GOOD STRATEGIES IN INFINITE GAMES

By

Jens Erik Fenstad
I. GENERAL NOTIONS FROM THE THEORY OF GAMES

The purpose of this report is to suggest a general approach to the study of "good" strategies in arbitrary two-person zero-sum games based upon fixed point theorems. Our contribution is to the "existence" part of the theory. We do not treat matters concerning calculation or characterization of optimal strategies, but are interested in an as general existence theorem for "good" strategies as possible, even if this theorem be highly non-effective. While the theorem we actually state is known, we do not believe that this result gives the limit of what is obtainable by this approach.

In this paper we shall always assume that a game is given in normal form, i.e. given as a triple

\[ G = \langle A, B, k \rangle, \]

where \( A \) and \( B \) are non-empty sets called sets of pure strategies and \( k \) is a real-valued function defined on the set \( A \times B \) and called the pay-off function.

In brief outline, the game is played as follows. There are two participants, \( P_A \) and \( P_B \). Each selects a strategy, i.e. \( P_A \) selects a point \( a \in A \) and \( P_B \) selects a point \( b \in B \). The outcome of the game is evaluated by calculating \( k(a, b) \). "Pay-off" then consists in \( P_A \) receiving the amount \( k(a, b) \) from \( P_B \). (If \( k(a, b) < 0 \), \( P_A \) receives a negative amount which means that \( P_B \) gets the amount \( -k(a, b) \) from \( P_A \).)

Somewhat imprecisely we may say that the purpose of the game considered from the viewpoint of \( P_A \) is to select a strategy \( a_0 \in A \) so as to maximize his expected pay-off. Conversely \( P_B \) wants to minimize the return to \( P_A \) by selecting a "good" \( b_0 \in B \).

This preliminary description can be made precise in the following way.
For each $a \in A$ define

$$v_A(a) = \min_b k(a,b)$$

(For the moment we assume that all entities entering into our calculations exist.) $v_A(a)$ represents the security level for $P_A$ using strategy $a$, i.e., playing against an intelligent (or rational) opponent $v_A(a)$ is the maximum pay-off $P_A$ may expect. A "good" strategy for $P_A$ in the game $G$ is then to select an $a_0 \in A$ so as to maximize $v_A(a)$, i.e., try to obtain the amount

$$v_A = \max_a v_A(a) = \max_a \min_b k(a,b)$$

Similarly we define

$$v_B(b) = \max_a k(a,b)$$

Then $v_B(b)$ represents the maximum loss $P_B$ may expect by choosing the strategy $b \in B$. The "good" thing for $P_B$ is then to choose a $b_0 \in B$ so as to minimize the maximum loss, i.e., try to hold $P_A$ down to the amount

$$v_B = \min_b v_B(b) = \min_b \max_a k(a,b)$$

It is easily seen that $v_A \leq v_B$. In an arbitrary game $G$, even if $v_A$ and $v_B$ exist, there may not exist strategies $a_0$ and $b_0$ such that $v_A = v_A(a_0)$ and $v_B = v_B(b_0)$, neither do we know whether $v_A < v_B$ or $v_A = v_B$. An investigation of these problems form the substance of this report.
It is necessary even for the simplest types of games to extend the concept of strategy. A "mixed" or "randomized" strategy is a probability distribution over the set of pure strategies. A randomized extension of a game $G$ is obtained in the following way.

Let $\Sigma_A$ and $\Sigma_B$ be $\sigma$-algebras of subsets in $A$ and $B$, respectively. Then the randomized extension of $G = \langle A, B, k \rangle$ with respect to the $\sigma$-algebras $\Sigma_A$ and $\Sigma_B$ is the triple

$$\Gamma = \langle \mathcal{A}, \mathcal{B}, k \rangle,$$

where $\mathcal{A}$ and $\mathcal{B}$ are the sets of all probability distributions with respect to the $\sigma$-algebras $\Sigma_A$ and $\Sigma_B$ and $K(\mu, \lambda)$, where $\mu \in \mathcal{A}$ and $\lambda \in \mathcal{B}$, is the extended pay-off function defined by

$$K(\mu, \lambda) = \iint k(a, b) \mu(\lambda),$$

where the assumption is made that the integral exists (and can be evaluated either as a double integral or as an iterated integral).

