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INVOLUTIVE ALGEBRAS OVER $\mathbb{C}$.

Part II.

By

Johan F. Aarnes

§1. INTRODUCTION

In this note we are concerned with pre-unitary or unitary $*$-algebras over the complex field. The study of these algebras was initiated by E.M. Alfsen in the preceding note ((1)). For definition and basic properties of pre-unitary algebras, we refer to this paper. Our goal in this note is to prove that a pre-unitary $*$-algebra $\mathfrak{A}$ can be imbedded in a C^*-algebra $\widehat{\mathfrak{A}}$, the imbedding being one to one onto a dense subset of $\widehat{\mathfrak{A}}$, and preserving all structure of $\mathfrak{A}$. In fact, $\widehat{\mathfrak{A}}$ will be the solution of a universal problem for $\mathfrak{A}$, in the sense that $\mathfrak{A}$ and $\widehat{\mathfrak{A}}$ will have the same (essentially, by canonical extensions) states, pure states, representations and topologically irreducible representations. $\widehat{\mathfrak{A}}$ will be called the enveloping C^*-algebra of $\mathfrak{A}$.

If $\mathfrak{A}$ is an involutive Banach-algebra with approximate identity, it is known that it has an enveloping C^*-algebra with the properties stated above. For an exposition of this, we refer to the book of Dixmier ((2)). However, involutive Banach-algebras are pre-unitary, so our results are strictly more general. Nevertheless, most of the methods in Dixmier's presentation can be carried over, due to the essential fact proved in ((1)), that a $*$-algebra $\mathfrak{A}$ is pre-unitary if and only if the set of states on $\mathfrak{A}$ is w^*-compact.

In the commutative case, the desired theorem is easily attainable by the function representation of a pre-unitary $*$-algebra as a dense subalgebra of $\mathcal{C}_0(M_R)$. In the general, non-commutative situation the adequate substitute for the multiplicative, real functionals are the topologically irreducible representations, so we have to establish a correspondance between irreducible representations and pure states, or more generally, between the states on $\mathfrak{A}$ and the representations of $\mathfrak{A}$.

§2. POSITIVE LINEAR FUNCTIONALS AND REPRESENTATIONS

In the first proposition we gather some information, needed for later reference. As in ((1)) , $\mathcal{P}^*$ will denote the set of positive, extendable linear functionals on a pre-unitary $*$-algebra $\mathfrak{A}$. For $p \in \mathcal{P}^*$, we put:

$$C(p) = \sup_{x \in \mathfrak{A}} \frac{|p(x)|^2}{p(x^*x)}$$

Correspondingly, for $x \in \mathfrak{A}$, we put:

$$B(x) = \sup_{\substack{C(p) \leq 1 \\ p \in \mathcal{P}^*}} p(x^*x)^{\frac{1}{2}}$$

For each $x \in \mathfrak{A}$, $B(x)$ is finite (((1)) , prop. 3).

2.1 Proposition

Let $\mathfrak{A}$ be a pre-unitary $*$-algebra, and suppose $p \in \mathcal{P}^*$. Then we shall have:

(i)

$$p(x^*) = \overline{p(x)} \quad ; \quad x \in \mathfrak{A}$$

$$|p(x)|^2 \leq C(p) \cdot p(x^*x) \quad ; \quad x \in \mathfrak{A}$$

$$|p(x)| \leq C(p) \cdot B(x) \quad ; \quad x \in \mathfrak{A}$$

(ii) The linear functional p_y defined by

$$p_y(x) = p(y^*xy) \quad ; \quad x,y \in \mathfrak{A}$$

is in $\mathcal{P}^*$, and $C(p_y) \leq p(y^*y)$.

(iii) $$C(p) \geq \sup_{\substack{B(x) \leq 1 \\ x \in \mathfrak{A}}} p(x^* x)$$

(iv) If $\{x_i\}_{i \in I}$ is a net in $\mathfrak{A}$, indexed by a directed set I ; such that $B(x_i) \leq 1$ and $p(x_i) \longrightarrow C(p)$, $i \in I$; then $p(x_i^* x_i) \longrightarrow C(p)$.

(v) p has a unique extension to a positive linear functional $\tilde{p}$ on $\tilde{\mathfrak{A}}$ ($= \mathfrak{A} \oplus \mathbb{C}$) such that $\tilde{p}(e) = C(p)$. $\tilde{p}$ is called the canonical extension of p , and every other positive linear functional on $\tilde{\mathfrak{A}}$ extending p majorizes $\tilde{p}$.

(vi) With the assumption of (iv) , we have $x_i \longrightarrow e$ in the N_p - topology; that is:

$$\tilde{p}\left[(x_i^* - e)(x_i - e)\right] \longrightarrow 0 .$$

Hence $\mathfrak{A}$ is N_p - dense in $\tilde{\mathfrak{A}}$ in this case.

