathbb{C}\, I$.

iii) There exists a minimal projection P' in the commutant $\mathcal{R}'$ of $\mathcal{R}$ with central carrier I such that $P'\mathcal{R} = \mathbb{Q}\, P'$.

iv) There exist two non zero projections P and Q with $P + Q = I$ such that $\mathcal{R} = \{ \lambda P + \bar{\lambda} Q : \lambda \in \mathbb{C} \}$.

In the notation of Lemma 3 we then obtain

L e m m a 10 . Let $\mathfrak{A}$ be a JW-factor of type I_n, $n \geqslant 3$, and with orthogonal minimal projections $\{E_\sigma\}_{\sigma \in J}$. Then the $\mathfrak{A}_\sigma$ are all spatially isomorphic, and each $\mathfrak{A}_\sigma$ is one of the following algebras:

i) $\mathfrak{A}_\sigma = \mathbb{R} E_\sigma$.

ii) $\mathfrak{A}_\sigma = \mathbb{C} E_\sigma$.

iii) There exists a projection $P' \in \mathfrak{A}'$ with central carrier I such that if $\mathfrak{A}$ is replaced by $P'\mathfrak{A}$ then $\mathfrak{A}_\sigma = \mathbb{Q} E_\sigma$.

iv) There exist two non zero orthogonal projections P_σ and Q_σ with sum E_σ such that $\mathfrak{A}_\sigma = \{ \lambda P_\sigma + \bar{\lambda} Q_\sigma : \lambda \in \mathbb{C} \}$.

It turns out that in case iv) $\sum P_\sigma$ and $\sum Q_\sigma$ belong to the center of $\mathfrak{A}'$. Hence, if $\mathfrak{A}$ is irreducible case iv) cannot occur, and we have

T h e o r e m 11 . Let $\mathfrak{A}$ be an irreducible JW-algebra of type I_n, $n \geqslant 3$. Let $\{E_\sigma\}_{\sigma \in J}$ be an orthogonal family of non zero abelian projections in $\mathfrak{A}$ with $\sum_{\sigma \in J} E_\sigma = I$. Let $\mathfrak{S}_{\sigma\rho} = E_\sigma \mathfrak{A} E_\rho$ for $\sigma \neq \rho$. Then every operator in $\mathfrak{S}_{\sigma\rho}$ is a scalar multiple of a partial isometry of E_ρ onto E_σ. If $W_{\sigma\rho}$ is a partial isometry in $\mathfrak{S}_{\sigma\rho}$ then one of three cases occur:

i) $\mathfrak{S}_{\sigma\rho} = \mathbb{R} W_{\sigma\rho}$ for all $\sigma \neq \rho$, and $\dim E_\sigma = 1$.

ii) $\mathfrak{S}_{\sigma\rho} = \mathbb{C} W_{\sigma\rho}$ for all $\sigma \neq \rho$, and $\dim E_\sigma = 1$.

iii) $\mathfrak{S}_{\sigma\rho} = \mathbb{Q} W_{\sigma\rho}$ for all $\sigma \neq \rho$, and $\dim E_\sigma = 2$.

We want to investigate case iv) of Lemma 10 further. We do this by classifying all JW-factors of type I_n, $n \geqslant 3$. This requires more analysis than the preceding. It is necessary to consider cyclic projections. This is done by classifying them in terms of abelian projections.

L e m m a 12 . Let $\mathfrak{A}$ be a JW-factor acting on a Hilbert space $\mathcal{H}$. Let E be a projection in $\mathfrak{A}$ and x a unit vector in E.

Assume $[(\mathfrak{A})x] = I$, where $(\mathfrak{A})$ denotes the C^*-algebra generated by $\mathfrak{A}$. Then E is abelian if and only if $E \leqslant [x] + I - [\mathfrak{A}x]$. Moreover, if $\mathfrak{A}$ is reversible then the above inequality is equality.

It is well known that a von Neumann algebra, which is a factor of type I, has a faithful representation as all bounded operators on a Hilbert space. An analogous result holds for JW-algebras.