An equilibrium pair $\langle \mu_o, \lambda_o \rangle$ in the game $\Gamma$ is a pair of mixed strategies $\mu_o \in \mathcal{A}$ and $\lambda_o \in \mathcal{B}$ such that $\mu_o$ is good against $\lambda_o$, i.e. $K(\mu_o, \lambda_o) \geq K(\mu, \lambda_o)$ for all $\mu \in \mathcal{A}$, and $\lambda_o$ is good against $\mu_o$, i.e. $K(\mu_o, \lambda_o) \leq K(\mu_o, \lambda)$ for all $\lambda \in \mathcal{B}$. Then $\mu_o$ and $\lambda_o$ satisfy the equations

$$\max_{\mu} K(\mu, \lambda_o) = K(\mu_o, \lambda_o) = \min_{\lambda} K(\mu_o, \lambda).$$

The existence of an equilibrium pair $\langle \mu_o, \lambda_o \rangle$ implies the validity of the equation $v_A = v_B$ (where the notions are suitably extended to $\Gamma$). In fact, one has that $v_A = v_A(\mu_o) = v_B(\lambda_o) = v_B$ which follows from the inequalities.
and the general inequality $v_A \leq v_B$. We note that if $K(\mu, \lambda)$ is defined for all $\mu \in \mathcal{A}$ and $\lambda \in \mathcal{B}$, then the existence of an equilibrium pair implies that all entities involved in our so far formal calculations are well defined.

Thus if $<\mu_0, \lambda_0>$ is an equilibrium pair in the extended game $\Gamma$, then $P_A$ by selecting the strategy $\mu_0$ can guarantee himself at least a pay-off equal to $v_A$, whereas $P_B$ by selecting $\lambda_0$ can hold $P_A$ down to $v_B$, and the game is "fair" or in equilibrium as $v_A = v_B$.

This concludes our introduction to some general notions from game theory. We refer the reader to basic treatises such as Karlin ((2)), Luce and Raiffa ((4)) and von Neumann and Morgenstern ((6)) for further informations.

Our problem is to determine whether any game $G$ has some randomized extension $\Gamma$ possessing an equilibrium point. In the next section we shall treat this problem by extending the fixed point technique given by Nash ((5)) for the case that both $A$ and $B$ are finite.

II. THE FIXED POINT THEOREM

We now consider some fixed extension $\Gamma = <\mathcal{A}, \mathcal{B}, K>$ of the game $G = <A, B, k>$ and assume that the pay-off function $k$ is bounded, i.e. there shall exist an $M_G$ such that

$$|k(a, b)| < M_G,$$

for all $a \in A$ and $b \in B$.

The restriction is further imposed that $k(a,b)$ is measurable with respect to the $\sigma$-algebra $\Sigma = \Sigma_A \times \Sigma_B$ on $A \times B$. Then $k(a,b)$ is integrable with respect to the product measure $\mu_1 \times \mu_2$ for all
\( \mu_1 \in \mathcal{A} \) and \( \mu_2 \in \mathcal{B} \), and we have unlimited access to the Fubini theorem.

