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MODEL THEORY, ULTRAPRODUCTS AND TOPOLOGY

By

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1. INTRODUCTION: BIRKHOFF's THEOREM

The purpose of this seminar report is to show how certain results in model theory can be proved using elementary set-theoretic topology.

In order to present some basic notions of model theory and to show a typical application we shall in this introductory section present a very simple proof of the Birkhoff theorem characterizing classes of algebras which are axiomatizable by sets of identities.

An *algebra* is a set A together with a finite sequence $f_1, \ldots, f_n$ of *operations* on A, i.e. each f_i is a map $f_i : A^{m_i} \dashrightarrow A$, where m_i is some natural number depending upon i. Algebras having the same type of operations are called *similar*.

The language of a *similarity type* consists of all formulas constructed in the usual way from the various operations f_i using variables and the connectives of logic. We note in passing that constants may be identified with constant operations.

An *identity* in the language, $\forall x_1 \ldots \forall x_n [t_1 = t_2]$, is a statement of unrestricted equality between terms t_i, where the class of terms is obtained from variables and operations by repeated substitutions.

Let K be a class of algebras of fixed similarity type. K is called *axiomatizable* if there exists some set of sentences Γ such that K is the class of models of Γ. K is called *equational* if Γ can be taken as a set of identities.

Birkhoff's theorem may now be stated as follows:

Theorem. Let K be a class of similar algebras. K is equational if and only if K is closed under the operations of taking homomorphic images, subalgebras and direct products.

This theorem is characteristic of model theory in that it shows a connection between syntactic and semantic properties: A purely syntactic condition, viz. the axioms can be given as identities, is related to semantic ones, viz. the set of models satisfying the axioms exhibit certain closure properties.

A proof of Birkhoff's theorem runs as follows, we indicate without being too careful about the details.

It is well known that if K is closed under subalgebras, homomorphisms and direct products, then for each set M there exists a free algebra F_M in K, defined uniquely as the solution of a certain universal mapping problem, and further that each $A \in K$ is the homomorphic image of some F_M in K.

Thus we can represent $K = \mathrm{Hom}(\{F_M\})$, where e.g. $\bar{M}$ ranges over the cardinalities of elements in K in order to escape set-theoretic difficulties.

We further remark that the set Γ of identities valid in all the free algebras F_M is precisely the set of identities valid throughout K.

To complete the proof we need show that any algebra B of the same similarity type as K and which satisfies the set Γ can be obtained as the homomorphic image of some F_M, specifically we shall construct a homomorphism $\sigma : F_B \longrightarrow B$, which concludes the proof as $F_B \in K$.

The construction is started by defining $\sigma(b) = b$ for all generators $b \in B$ of F_B. We must show that σ can be extended to all of F_B such that the extended map is a homomorphism onto B - a trivial fact if B were an element in K.

However, the proof is immediate on account of the following small remark: Let W_1 and W_2 be two words in F_B. If $W_1 = W_2$ in F_B, we have an equality $t_1 = t_2$ in elements of B (the generators) and the operations f_i. But as F_B is free, this equality implies the validity of an identity $\forall x_1 \ldots \forall x_n [t_1 = t_2]$ throughout F_M, where elements b_i has been replaced by variables x_i. (This is immediate from the notion of free algebra; the pedantic proof uses repeated applications of the universal mapping property.) Then by assumption this identity must also be valid in our algebra B . - Applications of this remark at once yields that σ can be extended as asserted.

For the remaining of this paper we shall observe the required standard of exactness and formalism. Hence in the next section we shall in some detail describe the languages and the models and define the notion of satisfaction. Thereafter we shall explain the model-theoretic construction of ultraproducts and give its main properties. In particular we shall obtain the compactness of the space of models as a consequence of the main theorem on ultraproducts, the topology being defined by letting closed sets correspond to axiomatizable classes of models. All of this is well known and we shall not give any proofs.