Lemma 13. Let $\mathfrak{A}$ be a JW-factor of type I_n, $n \geqslant 3$. Then there exists a representation of $(\mathfrak{A})$ which, when restricted to $\mathfrak{A}$, is a faithful normal representation as an irreducible JW-algebra of type I_n.

The proof consists of finding an irreducible representation φ of $(\mathfrak{A})$ due to a pure state extension of a pure vector state of $\mathfrak{A}$, and show φ is injective on $\mathfrak{A}$, which is easy, and then show φ is weakly continuous on the unit ball of $\mathfrak{A}$. For this we make use of Lemma 12 together with a result of Kadison ((3)). It is now an easy application of ((6)), in which all C^*-homomorphisms of C^*-algebras where shown to be sums of homomorphisms and anti-homomorphisms, to classify all JW-factors of type I_n, $n \geqslant 3$.

Theorem 14. Let $\mathfrak{A}$ be a JW-factor of type I_n, $n \geqslant 3$, acting on a Hilbert space $\mathcal{H}$. Let $\{E_\sigma\}_{\sigma \in J}$ be an orthogonal family of non zero abelian (i.e. minimal) projections in $\mathfrak{A}$ with $\sum_{\sigma \in J} E_\sigma = I$. For $\sigma \neq \rho$ let $\mathfrak{S}_{\sigma\rho} = E_\sigma \mathfrak{A} E_\rho$. Let $W_{\sigma\rho}$ be a partial isometry in $\mathfrak{S}_{\sigma\rho}$. Then one of the following four cases occurs:

i) $\mathfrak{S}_{\sigma\rho} = \mathbb{R}\, W_{\sigma\rho}$ for all $\sigma \neq \rho$.

ii) $\mathfrak{S}_{\sigma\rho} = \mathbb{C}\, W_{\sigma\rho}$ for all $\sigma \neq \rho$.

iii) There exists a projection $P' \in \mathfrak{A}'$ with central carrier I such that if $\mathfrak{A}$ is replaced by $P'\mathfrak{A}$ then $\mathfrak{S}_{\sigma\rho} = \mathbb{Q}\, W_{\sigma\rho}$ for all $\sigma \neq \rho$.

iv) There exist two non zero projections P_σ and Q_σ with $P_\sigma + Q_\sigma = E_\sigma$ such that $\mathfrak{S}_{\sigma\rho} = \{(\lambda P_\sigma + \bar{\lambda} Q_\sigma)W_{\sigma\rho} : \lambda \in \mathbb{C}\}$. In this case there exist a Hilbert space $\mathcal{K}$, a normal $*$-isomorphism ψ_1, and a normal $*$-anti-isomorphism ψ_2 of $\mathcal{B}(\mathcal{K})$ into $\mathcal{B}(\mathcal{H})$ such that $\psi_1(I)\psi_2(I) = 0$, and such that $\mathfrak{A}$ is the image of the C^*-isomorphism $\psi_1 + \psi_2$ of $\mathcal{B}(\mathcal{K})_{SA}$ into $\mathcal{B}(\mathcal{H})_{SA}$.

We complete this note by a discussion of the global case. If E is a projection in a JW-algebra $\mathfrak{A}$ its central carrier is the smallest central projection in $\mathfrak{A}$ greater than or equal to E. $\mathfrak{A}$ is of type I if and only if there exists an abelian projection E in $\mathfrak{A}$ with central carrier I (E is abelian means $E\mathfrak{A}E$ is an abelian JW-algebra). We have shown (Lemma 4) that every totally non reversible JW-algebra is of type I. From the developed techniques we can show an analogue for JC-algebras of Kadison's result ((4)), that every irreducible C^*-algebra is algebraically irreducible. Using this and techniques on CCR-algebras as developed by Kaplansky and Dixmier we can show the following relationship between the types of JW-algebras and their enveloping von Neumann algebras.

T h e o r e m 15. Let $\mathfrak{A}$ be a JW-algebra. If the double commutant $\mathfrak{A}''$ is a von Neumann algebra of type I then $\mathfrak{A}$ is of type I. Conversely, if $\mathfrak{A}$ is reversible and of type I then $\mathfrak{A}''$ is of type I.

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