In order to identify the pure strategies \( a \) and \( b \) with mixed ones, \( \mu_a \) and \( \mu_b \), we shall require that each one point set in \( A \) and \( B \) belongs to the \( \mathcal{G} \)-algebras \( \Sigma_A \) and \( \Sigma_B \). We may then define \( \mu_a \) and \( \mu_b \) by setting \( \mu_a(x) = 1 \) if \( a \in x \) and 0 otherwise, where \( x \in \Sigma_A \), and \( \mu_b(y) = 1 \) if \( b \in y \) and 0 otherwise, for \( y \in \Sigma_B \). With these assumptions we observe that the real-valued function

\[
\mu \to K(\mu_a, \mu_b),
\]

where \( \mu_2 \) is considered as a parameter, is integrable on \( A \) which follows from the equalities

\[
K(\mu_a, \mu_2) = \iint k(a', b)d\mu_a d\mu_2 = \int k(a, b)d\mu_2,
\]

applying the Fubini theorem. In the sequel we denote \( \mu_a \) and \( \mu_b \) simply by \( a \) and \( b \), respectively.

Next define for each \( \mu = \mu_1 \times \mu_2 \in \mathcal{A} \times \mathcal{B} \)

\[
c_\mu(a) = \left[ K(\mu_1, \mu_2) - K(\mu_1, \mu_2) \right] \vee 0,
\]

\[
d_\mu(b) = \left[ K(\mu_1, \mu_2) - K(\mu_1, \mu_2) \right] \vee 0.
\]

Both \( c_\mu \) and \( d_\mu \) are integrable and it is seen that they jointly measure how far \( \langle \mu_1, \mu_2 \rangle \) is from being a good strategy pair against the pure strategies \( a \) and \( b \).

**Regularity assumption on** \( k: A \times B \rightarrow \mathbb{R} \): For all \( \varepsilon > 0 \) there shall exist a finite covering of \( A \times B \) of the form \( U_i \times V_i \), \( i = 1, \ldots, n \), such that \( |k(a, b) - k(a', b')| < \varepsilon \) if both \( \langle a, b \rangle \) and \( \langle a', b' \rangle \) belong to the same set \( U_i \times V_i \).
This regularity condition is in particular satisfied if both $A$ and $B$ are compact spaces and $k$ continuous.

Using this regularity condition we shall construct two measures $\lambda_A$ and $\lambda_B$ suitable for measuring the average value of $c_\mu$ and $d_\mu$ over sets in $\Sigma_A$ and $\Sigma_B$, respectively. (The regularity condition is only sufficient. Below we shall make some remarks on other conditions which could equally well serve and which apply in some cases where the above condition fails.)

For each $\varepsilon > 0$ let $U_1^\varepsilon, \ldots, U_m^\varepsilon$ be a refinement of the covering $U_1, \ldots, U_n$ of $A$ such that if $U_j^\varepsilon \cap U_i \neq \emptyset$, then $U_j^\varepsilon \subseteq U_i$, and let $a_j^\varepsilon$ be some point in $U_j^\varepsilon$. Let $a \in U_j^\varepsilon$, for any $b \in B$ there is some $i$ such that $\langle a, b \rangle \in U_i \times V_i$. Hence $U_j^\varepsilon \cap U_i \neq \emptyset$. As then $U_j^\varepsilon \subseteq U_i$, this implies that $a_j^\varepsilon \in U_i$, therefore $\langle a_j^\varepsilon, b \rangle \in U_i \times V_i$. From the regularity condition we then may conclude that $|k(a, b) - k(a_j^\varepsilon, b)| < \varepsilon$.

And if $a \in U_j^\varepsilon$, this inequality holds for all $b \in B$.

Next observe that $|c_\mu(a) - c_\mu(a')| \leq |k(a, \mu_2) - k(a', \mu_2)|$ which entails that

$$|c_\mu(a) - c_\mu(a')| \leq \int |k(a, b) - k(a', b)| \, d\mu_2 \leq \max_b |k(a, b) - k(a', b)|.$$  

Thus for each $n \geq 1$ there exist points $a_1^n, \ldots, a_m^n$ with the property that given any $a \in A$ there is some $a_i^n$ such that $|c_\mu(a) - c_\mu(a_i^n)| < \frac{1}{n}$ for all $\mu = \mu_1 \times \mu_2$.