P r o o f : (i) is clear from ((1)) . To prove (ii) , let $y \in \mathfrak{A}$. p_y is positive, for if $x \in \mathfrak{A}$:

$$p_y(x^* x) = p(y^* x^* xy) = p((xy)^*(xy)) \geq 0$$

Next:

$$\frac{|p_y(x)|^2}{p_y(x^* x)} = \frac{|p(y^* xy)|^2}{p(y^* x^* xy)} \leq \frac{p(y^* y)p((xy)^*(xy))}{p((xy)^*(xy))}$$

$$\Longrightarrow \qquad C(p_y) \leq p(y^* y)$$

and $p_y \in \mathcal{P}^*$.

(iii) follows from formula (2.3) in ((1)) , and the definition of $B(x)$.

Now, suppose $\{x_i\}_{i\in I}$ is a net in $\mathfrak{A}$ such that $B(x_i) \leq 1$, $i\in I$, and $p(x_i) \to C(p)$. Then by

$$|p(x_i)|^2 \leq C(p) \cdot p(x_i^* x_i) \leq C(p)^2 \cdot B(x_i)^2$$
$$\leq C(p)^2$$

(iv) follows.

(v) is known.

To prove (vi) , consider:

$$\tilde{p}\left[(e - x_i)^*(e - x_i)\right] = \tilde{p}(e) - p(x_i^*) - p(x_i) + p(x_i^* x_i)$$

$$= C(p) - \overline{p(x_i)} - p(x_i) + p(x_i^* x_i)$$

$$\to C(p) - C(p) - C(p) + C(p) = 0 ,$$

according to (i) , (v) and (iv) . This proves the proposition.

2.2 We now turn to investigate the connection between the elements of $\mathcal{P}^*$ and the representations of $\mathfrak{A}$. As usual, a representation π of a $*$-algebra is a $*$-homomorphism into the $*$-algebra of bounded linear operators $\mathcal{L}(H)$ on a Hilbert-space H . The elementary properties of such representations will be assumed known.

First, let π be a representation of a pre-unitary $*$-algebra $\mathfrak{A}$ in the Hilbert-space H , and let ξ be an element of H . π and ξ then define a positive, linear functional $p = p_{\pi,\xi}$ on $\mathfrak{A}$ by:

$$p(x) = (\pi(x)\xi \mid \xi) \qquad ; \quad x \in \mathfrak{A}$$

Next, consider:

$$|p(x)|^2 = |(\pi(x)\xi|\xi)|^2 \leq (\pi(x)\xi|\pi(x)\xi)\cdot(\xi|\xi)$$

$$= (\pi(x^*x)\xi|\xi)\cdot(\xi|\xi) = p(x^*x)\cdot(\xi|\xi)$$

so $C(p) \leq (\xi|\xi)$ and $p \in \mathcal{P}^*$.

Now, both p and π have canonical extensions to $\widetilde{\mathfrak{A}}$ which preserves their essential properties, by putting

$$\tilde{p}(e) = C(p) \quad \text{and} \quad \tilde{\pi}(e) = 1 \in \mathcal{L}(H) .$$

In general, these extensions will not agree, on the contrary $\tilde{\pi}$ will define a positive, linear functional p° on $\widetilde{\mathfrak{A}}$ by

$$p^\circ(e) = (\tilde{\pi}(e)\xi|\xi) = (\xi|\xi)$$

p° is an extension of p , and in general $p^\circ \geq \tilde{p}$, according to prop. 2.1 (v) . If, however, ξ is cyclic in H with respect to $\pi(\mathfrak{A})$ so $\pi(\mathfrak{A})\xi$ is dense in H , we will have:

$$(\xi|\xi) = \sup_{x\in\mathfrak{A}} \frac{|(\pi(x)\xi|\xi)|^2}{(\pi(x)\xi|\pi(x)\xi)} = \sup_{x\in\mathfrak{A}} \frac{|p(x)|^2}{p(x^*x)} = C(p) \quad .$$

Therefore, in this case the canonical extensions $\tilde{p}$ and $\tilde{\pi}$ will agree:

$$\tilde{p}(x) = (\tilde{\pi}(x)\xi|\xi) \qquad ; \; x \in \widetilde{\mathfrak{A}}$$

Next, we turn to the converse problem:

2.3 P r o p o s i t i o n

Let $\mathfrak{A}$ be a pre-unitary $*$-algebra, and let $p \in \mathcal{P}^*$ be an arbitrarily chosen, fixed element. Let

$$\mathcal{N} = \{ x \mid x \in \tilde{\mathfrak{A}} , \tilde{p}(x^{*}x) = 0 \}$$

Then $\mathcal{N}$ is a left ideal of $\tilde{\mathfrak{A}}$, and define an inner-product.

$$(\bar{x} \mid \bar{y}) = \tilde{p}(y^{*}x)$$

on $\tilde{\mathfrak{A}}/_{\mathcal{N}}$; $x,y \in \tilde{\mathfrak{A}}$; $\bar{x},\bar{y}$ their respective equivalence-classes in $\tilde{\mathfrak{A}}/_{\mathcal{N}}$. With this inner-product $\tilde{\mathfrak{A}}/_{\mathcal{N}}$ becomes a pre-Hilbert space which will be denoted by $H_p^{\prime}$. Let $\xi \in H_p^{\prime}$ be the canonical image of $e \in \tilde{\mathfrak{A}}$. The Hilbert-space obtained by completing $H_p^{\prime}$ will be denoted H_p . For every $x \in \tilde{\mathfrak{A}}$ define an operator $\pi^{\prime}(x)$ in $H_p^{\prime}$ by left multiplication by $\bar{x}$ in $\tilde{\mathfrak{A}}/_{\mathcal{N}}$. Then:

(i) Every $\pi^{\prime}(x)$ can be extended to a continuous linear operator $\pi(x)$ in H_p .

