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# ON JORDAN ALGEBRAS OF TYPE I

by

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## 1.1. INTRODUCTION

The general theory of Jordan algebras of self-adjoint operators on a Hilbert space is still at its infancy. In ((3)) Jordan, von Neumann, and Wigner classified the finite dimensional irreducible ones, and found that they are either the real symmetric matrices, all self-adjoint matrices, or the Clifford algebras. A great deal of the von Neumann algebra theory, specially the comparison theory for projections, has recently been pushed through by Topping ((6)) . Add to this some ideal theory ((2)) , and more special theory directly influenced by quantum mechanics (for references see ((4)) ), and most of the known theory is covered.

In this note we shall discuss one approach to the theory. It turns out that a JW-algebra, i.e. a weakly closed Jordan algebra of self-adjoint operators, can be decomposed along its center into three parts, one part being the self-adjoint part of a von Neumann algebra, one part algebraically the same as in the first (i.e. is reversible) but more real in the sense that the real symmetric matrices are real, and a third part which behaves more like the Clifford algebras. (Theorem 2.4). We shall mainly be concerned with reversible JW-algebras of type I, and shall show that an Abelian projection in a reversible JW-algebra $\mathfrak{A}$ is also Abelian in the double commutant $\mathfrak{A}''$ of $\mathfrak{A}$ (Theorem 3.5), and that $\mathfrak{A}''$ is of type I if $\mathfrak{A}$ is of type I. (Theorem 3.8). We shall then classify the irreducible, reversible JW-algebras of type I as either all self-adjoint operators or all the real symmetric operators with respect to a basis (Theorem 4.4). As a consequence we can characterize the pure vector states of such JW-algebras (corollary 4.5) and show that every vector is cyclic (viz. $[\mathfrak{A}x] = I$ ) .

## 1.2. NOTATION

A <u>JW-algebra</u> (resp. <u>JC-algebra</u>) $\mathfrak{A}$ is a weakly (resp. uniformly) closed linear space of self-adjoint operators on a Hilbert space $\mathcal{H}$ , which is also a Jordan algebra over the reals, i.e. $A \circ B = AB + BA \in \mathfrak{A}$ for all $A,B \in \mathfrak{A}$. Then $ABA$ and $ABC + CBA \in \mathfrak{A}$ for all $A,B,C \in \mathfrak{A}$. A Jordan ideal $\mathfrak{J}$ in $\mathfrak{A}$ is a linear subset of $\mathfrak{A}$ such that $A \in \mathfrak{A}$, $B \in \mathfrak{J}$ implies $A \circ B \in \mathfrak{J}$. A JW-factor is a JW-algebra with center the scalars (with respect to operator multiplication). A JC-algebra $\mathfrak{A}$ is irreducible if its commutant is the scalars; $\mathfrak{A}$ is Abelian if all operators in $\mathfrak{A}$ operator commute. A projection $E$ in $\mathfrak{A}$ is Abelian if $E\mathfrak{A}E$ is Abelian. If $\mathfrak{A}$ is a JW-algebra then $\mathfrak{A}$ is of type I if every non-zero projection in $\mathfrak{A}$ majorizes a non-zero Abelian projection in $\mathfrak{A}$. A similar definition holds for von Neumann algebras, where by a von Neumann algebra we mean a self-adjoint weakly closed algebra $\mathfrak{B}$ of operators on $\mathcal{H}$. If $\mathfrak{B}$ contains the identity operator then $\mathfrak{B}$ equals its double commutant $\mathfrak{B}''$, hence if a JW-algebra $\mathfrak{A}$ contains the identity operator, then $\mathfrak{A}''$ equals the von Neumann algebra generated by $\mathfrak{A}$. Let $\mathcal{R}(\mathfrak{A})$ denote the uniformly closed real algebra generated by $\mathfrak{A}$, $\mathfrak{A}$ now a JC-algebra. We say $\mathfrak{A}$ is reversible if $\prod_{i=1}^{n} A_i + \prod_{i=n}^{1} A_i \in \mathfrak{A}$ whenever $A_1, \dots, A_n \in \mathfrak{A}$. Then $\mathfrak{A}$ equals the self-adjoint part of $\mathcal{R}(\mathfrak{A})$. Moreover, if $(\mathfrak{A})$ denotes the $C^*$-algebra generated by $\mathfrak{A}$ then $\mathcal{R}(\mathfrak{A}) \cap i\mathcal{R}(\mathfrak{A})$ is an ideal in $(\mathfrak{A})$ (see ((5)) ). If $\mathcal{V} \subset \mathcal{H}$ we denote by $[\mathfrak{A}\mathcal{V}]$ the subspace of $\mathcal{H}$ generated by vectors of the form $Ax$, $A \in \mathfrak{A}$, $x \in \mathcal{V}$. We identify subspaces of $\mathcal{H}$ and their projections. If $\mathfrak{M}$ is a set of operators $\mathfrak{M}^-$ is the weak closure of $\mathfrak{M}$, and $\mathfrak{M}_{SA}$ the self-adjoint operators in $\mathfrak{M}$.