We may now define $\lambda_A$ for each $X \in \Sigma_A$ by

$$\lambda_A(X) = \sum_n \sum_{1 \leq i \leq m} \frac{(n; i)}{2^n m_n},$$

where $(n; i) = 1$ if $a_i^n \in X$ and 0 otherwise. In the same way we define a measure $\lambda_B$ on $<B, \Sigma_B>$. 
We may establish the crucial property of the measure \( \mu_A \) must be identically 0 on \( A \) if \( \int c_\mu(a) d\lambda_A = 0 \). Suppose that \( c_\mu(a') > 0 \) for some \( a' \in A \), then there would exist some \( n \) and \( i \) such that \( c_\mu(a_i^n) > 0 \). By construction \( c_\mu \) is non-negative and \( \lambda_A(\{a_i^n\}) \geq 1/2^n m_0 \). Hence
\[
\int c_\mu(a) d\lambda_A \geq \int_{\{a_i^n\}} c_\mu(a) d\lambda_A \geq c_\mu(a_i^n)/2^n m_0 > 0.
\]

Similarly we may conclude that \( d_\mu \) is identically 0 on \( B \) if \( \int d_\mu(b) d\lambda_B = 0 \).

The extended Nash transformation may now be defined for \( X \in \Sigma_A \) and \( Y \in \Sigma_B \) by

\[
\mu_1'(X) = \frac{\mu_1(X) + \int_X c_\mu(a) d\lambda_A}{1 + \int_A c_\mu(a) d\lambda_A}
\]
and

\[
\mu_2'(Y) = \frac{\mu_2(Y) + \int_Y d_\mu(b) d\lambda_B}{1 + \int_B d_\mu(b) d\lambda_B}.
\]

It is immediate that \( \mu_1' \in \mathcal{U} \) and \( \mu_2' \in \mathcal{D} \). The transformation \( T : A \times B \to A \times B \) is obtained by setting

\[
T(\mu_1 \times \mu_2) = \mu_1' \times \mu_2'.
\]

**Proposition.** Let \( G = \langle A, B, k \rangle \) be any game such that \( k \) satisfies the above stated regularity assumption and let \( \Gamma \) be any mixed extension. Then \( \langle \mu_1, \mu_2 \rangle \) is an equilibrium pair for \( \Gamma \) if and only if \( \mu_1 \times \mu_2 \) is a fixed point for \( T \).

**Proof:** I. Let \( \langle \mu_1, \mu_2 \rangle \) be an equilibrium point for \( \Gamma \).

This means that
\[
\max_{\lambda_1} K(\lambda_1, \mu_2) = K(\mu_1, \mu_2) = \min_{\lambda_2} K(\mu_1, \lambda_2) \quad .
\]

It follows at once from the definitions of \( c_\mu \) and \( d_\mu \) that \( c_\mu(a) = 0 \) for all \( a \in A \) and \( d_\mu(b) = 0 \) for all \( b \in B \). But then \( \mu' = \mu_1 \) and \( \mu'' = \mu_2 \), i.e. \( \mu_1 \times \mu_2 \) is a fixed point for \( T \).

II. To prove the converse we first observe that there are sets \( X \in \Sigma_A \) and \( Y \in \Sigma_B \) such that \( \mu_1(X) > 0 \), \( \mu_2(Y) > 0 \) and \( K(\mu_1, \mu_2) \geq K(a, \mu_2) \) for all \( a \in X \) and \( K(\mu_1, \mu_2) \leq K(\mu_1, b) \) for all \( b \in Y \). If this were not the case we would e.g. have \( K(\mu_1, \mu_2) < K(a, \mu_2) \) for almost all \( a \in A \) (with respect to \( \mu_1 \)), hence

\[
K(\mu_1, \mu_2) = \int K(\mu_1, \mu_2) d\mu_1 < \int K(a, \mu_2) d\mu_1 = K(\mu_1, \mu_2) \quad ,
\]

a contradiction.