(ii) The map $x \rightsquigarrow \pi(x)$, $x \in \tilde{\mathfrak{A}}$; is a representation of $\tilde{\mathfrak{A}}$ in H_p .

(iii) ξ is cyclic with respect to $\pi(\tilde{\mathfrak{A}})$ in H_p .

(iv) $\tilde{p}(x) = (\pi(x)\xi \mid \xi)$ for every $x \in \tilde{\mathfrak{A}}$.

P r o o f . The statements in the head of the proposition are standard and readily verified. To prove (i) , let $\eta \in H_p^{\prime}$ be an arbitrary element. We have $\eta = \bar{y}$ for some $y \in \tilde{\mathfrak{A}}$, and hence

$$\begin{aligned} \| \pi^{\prime}(x)\eta \|^2 &= \tilde{p}((xy)^{*}(xy)) = \tilde{p}(y^{*}x^{*}xy) \\ &\leq B(x^{*}x)\tilde{p}(y^{*}y) = B(x^{*}x)\cdot \|\eta\|^2 , \end{aligned}$$

where prop. 2.1 (ii) is used. This proves that $\pi^{\prime}(x)$ is continuous on $H_p^{\prime}$, and hence extendable to all of H_p as required, so (i) is proved.

It follows immediately that π is an algebra-homomorphism, and moreover, for $x \in \widetilde{\mathfrak{A}}$, $\eta = \overline{y}$, $\gamma = \overline{z}$ in H_p' we have:

$$(\pi(x)\eta \mid \gamma) = \widetilde{p}(z^*(xy)) = \widetilde{p}((x^*z)^*y)$$

$$= (\eta \mid \pi(x^*)\gamma) = (\pi(x^*)^*\eta \mid \gamma)$$

so $\pi(x^*) = \pi(x)^*$, proving (ii).

(iii) is evident from the definition of ξ and H_p. Finally, let $x \in \widetilde{\mathfrak{A}}$:

$$(\pi(x)\xi \mid \xi) = (\overline{x}\,\overline{e} \mid \overline{e}) = \widetilde{p}(e^*xe) = \widetilde{p}(x) .$$

Q.e.d.

The representation π and the cyclic vector ξ are said to be associated with the given $p \in \mathcal{P}^*$.

§3. PURE STATES AND IRREDUCIBLE REPRESENTATIONS

We have already established a connection between elements of $\mathcal{P}^*$ and representations of $\mathfrak{A}$. In the case of C^*-algebras, it is further known that the pure states and irreducible representations correspond to each other. This can be seen to rely on the fact that a C^*-algebra has an approximate identity. In our situation with pre-unitary algebras we are left without any norm, so we have to be a bit roundabout when trying to obtain the connection mentioned above.

As a first step, we state the following proposition. The technique goes back to Grothendieck (1955) (see ((3)) for details).

If $p \in \mathcal{P}^*$, let $[p]$ be the linear space generated by the set $\{ q \mid 0 \leq q \leq p \}$

3.1 Proposition

Let $\mathfrak{A}$ be a unitary $*$-algebra, p a positive, linear functional on $\mathfrak{A}$, and π the representation associated with p, into H_p. Then there is a one to one linear, order-preserving map of $[p]$ onto $\pi(\mathfrak{A})'$ = the commutant of $\pi(\mathfrak{A})$ in $\mathcal{L}(H_p)$.

P r o o f. Let $q \in [p]$, and define a bilinear functional $(\cdot|\cdot)_q$ on $\pi(\mathfrak{A})\xi$ by

$$(\pi(x)\xi \,|\, \pi(y)\xi)_q = q(y^* x) \quad ; \quad x,y \in \mathfrak{A}$$

As $q \in [p]$, we must have

$$q = \sum_{i=1}^{4} \alpha_i q_i \quad ; \quad q_i \in [p] \quad ; \quad i = 1, \ldots, 4 .$$

Hence

$$\begin{aligned} |q_i(y^* x)| &\leq q_i(y^* y)^{\frac{1}{2}} q_i(x^* x)^{\frac{1}{2}} \\ &\leq K_i p(y^* y)^{\frac{1}{2}} \cdot p(x^* x)^{\frac{1}{2}} \end{aligned}$$

for some positive constant K_i (((1)) , 1.2). Thus:

$$|(\pi(x)\xi \,|\, \pi(y)\xi)_q| \leq \sum_{i=1}^{4} |\alpha_i| \cdot |q_i(y^* x)|$$

$$\leq \sum_{i=1}^{4} |\alpha_i| \cdot K_i \, p(y^* y)^{\frac{1}{2}} p(x^* x)^{\frac{1}{2}} = K \cdot N_p(y) \cdot N_p(x) \quad ,$$