Within the frame-work thus set up we shall present our proof of the Craig interpolation theorem which roughly says that if $\varphi \longrightarrow \psi$ is provable, then there exists a sentence φ_0 in the vocabulary common to both φ and ψ such that $\varphi \rightarrow \varphi_0$ and $\varphi_0 \longrightarrow \psi$ are provable.

2. THE LANGUAGE AND THE MODELS

In the introduction we treated algebraic systems with operations. As is well known operations can be reduced to relations, and for the sake of simplicity we shall in the sequel treat only relational systems.

A *relational system* is a sequence

$$\mathfrak{A} = \langle A, \ldots, R_p, \ldots \rangle ,$$

where A is some set and each R_p is a finitary relation in A , i.e. $R_p \subseteq A^{n_p}$ for some natural number n_p . Two relational systems are called *similar* if their sequences of relations have the same order type and for each p , the ranks n_p are the same. The class of all similar systems is called a *similarity type*. We assume the notions of subsystem, homomorphic and isomorphic image known. $\mathfrak{A}$ is said to be imbeddable in $\mathfrak{B}$ if $\mathfrak{A}$ is isomorphic to a subsystem of $\mathfrak{B}$.

Corresponding to a similarity class R of relational systems we may construct a first order language L(R) having an infinite denumerable sequence of variables, the usual logical connectives and a sequence of relational symbols P_p of ranks n_p corresponding to the sequence of relations in the type R.

The language L(R) and the systems $\mathfrak{A} \in R$ are connected through the all important notion of satisfaction. We are going to define the symbol complex

$$\mathfrak{A} \models \varphi(a)$$

meaning that the sequence $a = \langle a_1, a_2, \dots, a_n, \dots \rangle \in A^\omega$ satisfies the formula φ of L(R) in the model $\mathfrak{A} \in R$. The definition will be by recursion.

$\mathfrak{A} \models \varphi(a)$ if and only if either

i. φ is atomic, i.e. $\varphi = P(x_{i_1}, \dots, x_{i_{n_p}})$, and $\langle a_{i_1}, \dots, a_{i_{n_p}} \rangle \in R_p$, or

ii. $\varphi = \varphi_1 \vee \varphi_2$ and $\mathfrak{A} \models \varphi_1(a)$ or $\mathfrak{A} \models \varphi_2(a)$, or

iii. $\varphi = \neg \varphi_1$ and not $\mathfrak{A} \models \varphi_1(a)$, or

iv. $\varphi = (\exists x_j)\varphi_1$ and there exists a $b \in A^\omega$ such that $b_i = a_i$ for $i \neq j$ and $\mathfrak{A} \models \varphi_1(b)$.

Thus $\mathfrak{A} \models \varphi(a)$ is defined recursively with respect to the length of the formula φ. The definition is unique because each formula can be uniquely decomposed in a sequence of subformulas ending up with atomic ones.

A *sentence* is a formula without free variables. It is easily seen that given a sentence φ and a system $\mathfrak{A}$, either $\mathfrak{A} \models \varphi(a)$ for all $a \in A^{\omega}$, or else there does not exist any $a \in A^{\omega}$ such that a satisfies φ in $\mathfrak{A}$. For sentences we simply write $\mathfrak{A} \models \varphi$ if $\mathfrak{A} \models \varphi(a)$ for some $a \in A^{\omega}$, and say that φ is *true* in $\mathfrak{A}$, or that $\mathfrak{A}$ is a *model* for φ. If Γ is a set of sentences in L(R) then $\mathfrak{A}$ is a model for Γ if all $\varphi \in \Gamma$ is true in $\mathfrak{A}$.

Two systems $\mathfrak{A}$ and $\mathfrak{B}$ are called *elementary equivalent*, in symbols $\mathfrak{A} \equiv \mathfrak{B}$, if $\mathfrak{A} \models \varphi$ iff $\mathfrak{B} \models \varphi$ for all sentences φ in L(R). It is trivial that isomorphic systems are elementary equivalent. The converse, however, is not true.