Using now the fact that \( \mu_1 \times \mu_2 \) is a fixed point for \( T \) we have

\[
\mu_1(X) = \mu_1(X)/(1 + \int c_\mu(a) d\lambda_A)
\]
and

\[
\mu_2(Y) = \mu_2(Y)/(1 + \int d_\mu(b) d\lambda_B) \quad .
\]

But \( \mu_1(X) > 0 \) and \( \mu_2(Y) > 0 \), thus \( \int c_\mu(a) d\lambda_A = 0 \) and \( \int d_\mu(b) d\lambda_B = 0 \), from which we conclude that both \( c_\mu \) and \( d_\mu \) are identically \( 0 \). Hence from the definitions of \( c_\mu \) and \( d_\mu \) we obtain the inequalities, valid for all \( a \in A \) and \( b \in B \),

\[
K(\mu_1, \mu_2) \geq K(a, \mu_2)
\]
and

\[
K(\mu_1, \mu_2) \leq K(\mu_1, b) \quad .
\]
Integrating with respect to arbitrary measures \( \lambda_1 \in \mathcal{A} \) and \( \lambda_2 \in \mathcal{B} \) this gives

\[
K(\lambda_1, \mu_2) \leq K(\mu_1, \mu_2) \leq K(\mu_1, \lambda_2)
\]

i.e. \( \langle \mu_1, \mu_2 \rangle \) is an equilibrium point for \( \Gamma \).

This completes the proof.

To conclude this section we shall make some remarks on the pay-off function \( k \).

In the topological case there is another reasonable candidate for the averaging measure \( \lambda_A \), viz. a measure which is strictly positive on non-empty open sets. (The Lebesgue measure is one such example.) Such measures exist in every locally compact and separable space: If \( \{ a_i \} \) is a countable dense subset define \( \lambda_A(x) = \sum \frac{1}{2^i}, x \in \sum A \), where the sum is taken over those \( i \) such that \( a_i \in X \). As \( c_\mu \) is non-negative, the continuity of \( c_\mu \) implies that \( c_\mu \) is identically 0 on \( A \) if \( \int c_\mu(a) d\lambda_A = 0 \) (see Halmos ((3)), Ch. X).

However, it may be of interest to remark that not every compact space admits a measure \( \lambda \) (on the Borel sets) which is strictly positive on non-empty open sets. To prove this let \( A_0 \) be a non-denumerable discrete space and \( A \) the one-point compactification. For each \( a \in A_0 \), \( \{ a \} \) is open, hence we assume that \( x_a = \lambda(\{ a \}) > 0 \). ( \( x_\omega = \lambda(\{ \omega \}) \), \( \omega \in A - A_0 \), need not be positive.) If there is no countable subset \( K \) of \( A \) such that \( \sum_{a \in K} x_a = 1 \), then there is a least positive number \( \varepsilon_0 \in [0, 1] \) such that \( \sum_{a \in K} x_a \leq \varepsilon_0 \) for all countable \( K \subseteq A \). If \( \sum_{a \in K} x_a = \varepsilon_0 \) for some \( K \), we pick an \( a' \) in \( A_0 - K \), hence \( x_{a'} > 0 \), and adding \( x_{a'} \) to the sum the result will be \( > \varepsilon_0 \), a contradiction. Thus \( \sum_{a \in K} x_a < \varepsilon_0 \) for all \( K \).
But this is also impossible: Pick a sequence of positive reals \( \varepsilon_n \uparrow \varepsilon_0 \).

For each \( n \) let \( K_n \) be a countable subset of \( A \) such that \( \sum_{a \in K_n} x_a > \varepsilon_0 - \varepsilon_n \).

Then the sum over \( K = \bigcup_{n} K_n \) will be \( \geq \varepsilon_0 \). From this we may conclude that \( \lambda(\{a\}) \) can be strictly positive for at most a countable subset of \( A_0 \), i.e. no Borel measure exists on the compact space \( A \) giving each non-empty open set positive measure.