## 2. DECOMPOSITION OF JW-ALGEBRAS

We show the decomposition result announced in the introduction.

L e m m a 2.1. Let $\mathfrak{A}$ be a reversible JW-algebra. Then there exist central projections $E$ and $F$ in $\mathfrak{A}$ with $E + F = I$ such that $E\,\mathfrak{A}$ is the self-adjoint part of a von Neumann algebra, and $\mathcal{R}(F\mathfrak{A}) \cap i\,\mathcal{R}(F\mathfrak{A}) = \{0\}$.

P r o o f . Let $\mathfrak{J} = \mathcal{R}(\mathfrak{A}) \cap i\,\mathcal{R}(\mathfrak{A})$. Then $\mathfrak{J}$ is an ideal in $(\mathfrak{A})$, hence its weak closure $\mathfrak{J}^-$ is an ideal in $(\mathfrak{A})^-$. Thus there exists a central projection $E$ in $(\mathfrak{A})^-$ such that $\mathfrak{J}^- = E(\mathfrak{A})^-$ ((1, p. 45)), and $E \in \mathfrak{J}^-$. Now $\mathfrak{A}$ is reversible, hence $\mathfrak{J}_{SA} \subset \mathfrak{A}$, and $(\mathfrak{J}^-)_{SA} = (\mathfrak{J}_{SA})^- \subset \mathfrak{A}$. Thus $E \in \mathfrak{A}$, and $\mathfrak{J}^-_{SA} = E\,\mathfrak{A}$. Moreover, $\mathfrak{J}^-$ is a von Neumann algebra. Let $F = I - E$. Then $F$ is a central projection in $\mathfrak{A}$. Thus $\mathcal{R}(F\mathfrak{A}) \cap i\,\mathcal{R}(F\mathfrak{A}) = F(\mathcal{R}(\mathfrak{A}) \cap i\,\mathcal{R}(\mathfrak{A})) = F\,\mathfrak{J} \subset F(E(\mathfrak{A})^-) = \{0\}$. The proof is complete.

L e m m a 2.2. Let $\mathfrak{A}$ be a JC-algebra. Let $\mathfrak{J} = \{A \in \mathfrak{A} : BAC + C^*AB^* \in \mathfrak{A}$ for all $B, C \in \mathcal{R}(\mathfrak{A})\}$. Then $\mathfrak{J}$ is a uniformly closed Jordan ideal in $\mathfrak{A}$. Moreover, $\mathfrak{J}$ is a reversible JC-algebra.