The notion of *elementary extension* is important. $\mathfrak{B}$ is called an elementary extension of $\mathfrak{A}$ ($\mathfrak{A}$ an elementary subsystem of $\mathfrak{B}$) if $\mathfrak{A}$ is a subsystem of $\mathfrak{B}$ and for each φ in L(R) and $a \in A^{\omega}$ $\mathfrak{A} \models \varphi(a)$ iff $\mathfrak{B} \models \varphi(a)$. (Here the equivalence could be replaced by an implication in either direction.) $\mathfrak{A}$ is said to be elementary imbeddable in $\mathfrak{B}$ if $\mathfrak{A}$ is isomorphic to an elementary subsystem of $\mathfrak{B}$.

The notions of this section are due to A. Tarski and they are indispensable for any treatment of formal languages and their interpretations.

3. ULTRAPRODUCTS

Recently there has emerged a model theoretic construction of great power and versatility, viz. the ultraproduct construction which perhaps has its root in Skolem's construction of non-standard models of elementary arithmetic in 1935.

In order to give the definition we shall recall the necessary properties of filters. Let D be a class of subsets of some non-empty set I. Then D is called a *filter* if (i) $s,t \in D \Rightarrow s \cap t \in D$ and

(ii) $S \in D$ and $s \subseteq t \subseteq I \Rightarrow t \in D$. The maximal elements in the class of filters on I are called ultrafilters and are characterized by the equivalence $s \cup t \in D \Longleftrightarrow s \in D$ or $t \in D$.

Perhaps the main result on the existence of ultrafilters is that every class F having the finite intersection property can be extended to an ultrafilter. (F has the finite intersection property if every finite subset of elements of F have a non-empty intersection.)

Let $\{ \mathfrak{A}_i \mid i \in I \}$ be a family of relational systems of same type, $\mathfrak{A}_i = \langle A_i, \ldots, R_p^i, \ldots \rangle$. Let D be an ultrafilter in the index set I. By the u l t r a p r o d u c t of the systems $\mathfrak{A}_i$ with respect to the filter D we shall understand the system

$$\prod_{i \in I} \mathfrak{A}_i / D = \langle \prod_{i \in I} A_i/D, \ldots, R_p, \ldots \rangle ,$$

where $\prod A_i$ is the cartesian product of the various A_i consisting of all functions $f : I \to \cup A_i$ such that $f(i) \in A_i$ and $\prod A_i/D$ the set of equivalence classes araising from the relation

$$f \sim_D g \Longleftrightarrow \{ i \in I \mid f(i) = g(i) \} \in D .$$

The relations R_p is defined by the condition that

$$\langle f_1/D, \ldots, f_n/D \rangle \in R_p \Longleftrightarrow \{ i \in I \mid \langle f_1(i), \ldots, f_n(i) \rangle \in R_p^i \} \in D ,$$

where f_i/D denotes elements in $\prod A_i/D$ and the rank of each R_p^i is assumed to be n. It is easily verified that R_p is well defined.

The chief interest of the ultraproduct as a model theoretic, and not only as an algebraic construction, stems from the following theorem.

M a i n t h e o r e m o n u l t r a p r o d u c t s. <u>Let $\{ \mathfrak{A}_i \mid i \in I \}$ be a class of models of type R and let φ be a formula in $L(R)$. Denote by f/D an element in $(\prod A_i/D)^{\omega}$ and by $f(i)$ the</u>

corresponding elements in A_i^{ω} . Then the following equivalence holds:

$$\prod_{i \in I} \mathfrak{A}_i/D \models \varphi(f/D) \iff \{ i \in I \mid \mathfrak{A}_i \models \varphi(f(i)) \} \in D .$$

This theorem was first stated by Łoś in 1955 and the proof proceeds by induction on the length of the formula φ , using at appropriate places the properties of ultrafilters. Assume e.g. that $\varphi = \neg \varphi_1$ and that the equivalence is proved true for φ_1 . Then $\prod \mathfrak{A}_i/D \models \neg\varphi_1(f/D)$ means that not $\prod \mathfrak{A}_i/D \models \varphi_1(f/D)$, which by the assumption on φ_1 is equivalent to $\{ i \in I \mid \mathfrak{A}_i \models \varphi_1(f(i)) \} \notin D$. This means by the characteristic property of ultrafilters that $\{ i \in I \mid \mathfrak{A}_i \models \neg\varphi_1(f(i)) \} \in D$, remembering that for all $i \in I$ $f(i)$ either satisfies φ_1 or $\neg \varphi_1$ in $\mathfrak{A}_i$.