so the bilinear form is continuous on the dense set $\pi(\mathfrak{A})\xi$ in H_p, and may therefore be extended to all of H_p in a continuous way. We can then find a bounded linear operator $s = s_q$ on H_p such that:

$$q(y^* x) = (s \cdot \pi(x)\xi \,|\, \pi(y)\xi)$$

Now, let $z \in \mathfrak{A}$, and $x,y \in \mathfrak{A}$ arbitrarily given. Then:

$$((s \cdot \pi(z))\pi(x)\xi \mid \pi(y)\xi) = q(y^*(zx))$$

$$= q((z^*y)^*x) = (s \cdot \pi(x)\xi \mid \pi(z^*y)\xi)$$

$$= ((\pi(z)s)\pi(x)\xi \mid \pi(y)\xi) \quad , \qquad \text{which}$$

proves that $s \in \pi(\mathfrak{A})'$, again applying the density of $\pi(\mathfrak{A})\xi$ in H_p. The linearity and order-properties of the map $q \rightsquigarrow s = s_q$ are immediate. It is injective, for suppose $q \neq 0$. Then there is an element $x \in \mathfrak{A}$ such that $q(x^*x) \neq 0$. This implies that

$$(s_q \pi(x)\xi \mid \pi(x)\xi) = q(x^*x) \neq 0$$

so $s_q \neq 0$. Finally, we prove that this map is onto $\pi(\mathfrak{A})'$. To do this, let $s \in \pi(\mathfrak{A})'$, and define

$$q(x) = (s\pi(x)\xi \mid \xi)$$

Then:

$$|q(x^*x)| = |(s\pi(x^*x)\xi \mid \xi)|$$

$$= |(s\pi(x)\xi \mid \pi(x)\xi)|$$

$$\leq \|s\| \cdot \|\pi(x)\xi\|^2 = \|s\| \cdot p(x^*x)$$

which proves that $q \in [p]$.

Q.e.d.

3.2 C o r o l l a r y

An element $p \in \mathcal{P}^*$ is a pure state on a unitary $*$-algebra if and only if the associated representation π is topologically irreducible, that is: $\pi(\mathfrak{A})' \cong \mathbb{C}$.

P r o o f : p is a pure state on a unitary $*$-algebra if and only if $[p] \cong \mathbb{C}$. This fact, together with prop. 3.1 proves the corollary.

We want to extend this connection to pre-unitary $*$-algebras, and will then need the following

3.3 L e m m a

Let $\mathfrak{A}$ be a $*$-algebra, $\tilde{\mathfrak{A}}$ the $*$-algebra obtained by adjoining a unit to $\mathfrak{A}$. Then a representation $\tilde{\pi}$ of $\tilde{\mathfrak{A}}$ is irreducible if and only if its restriction to $\mathfrak{A}$ is irreducible.

P r o o f : Suppose $\tilde{\pi} : \tilde{\mathfrak{A}} \longrightarrow \mathcal{L}(H)$, H some Hilbert-space. As $\tilde{\mathfrak{A}} \supset \mathfrak{A}$, we will obviously have

$$\tilde{\pi}(\mathfrak{A})' \supset \tilde{\pi}(\tilde{\mathfrak{A}})'$$

(the $'$ denotes the commutant-operation in $\mathcal{L}(H)$.) The lemma will be proved if we can establish the converse inclusion. Suppose, therefore, that $s \in \tilde{\pi}(\mathfrak{A})'$, and that $x \in \tilde{\mathfrak{A}}$ is arbitrarily chosen. $x = x_0 + \lambda e$, $x_0 \in \mathfrak{A}$. Hence:

$$s \cdot \tilde{\pi}(x) = s \cdot \tilde{\pi}(x_0) + \lambda s = \tilde{\pi}(x_0) \cdot s + \lambda s$$

$$= \tilde{\pi}(x) \cdot s \quad ,$$

which proves that $s \in \tilde{\pi}(\tilde{\mathfrak{A}})'$. Q.e.d.

3.4 C o r o l l a r y

An element $p \in \mathcal{P}^*$ is a pure state on a pre-unitary $*$-algebra if and only if the associated representation is topologically irreducible.

P r o o f : p is pure on $\mathfrak{A}$ if and only if the canonical extension $\tilde{p}$ is pure on $\tilde{\mathfrak{A}}$, which by cor. 3.2 is the case if and only if the associated representation to $\tilde{p}$, say $\tilde{\pi}$, is topologically irreducible on $\tilde{\mathfrak{A}}$, and this is by the lemma equivalent to that the restriction $\pi = \tilde{\pi}/\mathfrak{A}$ is topologically irreducible on $\mathfrak{A}$. As this restriction π is the representation of $\mathfrak{A}$ associated with p, the proof is complete.

§4. THE ENVELOPING C^{*}-ALGEBRA OF A PRE-UNITARY $*$-ALGEBRA

In this and the next section we will assume that if $\mathfrak{A}$ is a pre-unitary $*$-algebra, then $\mathcal{P}^{*}$ separates the points of $\mathfrak{A}$.