One could also try to approach this problem by way of product measures as each compact space is embeddable in a product of intervals \([0,1]\). And, in fact, if \( A \) is sufficiently "thick" in the product space, i.e. if \( 0 \cap A \) contains an open set from the base for each open \( 0 \) in the product space, then a suitable \( \lambda_A \) exists. But this is a somewhat restrictive condition, and as the problem of relativizing measures is rather complicated, we leave the matter here and in the sequel stick to our non-topological regularity condition on the pay-off function \( k \).

III. TOPOLOGIES ON THE SPACES OF STRATEGIES

The set \( \mathcal{E} \) of all bounded \( \sigma \)-measures on \( \langle A \times B, \Sigma \rangle \) is a linear space. The subset \( \mathcal{C} \times \mathcal{B} \) is not convex, but it is easy to extend \( T \) to the set of all probability measures, \( \mathcal{C} \), on \( \langle A \times B, \Sigma \rangle \) and \( \mathcal{C} \) is a convex subset of \( \mathcal{B} \). The extension \( T : \mathcal{C} \to A \times B \) is obtained by setting for each \( \mu \in \mathcal{C} \)

\[
T(\mu) = \mu_1' \times \mu_2',
\]

where \( \mu_1 \) and \( \mu_2 \) are the projections on \( \mathcal{C} \) and \( \mathcal{B} \), respectively, i.e. \( \mu_1(x) = \mu(x \times B), x \in \Sigma_A \), and \( \mu_2(y) = \mu(A \times y), y \in \Sigma_B \), and then compose with the map \( \mu_1 \times \mu_2 \to \mu_1' \times \mu_2' \) as defined in the previous section.
Our task is now to set the stage for an application of the Schauder-
Tychonoff fixed point theorem, \((1)\), p. 456, by searching for some
topology on \(\mathcal{D}\) which (i) makes \(\mathcal{D}\) into a locally convex linear
space such that (ii) \(\mathcal{C}\) will be compact and (iii) \(T\) continuous.
Then \(T\) will have a fixed point, necessarily in the set \(\mathcal{A} \times \mathcal{B}\).

A most natural topology, taking into regard the definition of \(\mu_1^1\) and
\(\mu_2^2\), would be obtained by imbedding \(\mathcal{C}\) (and \(\mathcal{B}\)) into a product of
real lines by using the set of linear maps \(f: \mu \rightarrow \mu(X), X \in \Sigma\).
It is immediate that \(\Phi(\mu) = \langle f(\mu) \rangle\) imbeds \(\mathcal{C}\) as a convex sub-
set of a compact set in a locally convex linear topological space. Hence
the first thing would be to show that \(\mathcal{C}\) is closed.

However, \(\mathcal{C}\) is not in general closed as the following example shows:
Let \(A = \{a_1^1, a_2^2, \ldots, a_n^n, \ldots\}\) and \(B = \{b_1^1, b_2^2, \ldots\}\) and let \(\Sigma_A\) and
\(\Sigma_B\) be the sets of all subsets of \(A\) and \(B\), respectively. Then \(\mathcal{C}\)
essentially reduces to the set of all probability measures on \(A\). We shall
construct a \(\lambda\) in the closure of \(\mathcal{C}\) (with respect to the above imbedding
\(\Phi\)) which is not \(\sigma\)-additive.