We shall not expand upon the theory of ultraproducts in this report, but we cannot resist including the following small result having a proof so characteristic of the "metamathematical" approach in this field. Let $|\mathfrak{A}| = |A|$ denote the cardinality of the set A . Assume that for all $i \in I$, $|A_i| \leq m$. We shall show that $|\prod \mathfrak{A}_i/D| \leq m$ for all ultrafilters D in I . Note that I may have any cardinality. The proof, using the main theorem, is trivial. Assume that equality $=$ is included in the type and that it is interpreted as identity in models. Consider the following sentence

$$(\exists x_1)(\exists x_2) \ldots (\exists x_m)(\forall y) \left[y = x_1 \vee \ldots \vee y = x_m \right]$$

which asserts that there are at most m individuals. This sentence is true in each $\mathfrak{A}_i$, hence by the theorem it is also valid in the ultraproduct, which thus has cardinality $\leq m$.

4. THE SPACE OF MODELS

Let R be a similarity class and $L(R)$ the corresponding language.

For every sentence φ in L(R) we define a subset K_φ of R by

$$K_\varphi = \{ \mathfrak{A} \in R \mid \mathfrak{A} \models \varphi \}.$$

On account of the formulas $K_{\varphi_1} \cap \ldots \cap K_{\varphi_n} = K_{\varphi_1 \wedge \ldots \wedge \varphi_n}$ and $K_{\varphi_1} \cup \ldots \cup K_{\varphi_n} = K_{\varphi_1 \vee \ldots \vee \varphi_n}$ we see that the class K is both closed under finite intersections and unions, hence can be taken simultaneously as a base for open and closed sets for a topology on R . (Note that because of the formula $R - K_\varphi = K_{\neg\varphi}$, each K_φ is both open and closed.)

In logic it is usual to denote the class $\{K_\varphi\}$ by EC , the e l e m e n t a r y class or the class of finitely axiomatizable theories. The collection of closed sets will be denoted by EC_Δ . It consists of arbitrary intersections of elementary classes and corresponds to theories axiomatizable by some set of sentences Γ in the language L(R) . The open sets will be denoted by EC_Σ .

R with the topology defined by EC is not Hansdorff, we have elementary equivalent models which are not isomorphic, and elementary equivalent models cannot be separated by open sets, in fact $\mathfrak{A} \equiv \mathfrak{B}$ if and only if $\mathfrak{A} \in K_\varphi \iff \mathfrak{B} \in K_\varphi$, for all sentences φ in L(R) . To produce non-isomorphic elementary equivalent models we use the same argument that Skolem used in constructing non-standard models of arithmetic. Take any infinite system $\mathfrak{A} \in R$ and an infinite index set I . Let D be a non-principal ultrafilter in I , then $\mathfrak{A}^I/D \equiv \mathfrak{A}$ by the main theorem on ultraproducts, but they are n o t isomorphic.

(R,EC) is, however, quasi-compact. This follows from the following result.

C o m p a c t n e s s t h e o r e m . Let Γ be a set of sentences in L(R) . Then there exists a model $\mathfrak{A}$ for Γ in R if and only if every finite subset of Γ has a model in R .