Now, let K denote the set of states on a pre-unitary $*$-algebra ,

that is:

$$K = \{ p \mid p \in \mathcal{P}^{*} , \ C(p) \leq 1 \}$$

Let $\partial_e K$ be the set of extreme points in K, i.e. the set of pure states on $\mathfrak{A}$. Furthermore, let R be the set of representations of $\mathfrak{A}$, and R' the set of topologically irreducible representations of $\mathfrak{A}$.

4.1 P r o p o s i t i o n

Let $\mathfrak{A}$ be a pre-unitary $*$-algebra, and let $x \in \mathfrak{A}$ be an arbitrary element. Then:

$$\sup_{\pi \in R} \| \pi(x) \| = \sup_{\pi \in R'} \| \pi(x) \|$$

$$= \sup_{p \in K} p(x^{*}x)^{\frac{1}{2}} = \sup_{p \in \partial_e K} p(x^{*}x)^{\frac{1}{2}}$$

P r o o f : Let a, b, c, d denote the four numbers considered successively above.

$d \leq b$. Let $p \in \partial_e K$ be given. By cor. 3.4 the representation associated to p is in R' , and

$$\begin{aligned} p(x^{*}x) &= (\pi(x^{*}x)\xi \mid \xi) = (\pi(x)\xi \mid \pi(x)\xi) \\ &\leq \|\pi(x)\|^{2} (\xi \mid \xi) = \|\pi(x)\|^{2}\, \tilde{p}(e) \\ &= \|\pi(x)\|^{2} \end{aligned}$$

so $$p(x^{*}x)^{\frac{1}{2}} \leq \sup_{\pi \in R'} \|\pi(x)\| \implies d \leq b .$$

$b \leq a$; evident.

$a \leq c$. Let $\pi \in R$ be given, and H the Hilbert-space in question. For $\eta \in H$:

$$\begin{aligned} \|\pi(x)\eta\|^{2} &= (\pi(x)\eta \mid \pi(x)\eta) = (\pi(x^{*}x)\eta \mid \eta) \\ &= p_{\pi,\eta}(x^{*}x) \leq C(p_{\pi,\eta}) \cdot B(x)^{2} \\ &\leq (\eta \mid \eta) \cdot B(x)^{2} \end{aligned}$$

by 2.2. Hence $\|\pi(x)\| \leq B(x) = \sup_{p \in K} p(x^{*}x)^{\frac{1}{2}}$, so $a \leq c$ follows.

$c \leq d$. By prop. 4 in ((1)) , K is w^{*}-compact, so this follows from the Krein-Milman theorem.

Q.e.d.

4.2 C o r o l l a r y

The norm $\|\cdot\|$ on $\mathfrak{A}$ defined by $\|x\| = B(x)$; $x \in \mathfrak{A}$, has the properties:

$$\|x^{*}\| = \|x\| , \quad \|x^{*}x\| = \|x\|^{2} ,$$

$$\|xy\| \leq \|x\| \cdot \|y\|$$

$x, y \in \mathfrak{A}$.

P r o o f : This follows immediately from the proposition above, via the fact that for each $\pi \in R$ we will have:

$$\| \pi(x^*) \| = \| \pi(x) \| \quad , \quad \| \pi(x^*x) \| = \| \pi(x) \|^2$$

$$\| \pi(xy) \| \leq \| \pi(x) \| \cdot \| \pi(y) \| \leq \| x \| \cdot \| y \| \quad .$$

Q.e.d.

Now, let $\widehat{\mathfrak{A}}$ be the completition of $\mathfrak{A}$ with respect to this norm. Then $\widehat{\mathfrak{A}}$ is a C^*-algebra which contains $\mathfrak{A}$ as a dense $*$-subalgebra. $\widehat{\mathfrak{A}}$ will be called the enveloping C^*-algebra of $\mathfrak{A}$ (ref. ((2)) , 2.7.2).

Next, we are going to study the relationship between elements of $\mathcal{P}^*$ and positive linear functionals on $\widehat{\mathfrak{A}}$. This is quite simple.

2.3 P r o p o s i t i o n

Let $\mathfrak{A}$ be a pre-unitary $*$-algebra and suppose that p is a linear functional on $\mathfrak{A}$. Then the following statements are equivalent:

(i) $p \in \mathcal{P}^*$.

(ii) p is positive on $\mathfrak{A}$ and $\| \cdot \|$ -continuous.

(iii) p has a unique extension to a positive linear functional on $\widehat{\mathfrak{A}}$.

Moreover, if one of these conditions are satisfied, then

$$C(p) = \| p \| \quad .$$

P r o o f : (i) $\Longrightarrow$ (ii) . If $x \in \mathfrak{A}$, then by prop. 2.1 (i) :

$$|p(x)| \leq C(p) \cdot B(x) = C(p) \cdot \| x \| \quad ,$$

so $p \in \mathcal{P}^*$ implies norm-continuity.