It is easy to show that each \(\lambda \in \overline{\mathcal{C}}\) is a finitely additive measure
on \((A, \Sigma_A)\). Let \(F\) be some ultrafilter refining the Fréchet filter on
\(A\). Define \(\lambda\) by \(\lambda(X) = 1\) if \(X \in F\) and \(0\) otherwise. \(\lambda\) is
finitely additive but not \(\sigma\)-additive:

\[1 = \lambda(A) = \lambda(U \{a_k^k\}) > \Sigma \lambda(\{a_k^k\}) = 0.\]

Let \(X_1, \ldots, X_n \in \Sigma_A\) and assume that \(X_1, \ldots, X_k \in F\),
\(X_{k+1}, \ldots, X_n \notin F\). As \(F\) has the finite intersection property, there is
some point \(a_0 \in X_1 \cap \ldots \cap X_k \cap X_{k+1} \cap \ldots \cap X_n\). Let \(\mu \in \mathcal{C}\) be
defined by the condition \(\mu(\{a_0^0\}) = 1\). Then \(\mu\) approximates \(\lambda\)
at \(X_1, \ldots, X_n\) for any \(\varepsilon > 0\); thus \(\lambda \in \overline{\mathcal{C}}\).
It is therefore necessary to impose restrictions on our games G and \( \Gamma \) in order to obtain equilibrium points. Thus we now assume that \( A \) and \( B \) are compact spaces and \( k \) continuous. For \( \Sigma_A \) and \( \Sigma_B \) we take the Baire sets in \( A \) and \( B \), respectively. (In this case not every one-point set need belong to the \( \Sigma \)-algebras, but the reasoning above remains valid: e.g. to show that \( K(\mu_a, \mu_2) = \int k(a, b) d\mu_2 \) one extends \( \mu_a \) to its uniquely associated regular Borel measure and evaluate the integral with respect to this measure. And as we have an extension, we obtain the correct value.)

It is well known that the space of all finite signed Baire measures is the dual of \( C(A \times B) \), the set of continuous functions on \( A \times B \). Further \( \mathcal{C} \) is a convex, compact subset of this space in the topology induced by the maps \( \mu \rightarrow \mu(f) = \int f d\mu \), \( f \in C(A \times B) \). Hence in order to apply the Schauder-Tychonoff fixed point theorem, it now remains to verify that \( T : \mathcal{C} \rightarrow \mathcal{C} \) is continuous in the "vague" topology.

In order to carry out this verification it will prove convenient to modify the definition of \( T \) somewhat by setting for each \( f \in C(A) \) and \( g \in C(B) \)

\[
\mu_1'(f) = (\mu_1(f) + \int f(a) c_\mu(a) d\lambda_A)/(1 + \int c_\mu(a) d\lambda_A), \\
\mu_2'(g) = (\mu_2(g) + \int g(b) d\mu(b) d\lambda_B)/(1 + \int d\mu(b) d\lambda_B).
\]

Here \( \mu_1 ' \) and \( \mu_2 ' \) must initially be conceived to be positive linear functionals, but by the well known duality already referred to, they correspond to uniquely defined measures \( \mu_1 ' \) and \( \mu_2 ' \) such that \( \mu_1'(f) = \int f(a) d\mu_1 \) and \( \mu_2'(g) = \int g(b) d\mu_2 \). As \( \mu_1'(1) = 1 \) and \( \mu_2'(1) = 1 \) we have \( \mu_1 ' \in \mathcal{C} \) and \( \mu_2 ' \in \mathcal{B} \).

The proof of the proposition of section II goes through with small
modifications. In proving the sufficiency we now e.g. obtain a compact
Baire set $X$ such that $\mu_1(X) > 0$ and $c_\mu(a) = 0$ for all $a \in X$.
Let then $f_n$ be a decreasing sequence of continuous functions converging
pointwise to the characteristic function of $X$ (see Halmos ((3)), Ch. X).
Observe that $f_n \cdot c_\mu \to 0$, hence going to the limit we have
\[
\mu_1(X) = \mu_1(X)/(1 + \int c_\mu(a) d\lambda_a),
\]
and the proof is completed as above.

We may now state the following result on the existence of good strategies.

**Theorem.** Let $A$ and $B$ be compact spaces and $k$ a continuous
real-valued function on $A \times B$. Let $\Gamma = \langle \mathcal{A}, \mathcal{B}, K \rangle$ be the mixed
extension of the game $G = \langle A, B, k \rangle$ obtained by letting $\mathcal{A}$ and $\mathcal{B}$
be the sets of probability measures on the Baire set in $A$ and $B$, respectively. Then $\Gamma$ has an equilibrium pair.