The proof is an easy application of ultraproducts. We may assume (for simplicity) that Γ is countable, $\Gamma = \{ \varphi_1, \varphi_2, \ldots, \varphi_n, \ldots \}$. Define $\psi_1 = \varphi_1$ and inductively $\psi_{n+1} = \psi_n \wedge \varphi_{n+1}$. By assumption for any n there exists an $\mathfrak{A}_n \in R$ such that $\mathfrak{A}_n \models \psi_n$. Let $I = \{ 1, 2, \ldots, n, \ldots \}$ and let F be the class of sets $s_n = \{ n, n+1, \ldots \}$, $n > 1$. Then F has finite intersection property, hence there exists an ultrafilter D extending F. Our model $\mathfrak{A}$ is then

$$\mathfrak{A} = \prod_{i \in I} \mathfrak{A}_i / D .$$

We must show that $\mathfrak{A} \models \varphi_n$ for all $\varphi_n \in \Gamma$. But this is immediate by the main theorem on ultraproduct noting that each φ_n is a conjunct in all but a finite number of the sentences ψ_m.

T h e o r e m . <u>(R, EC) is quasi-compact.</u>

This is a corollary of the compactness theorem, the proof is by the usual "dual" argument. We also note another immediate corollary: $EC = EC_\Sigma \cap EC_\Delta$, i.e. EC consists of exactly the open-closed sets, or, expressed in logical terms, a class K is finitely axiomatizable if and only if both K and $R - K$ are axiomatizable. The proof is the usual compactness argument that if a closed set in a quasi-compact space is a union of open sets, it is a union of a finite number of open sets. And this finite collection of open sets yields the finite axiom system.

Let R_1 and R_2 be two similarity classes. We say that R_1 is a s u b t y p e of R_2 if the order type of relations in R_1 is (in a suitable sense) a subtype of the ordertype of relations in R_2, or equivalently if the language $L(R_2)$ is an extension of the language $L(R_1)$ obtained by adding relational symbold. If R_1 is a subtype of R_2 we may introduce a map

$$\mathrm{red}_{R_2,R_1}: \quad R_2 \longrightarrow R_1 \quad ,$$

the reduction map of R_2 to R_1 , defined in the following way: $\mathrm{red}_{R_2,R_1}(\mathfrak{A})$ is obtained from $\mathfrak{A}$ by deleting the relations not occurring in the subtype of relations in R_2 corresponding to the relations in R_1 , or said more simply but less exact, $\mathrm{red}(\mathfrak{A})$ is obtained from $\mathfrak{A}$ by "throwing away" the relations in $R_2 - R_1$.

P r o p o s i t i o n . <u>red_{R_2,R_1} is continuous with respect to the topologies defined by elementary classes.</u>

Again the proof is very simple, being based upon the observation that if R_1 is a subtype of R_2 , then $L(R_1)$ is a sublanguage of $L(R_2)$, i.e. every sentence φ in $L(R_1)$ is also a sentence in $L(R_2)$. It suffices to show that $\mathrm{red}^{-1}_{R_2,R_1}(K^{R_1}_{\varphi})$ is open in R_2 . But it is easy to show that

$$\mathrm{red}^{-1}_{R_2,R_1}(K^{R_1}_{\varphi}) = K^{R_2}_{\varphi}$$

for all sentences φ in $L(R_1)$.

In every similarity type R we may introduce an equivalence relation by identifying two systems if they are elementary equivalent. In this way we obtain the reduced type R' which in the topology defined by the projection map is compact and Hausdorff. It is Hausdorff because non-equivalent models can be separated by a sentence, and the compactness of R' is most easily inferred by use of the compactness theorem stated above. For some applications it is necessary, as will be seen in the next section, to work with reduced types.

5. CRAIG's INTERPOLATION THEOREM

This result can be stated as follows.

T h e o r e m . Let φ and ψ be two sentences within a language $L(R)$ and suppose that $\varphi \rightarrow \psi$ is provable.

1. Assume that φ and ψ have at least one relational symbol in common and let $L(R_o)$ be the language based upon the relational symbols common to both φ and ψ . Then there exists a sentence θ in $L(R_o)$ such that θ interpolates between φ and ψ , i.e. such that both $\varphi \rightarrow \theta$ and $\theta \rightarrow \psi$ are provable.