(ii) $\Longrightarrow$ (iii). Suppose $x \in \hat{\mathfrak{A}}$. Then we can find a sequence $\{x_n\}_{n\in N} \subset \mathfrak{A}$, $x_n \longrightarrow x$ in norm. Then

$$\|x^*x - x_n^*x_n\| \leq \|x^*x - x^*x_n\| + \|x^*x_n - x_n^*x_n\|$$

$$\leq \|x^*\| \cdot \|x - x_n\| + \|x^* - x_n^*\| \cdot K$$

$$\leq K' \cdot \|x - x_n\| \longrightarrow 0 \quad , \quad \text{where}$$

K and K' are positive constants. Therefore, if p is positive on $\mathfrak{A}$ and norm-continuous, then

$$0 \leq p(x_n^* x_n) \longrightarrow \hat{p}(x^*x) \quad ; \quad n \in N \quad ,$$

when $\hat{p}$ is the unique continuous linear functional on $\hat{\mathfrak{A}}$ extending p. Hence $\hat{p}$ is positive on $\hat{\mathfrak{A}}$. Since a positive, linear functional on a C^*-algebra is automatically continuous, $\hat{p}$ is also unique as a positive extension of p.

(iii) $\Longrightarrow$ (i). Suppose $\hat{p}$ is the positive extension of p to $\hat{\mathfrak{A}}$. $\hat{\mathfrak{A}}$ is a C^*-algebra, and has in particular an approximate unit, so every positive, linear functional on $\hat{\mathfrak{A}}$ is extendable to $\hat{\mathfrak{A}} \oplus \mathbb{C}$ ($= \hat{\mathfrak{A}}$ with adjoined unit) as a positive linear functional. Clearly $\tilde{\mathfrak{A}} \subset \hat{\mathfrak{A}} \oplus \mathbb{C}$, so by restricting the last extension of p to $\tilde{\mathfrak{A}}$, it follows that $p \in \mathcal{P}^*$.

Finally, if $p \in \mathcal{P}^*$, we have already noted that $\|p\| \leq C(p)$. Now,

$$\|p\| = \|\hat{p}\| = \sup_{x\in\hat{\mathfrak{A}}} \frac{|\hat{p}(x)|^2}{\hat{p}(x^*x)} \geq \sup_{x\in\mathfrak{A}} \frac{|p(x)|^2}{p(x^*x)} = C(p) \quad ,$$

where the second equality is known for positive, linear functionals on C^*-algebras. Hence we have $C(p) = \| p \|$. Q.e.d.

4.4 Corollary

A linear functional q on a pre-unitary $*$-algebra $\mathfrak{A}$ is in $[\mathcal{P}^*] =$ The linear space determined by $\mathcal{P}^*$ if and only if q is norm-continuous on $\mathfrak{A}$.

Proof: Suppose $q \in [\mathcal{P}^*]$, so

$$q = \sum_{i=1}^{4} \alpha_i q_i \quad , \quad q_i \in \mathcal{P}^* \quad , \quad \alpha_i \in \mathbb{C} \quad , \quad i = 1, \dots, 4 \ .$$

Then each q_i is norm-continuous on $\mathfrak{A}$ by prop. 4.3, and hence also q . Conversely, suppose q is norm-continuous on $\mathfrak{A}$. Then it can be extended by continuity to $\hat{\mathfrak{A}}$. $\hat{\mathfrak{A}}$ is a C^*-algebra, so we may decompose $\hat{q}$ into positive parts. The corollary then follows from the implication (iii) $\Longrightarrow$ (i) in prop. 4.3. Q.e.d.

At this point we take time to pick up some further information about the structure of $\mathcal{P}^*$, now easily available.

4.5 Corollary

If $\mathfrak{A}$ is a pre-unitary $*$-algebra and $p \in \mathcal{P}^*$, then

$$C(p) = \sup_{\substack{x \in \mathfrak{A} \\ \| x \| \leq 1}} p(x^* x)$$

Note: This improves the inequality of prop. 2.1 (iii) .

Proof: By prop. 4.3 we know that

$$C(p) = \| p \| = \sup_{\substack{x \in \mathfrak{A} \\ \| x \| \leq 1}} | p(x) |$$

so that we can find a sequence $\{x_n\}_{n \in N} \subset \mathfrak{A}$ satisfying $\| x_n \| \leqslant 1$; $n \in N$, and

$$| p(x_n) | \longrightarrow C(p)$$

We may now apply the inequality:

$$| p(x_n) |^2 \leq C(p) p(x_n^* x_n) \leq C(p)^2 \cdot \| x_n \|^2$$

(see the proof of prop. 2.1 (iv)), which proves the corollary.

4.6 Corollary

If $\mathfrak{A}$ is a pre-unitary $*$-algebra and $p \in \mathcal{P}^*$, then $\mathfrak{A}$ is N_p - dense in $\tilde{\mathfrak{A}}$.

Proof : By the corollary above, we may find a sequence $\{x_n\}_{n \in N} \subset \mathfrak{A}$ such that $p(x_n^* x_n) \longrightarrow C(p)$; $\| x_n \| \leq 1$; $n \in N$. Now, put $y_n = x_n^* x_n$. Then $\| y_n \| \leq 1$; $n \in N$, and $p(y_n) \longrightarrow C(p)$, so that by prop. 2.1 (vi) :

$$N_p(e - y_n)^2 = \tilde{p}\left[(e - y_n)^*(e - y_n)\right] \longrightarrow 0 .$$

The proposition follows.