It remains to show that $T$ is continuous, i.e. we must show that
$\mu \rightarrow \mu_1 \cdot \mu_2(f)$ is continuous for all $f \in C(A \times B)$. It suffices
to show that $\mu \rightarrow \mu_1(f)$, $f \in C(A)$, and $\mu \rightarrow \mu_2(g)$, $g \in C(B)$,
are continuous: If this is proved, then $\mu \rightarrow (\mu_1 \cdot \mu_2)(f \cdot g) =
\mu_1(f) \cdot \mu_2(g)$ is continuous, hence also all the maps
$\mu \rightarrow (\mu_1 \cdot \mu_2)(\sum_{i=1}^n f_ig_i)$. But the set of maps $\sum_{i=1}^n f_ig_i$, $f_i \in C(A)$
and $g_i \in C(B)$, is uniformly dense in $C(A \times B)$ by virtue of the Stone-
Weierstrass theorem. Thus the continuity of $\mu \rightarrow \mu_1 \cdot \mu_2(f)$ for
an arbitrary $f \in C(A \times B)$ follows by the inequality:
\[
|\mu_1 \cdot \mu_2(f) - \lambda_1 \cdot \lambda_2(f)| \leq 2 \cdot \|
\sum_{i=1}^n f_ig_i \|_1 + |\mu_1 \cdot \lambda_2(\sum_{i=1}^n f_ig_i) - \lambda_1 \cdot \lambda_2(\sum_{i=1}^n f_ig_i)|.
\]
The continuity of \( \mu \rightarrow \mu_1(f) \) and \( \mu \rightarrow \mu_2(g) \) is proved in exactly the same way, hence we treat only the first map. From the definition of \( \mu_1(f) \) it follows that we must verify that the maps \( \mu \rightarrow \mu_1(f) \), \( \mu \rightarrow \int f(a)c\mu(a)d\lambda_\alpha \) and \( \mu \rightarrow \int c\mu(a)d\lambda_\alpha \) are continuous. This readily reduced to show that the map \( \mu \rightarrow \mu_1 \times \mu_2 \) is continuous, which is straight forward, and that the family of maps \( \mu \rightarrow c\mu(a) \), \( a \in A \), is equicontinuous.

To verify this last assertion we need the compactness of \( A \times B \) and the continuity of \( k : \) We first note that for all \( \varepsilon > 0 \) there exists a finite covering \( U_1, \ldots, U_m \) of \( A \) and points \( a_i \in U_1 \) such that \( |k(a,b) - k(a_1,b)| < \varepsilon \) for all \( b \in B \), provided \( a \in U_1 \). We also note that the map \( \mu \rightarrow K(a,\mu_2) = \int k(a,b)d\mu_2 \) is continuous, \( a \) is here a fixed parameter. Making then use of the inequality

\[
|K(a,\mu_2) - K(a,\lambda_2)| \leq 2\max_b |k(a,b) - k(a_1,b)| + |K(a_1,\mu_2) - K(a_1,\lambda_2)|
\]

we may conclude that the family of maps \( \mu \rightarrow K(a,\mu_2) \), \( a \in A \), is equicontinuous, because for each \( \varepsilon > 0 \) there is a finite number of points \( a_1 \) for which we need to have the continuity of the map \( \mu \rightarrow K(a_1,\mu_2) \). By considering the inequality

\[
|c\mu(a) - c\lambda(a)| \leq |K(a,\mu_2) - K(a,\lambda_2)| + |K(\mu_1,\mu_2) - K(\mu_1,\lambda_2)|
\]

we easily obtain that the maps \( \mu \rightarrow c\mu(a) \), \( a \in A \), are equicontinuous. This completes the proof.
References