2. If φ and ψ have no common relational symbol, then either $\neg \varphi$ is provable or ψ is provable.

The intuitive idea is that this theorem helps us get rid of unnecessary hypotheses in proofs, and the result has emerged as a rather important one in recent research within logic. One may mention that Beth's theorem on how to convert implicit definitions to explicit ones as well as Robinson's consistency lemma are easily obtainable from Craig's theorem. We may further refer the reader to Addison's report to the 1960 Stanford Congress.

Case 2 of the theorem is an easy exercise in model theory. Assume that φ and ψ have no common vocabulary and that $\neg \varphi$ is not provable. If ψ is not a theorem, we have by Gödel's completeness theorem a model $\mathfrak{A}_o \in R$ such that $\mathfrak{A}_o \models \neg \psi$. φ is also consistent, hence with a little bit of care we may modify $\mathfrak{A}_o$ to a model $\mathfrak{A}$ of both φ and $\neg \psi$, essentially because the interpretations of the relational symbols in φ and $\neg \psi$ do not interfere. But this is a contradiction as $\varphi \rightarrow \psi$ is assumed provable.

The proof of part 1 uses the machinery developed in previous sections. Define

$$K_1 = \{ \mathfrak{A} \in R \mid \mathfrak{A} \models \varphi \} ,$$

$$K_2 = \{ \mathfrak{A} \in R \mid \mathfrak{A} \models \neg \psi \}.$$

For simplicity of notation we drop the subscripts on red_{R,R_o} in the sequel. Let $N_1 = \mathrm{red}(K_1)$ and $N_2 = \mathrm{red}(K_2)$. We have that $N_1 \cap N_2 = \emptyset$. This follows from the assumption that $\varphi \to \psi$ is provable which at once gives that $K_1 \cap K_2 = \emptyset$. And if there existed a model $\mathfrak{A} \in N_1 \cap N_2$, we would have models $\mathfrak{A}_1 \in K_1$ and $\mathfrak{A}_2 \in K_2$ such that $\mathrm{red}(\mathfrak{A}_1) = \mathrm{red}(\mathfrak{A}_2) = \mathfrak{A}$. But as $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are defined in a common set A of individuals and we have in $\mathfrak{A}$ an interpretation of the common relational symbols of φ and $\neg\psi$, it is obvious that we can form a common model $\mathfrak{A}' \in R$ of both φ and $\neg\psi$. But then $\mathfrak{A}' \in K_1 \cap K_2$ which contradicts the fact that $K_1 \cap K_2 = \emptyset$.

The proof would then be finished if we could find a class $K \in EC$ in R_o which separates N_1 and N_2, i.e. such that $N_1 \subseteq K$ and $N_2 \cap K = \emptyset$. This implies that $K_1 \subseteq \mathrm{red}^{-1}(K)$ and $K_2 \cap \mathrm{red}^{-1}(K) = \emptyset$. But $\mathrm{red}^{-1}(K) = K^R_\theta$ for some sentence θ in $L(R_o) \subseteq L(R)$, thus we have by Gödel's completeness theorem that both $\varphi \to \theta$ and $\theta \to \psi$ are provable.

To separate N_1 and N_2 by an EC-class K in R_o we have to go to the reduced type R_o' obtained by identifying elementary equivalent systems in R_o. R_o' is compact and Hausdorff in the quotient topology. $K_1, K_2 \in EC^R$, which means that K_1 and K_2 are closed in R. But R is quasi-compact, hence both K_1 and K_2 are compact sets in R. The reduced classes N_1' and N_2' are continuous images of K_1 and K_2 as red is a continuous map between types. R_o' is Hausdorff, hence N_1' and N_2' are disjoint closed subsets of R_o'. (N_1' and N_2' are disjoint for the same reason that N_1 and N_2 are disjoint.) R_o' is compact, hence normal, and thus N_1' and N_2' can be separated by an open set K_o', i.e.

$$N_1' \subseteq K_o' \qquad \text{and} \qquad N_2' \cap K_o' = \emptyset .$$

Going back to R_o, we have an open set K_o separating N_1 and N_2.