4.7 Proposition

Let $\mathfrak{A}$ be a pre-unitary $*$-algebra, and suppose $p \in \mathcal{P}^*$. If q is a linear functional on $\mathfrak{A}$ satisfying: $0 \leq q \leq p$, then $q \in \mathcal{P}^*$ and $C(q) \leq C(p)$.

Proof : Let x be an arbitrary element of $\hat{\mathfrak{A}}$. $\mathfrak{A}$ is norm-dense in $\hat{\mathfrak{A}}$, so that we may evidently find a sequence

$\{x_n\}_{n \in N} \subset \mathfrak{A}$ converging in the N_p -topology to x :

$$\hat{p}\left[(x - x_n)^*(x - x_n)\right] \leq C(p) \cdot \| x - x_n \|^2 \longrightarrow 0 ,$$

where $\hat{p}$ is the unique positive extension of p to $\hat{\mathfrak{A}}$ (prop. 4.3). In particular,

$$0 \leq q\left[(x_n^* - x_m^*)(x_n - x_m)\right] \leq p\left[(x_n^* - x_m^*)(x_n - x_m)\right] \longrightarrow 0 ,$$

so $N_q(x_n - x_m) \longrightarrow 0$. As

$$| N_q(x_n) - N_q(x_m) | \leq N_q(x_n - x_m) ,$$

the sequence $\{ N_q(x_n) \}_{n \in N}$ must be Cauchy, so we may define

$$\hat{q}(x^* x) = \lim_{n \to \infty} N_q(x_n)^2 = \lim_{n \to \infty} q(x_n^* x_n)$$

Now, $0 \leq q(x_n^* x_n) \leq p(x_n^* x_n)$, so we must have

$$0 \leq \hat{q}(x^* x) \leq \hat{p}(x^* x)$$

It is readily verified that the value of $\hat{q}(x^* x)$ is independent of the particular sequence used to define it, so $\hat{q}$ becomes a positive, additive, homogenous functional on the positive cone in $\hat{\mathfrak{A}}$. (Every positive element of $\hat{\mathfrak{A}}$ can be written in the form $x^* x$ for some $x \in \hat{\mathfrak{A}}$). We can then extend $\hat{q}$ to a positive, linear functional on $\hat{\mathfrak{A}}$. By prop. 4.3, the restriction $q = \hat{q} \mid \mathfrak{A}$ is then in $\mathcal{P}^*$. By cor. 4.5, we shall have:

$$C_q = \sup_{\substack{x \in \mathfrak{A} \\ \| x \| \leq 1}} q(x^* x) \leq \sup_{\substack{x \in \mathfrak{A} \\ \| x \| \leq 1}} p(x^* x) = C_p$$

The proof is finished.

4.8 Proposition

If $\mathfrak{A}$ is a pre-unitary $*$-algebra, and p, q are elements of $\mathcal{P}^*$, then $p + q \in \mathcal{P}^*$, and

$$C(p + q) = C(p) + C(q) \quad ; \quad \tilde{p} + \tilde{q} = \widetilde{p + q}$$

Proof: Clearly $p + q \in \mathcal{P}^*$, and if extended to $\hat{\mathfrak{A}}$, we have

$$\widehat{p + q} = \hat{p} + \hat{q}$$

so

$$\|\widehat{p + q}\| = \|\hat{p}\| + \|\hat{q}\|$$

The proposition then follows from prop. 4.3, last statement.

We recall that for a unitary $*$-algebra $\mathfrak{A}$, and p a positive, normalized, linear functional on $\mathfrak{A}$, then $[p]$ = the linear space generated by the set $\{ q \mid 0 \leq q \leq p \}$ is isomorphic to $\mathbb{C}$ if and only if p is a pure state. We are now able to extend this to pre-unitary $*$-algebras.

4.9 Corollary

If $\mathfrak{A}$ is a pre-unitary $*$-algebra, and $p \in \mathcal{P}^*$, $C(p) = 1$, then $p \in \partial_e K$ if and only if $[p] \cong \mathbb{C}$.

Proof: 1) Suppose $p \in \partial_e K$, $p \neq 0$, and $p \geqslant q_1 \geqslant 0$; q_1 a linear functional on $\mathfrak{A}$. Put $q_2 = p - q_1$, $p \geqslant q_2 \geqslant 0$ and $p = q_1 + q_2$. Then, by props. 4.7 and 4.8, $q_1, q_2 \in \mathcal{P}^*$, and

$$1 = C(p) = C(q_1) + C(q_2) \quad ,$$

so we may put $\lambda = C(q_1) \Longrightarrow C(q_2) = 1 - \lambda$, and $r_1 = \frac{1}{\lambda} q_1$, $r_2 = \frac{1}{1-\lambda} q_2$. Thus $r_1, r_2 \in K$ and $p = \lambda r_1 + (1-\lambda) r_2$. p is extreme i K $\Longrightarrow p = r_1 = r_2$, so $q_1 = \lambda p$, which proves the first part.