Now $K_o = \bigcup K_i$ with each $K_i \in EC^{R_o}$, and as N_1 is quasi-compact, we obtain that

$$N_1 \subseteq K_{i_1} \cup \dots \cup K_{i_K} \quad ,$$

for some indices $i_1, \dots, i_k$. Let $K = K_{i_1} \quad \dots \quad K_{i_k}$, then $K \in EC^{R_o}$ and separates N_1 and N_2. This completes the proof.

6. REFERENCES

We have not found it necessary to include bibliographic references in the main part of this report. However, we ought to indicate our sources, and perhaps someone would be interested in further references to the literature.

Birkhoff's theorem was given in ((2)). The conceptual framework of model theory is due to Tarski ((10)). Skolem's construction of non-standard models can be found in ((9)) and the first explicit construction of ultraproducts in Łoś ((5)) where he proves the "main theorem" on ultraproducts. Two recent introductory papers on ultraproducts are ((4)) and ((7)), the compactness theorem being proved by ultraproducts for the first time in ((4)), previously one had to invoke a metamathematical argument via Gödel's theorem. A most valuable paper on ultraproducts is Keisler ((6)) in which he proves that two systems are elementary equivalent if and only if they have isomorphic ultrapowers (using the continuum hypothesis for the proof).

Craig's theorem was first stated in ((3)), the proof being heavily syntactic using the Herbrand-Gentzen machinery. An exposition via Robinson's consistency lemma is found in ((8)), a book which can serve as a general introduction to model theory. A recent discussion of Craig's theorem is found in Addison ((1)).

((1)) Addison; The theory of hierarchies. Proceedings of the 1960 International Congress on Logic, Methodology and Philosophy of Science, Stanford 1962.

((2)) Birkhoff; On the structure of abstract algebras. Proc. Cambridge Philos. Soc. vol. 31 (1935).

((3)) Craig. Linear reasoning. A new form of the Herbrand-Gentzen theorem. J. Symbolic Logic vol. 22 (1957).

((4)) Frayne, Morel, Scott. Reduced direct products. Fund. Math. vol. 51 (1962).

((5)) Loś. Quelques remarques, théorèmes, et problèmes sur les classes définissable d'algèbres. Mathematical interpretation of formal systems, Amsterdam 1955.

((6)) Keisler. Ultraproducts and elementary classes. Indag. Math. vol. 23 (1961).

((7)) Kochen. Ultraproducts in the theory of models. Ann. Math. vol. 74 (1961).

((8)) Robinson. Introduction to model theory and the metamathematics of algebra. Amsterdam 1963.

((9)) Skolem. Über die Nicht-characterisierbarkeit der Zahlenreihe mittels endlich oder abzählbar unendlich vieler Aussagen mit ausschliesslich Zahlenvariablen. Fund. Math., vol. 23 (1934).

((10)) Tarski. Contributions to the theory of models. Indag. Math. vol. 16,17 (1954-5).

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There is a gap in the proof offered for Craig's theorem. On page 12 the assertion in parantheses "N_1' and N_2' are disjoint for the same reason that N_1 and N_2 are disjoint" is not quite true and not quite sufficient for our purpose.

In fact, N_1' and N_2' <u>are</u> disjoint and the proof is as follows: Suppose $N_1' \cap N_2' \neq \emptyset$. Then there are systems $\mathfrak{A}$ and $\mathfrak{B}$, such that $\mathfrak{A} \equiv \mathfrak{B}$, $\mathfrak{A} \in N_1$ and $\mathfrak{B} \in N_2$. Using the characterization of elementary equivalence in terms of ultralimit (see Kochen ((7))) we may infer that $\mathfrak{A}$ and $\mathfrak{B}$ have isomorphic ultralimits. If N_1 and N_2 are closed under ultralimits, it follows that $N_1 \cap N_2 \neq \emptyset$, and we may obtain a contradiction as in the text.

Now N_i = red K_i and obviously each K_i is closed under ultralimits. Noticing that the ultralimit construction is co-ordinatewise defined, it easily follows that every ultra-limit in N_i is the reduction of some ultralimit in K_i. Hence each N_i is closed under ultralimits. This fills the gap and concludes the proof.