2) Suppose $[p] \cong \mathcal{C}$, and suppose $p = \lambda q_1 + (1-\lambda) q_2$; $0 < \lambda < 1$; $q_1, q_2 \in K \Longrightarrow 0 \leq \lambda q_1 \leq p$, which by the assumption implies $\lambda q_1 = \mu p$, $0 \leq \mu \leq 1$. Now

$$1 = C(p) = \lambda C(q_1) + (1-\lambda) C(q_2)$$

and $C(q_1)$, $C(q_2) \leq 1$, so we must necessarily have $C(q_1) = C(q_2) = 1$. Hence

$$\lambda = \lambda C(q_1) = \mu C(p) = \mu$$

$$\Longrightarrow \lambda q_1 = \lambda p \quad \text{so} \quad q_1 = p \Longrightarrow q_2 = p \ .$$

Q.e.d.

§5. The universal problem.

Theorem 1. Let $\mathfrak{A}$ be a pre-unitary *-algebra, $\widehat{\mathfrak{A}}$ its enveloping C^*-algebra, and τ the canonical map of $\mathfrak{A}$ into $\widehat{\mathfrak{A}}$. If π is a *-homomorphism of $\mathfrak{A}$ into a C^*-algebra $\mathfrak{B}$, then there exists a unique *-homomorphism $\hat{\pi}$ of $\widehat{\mathfrak{A}}$ into $\mathfrak{B}$ such that $\pi = \hat{\pi} \circ \tau$. $\pi(\mathfrak{A})$ is pre-unitary, and $\hat{\pi}(\widehat{\mathfrak{A}})$ is the enveloping C^*-algebra of $\pi(\mathfrak{A})$.

Proof: First we observe that $\| \pi(x) \| \leq \| x \|$; $x \in \mathfrak{A}$, so π has a unique extension $\hat{\pi}$ taking $\widehat{\mathfrak{A}}$ into $\mathfrak{B}$. $\hat{\pi}(\widehat{\mathfrak{A}})$ contains $\pi(\mathfrak{A})$ as a dense sub-algebra and $\pi = \hat{\pi} \circ \tau$. Moreover,

$\hat{\pi}$ is an open mapping and $\hat{\pi}(\hat{\mathfrak{A}}) = \overline{\pi(\mathfrak{A})}$ is a C^*-algebra contained in $\mathcal{B}$ (ref. ((2)), 1.8.3). To see that $\pi(\mathfrak{A})$ is pre-unitary, let q be a linear functional on $\pi(\mathfrak{A})$, satisfying $q(y^*y) \geq 0$ for all $y \in \pi(\mathfrak{A})$; and with $C(q) = \sup_{y\in\pi(\mathfrak{A})} \frac{|q(y)|}{q(y^*y)} < \infty$. Then $p = q \circ \pi$ is an element of $\mathcal{P}^*$, satisfying $C(p) = C(q) < \infty$. Hence p is norm-continuous on $\mathfrak{A}$ (prop. 4.3). As $\hat{\pi}$ is open, q must be continuous on $\pi(\mathfrak{A})$, and therefore has a unique positive extension to $\overline{\pi(\mathfrak{A})}$. Consequently the set of states on $\pi(\mathfrak{A})$ is w^*-compact, so $\pi(\mathfrak{A})$ is preunitary (ref. ((1)); prop.4). Now any state on a sub-C^*-algebra of $\mathcal{B}$ may be extended to a state on $\mathcal{B}$ (ref. ((2)); 2.10.1). This proves that the norm constructed on $\pi(\mathfrak{A})$ as a pre-unitary algebra will coalesce with the norm it inherits from $\mathcal{B}$. Hence $\hat{\pi}(\hat{\mathfrak{A}})$ is the enveloping C^*-algebra of $\pi(\mathfrak{A})$.

Consequently, $\hat{\mathfrak{A}}$ represents the solution of a universal problem for $\mathfrak{A}$. In particular, $\hat{\mathfrak{A}}$ is the only C^*-algebra containing $\mathfrak{A}$ as a dense subalgebra.

Corollary. Let π be a representation of the pre-unitary $*$-algebra $\mathfrak{A}$. Then there is a unique representation $\hat{\pi}$ of $\hat{\mathfrak{A}}$ such that $\pi = \hat{\pi} \circ \tau$. π is topologically irreducible if and only if $\hat{\pi}$ is topologically irreducible.

Proof: The first statement is immediate from the theorem, and the second follows from the equality $\pi(\mathfrak{A})' = \hat{\pi}(\hat{\mathfrak{A}})'$.

Theorem 2. Let $\mathfrak{A}$ be a pre-unitary *-algebra. Then there is a faithful representation π of $\mathfrak{A}$ in a Hilbert-space H . Moreover $\pi(\mathfrak{A})$ is pre-unitary, and the map $\mathfrak{A} \longrightarrow \pi(\mathfrak{A}) \subseteq \mathcal{L}(H)$ is isometric.

Proof: This theorem is valid for C^*-algebras, so we may just apply this fact together with Theorem 1.

Hence, the most general kind of pre-unitary *-algebras are the involutive subalgebras of $\mathcal{L}(H)$, H some Hilbert-space.

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