P r o o f . Let $A, B \in \mathfrak{J}$, $S, T \in \mathcal{R}(\mathfrak{A})$. Then $S(A + B)T + T^*(A + B)S^* = (SAT + T^*AS^*) + (SBT + T^*BS^*) \in \mathfrak{A}$, so $\mathfrak{J}$ is linear. Let $A \in \mathfrak{J}$, $B \in \mathfrak{A}$, $S, T \in \mathcal{R}(\mathfrak{A})$. Then $S(AB + BA)T + T^*(AB + BA)S^* = (SA(BT) + (BT)^*AS^*) + ((SB)AT + T^*A(SB)^*) \in \mathfrak{A}$, so $A \circ B \in \mathfrak{J}$, and $\mathfrak{J}$ is a Jordan ideal in $\mathfrak{A}$. Since multiplication is uniformly continuous $\mathfrak{J}$ is uniformly closed. Let $A_1 \in \mathfrak{J}$, $A_2, \dots, A_n \in \mathfrak{A}$. Let $A = \prod_{i=2}^{n} A_i$. Then $A_1A + A^*A_1 \in \mathfrak{A}$ by definition of $\mathfrak{J}$. We show $A_1A + A^*A_1 \in \mathfrak{J}$, hence $\mathfrak{J}$ is in particular reversible (with $A_2, \dots, A_n \in \mathfrak{J}$). Let $B, C \in \mathcal{R}(\mathfrak{A})$. Then $B(A_1A + A^*A_1)C + C^*(A_1A + AA_1)B^*$ $= (BA_1(AC) + (AC)^*A_1B^*) + ((BA^*)A\,C + C^*A_1(BA^*)^*) \in \mathfrak{A}$.

The proof is complete.

D e f i n i t i o n 2.3. Let $\mathfrak{A}$ be a JC-algebra. We say is <u>totally non-reversible</u> if the ideal $\mathfrak{J}$ in Lemma 2.2 is zero.

T h e o r e m 2.4. Let $\mathfrak{A}$ be a JW-algebra. Then there exist three central projections $E, F, G$ in $\mathfrak{A}$ with $E + F + G = I$ such that

$E\mathfrak{A}$ is the self-adjoint part of a von Neumann algebra,

$F\mathfrak{A}$ is reversible and $\mathcal{R}(F\mathfrak{A}) \cap i\,\mathcal{R}(F\mathfrak{A}) = \{0\}$,

$G\mathfrak{A}$ is totally non-reversible.

P r o o f . Let $\mathfrak{J}$ be the ideal defined in Lemma 2.2. $\mathfrak{J}$ is weakly closed, in fact, if $A_\alpha \in \mathfrak{J}$, $A_\alpha \rightarrow A$ weakly, then for all $S, T \in \mathcal{R}(\mathfrak{A})$, $SA_\alpha T + T^*A_\alpha S^* \longrightarrow SAT + T^*AS^*$ weakly. Since $\mathfrak{A}$ is weakly closed $SAT + T^*AS^* \in \mathfrak{A}$, hence $A \in \mathfrak{J}$ as asserted. Let $H$ be the central projection in $\mathfrak{A}$ such that $H\mathfrak{A} = \mathfrak{J}$ (the existence of such an $H$ is shown by an easy modification of ((1, Cor. 3, p. 45)) ). Then $H\mathfrak{A}$ is reversible, and the existence of $E$ and $F$ follows from Lemma 2.1. Let $G = I - H$. We must show $G\mathfrak{A}$ is totally non-reversible. Let $A \in G\mathfrak{A}$. If for all $B, C \in \mathcal{R}(G\mathfrak{A}) = G\mathcal{R}(\mathfrak{A})$, $BAC + C^*AB^* \in G\mathfrak{A}$, then since $B = GS$, $C = GT$, $S, T \in \mathcal{R}(\mathfrak{A})$, $BAC + C^*AB^* = G(SAT + T^*AS^*) \in G\mathfrak{A}$, or, since $A = GA$, $SAT + T^*AS^* \in G\mathfrak{A} \subset \mathfrak{A}$ for all $S, T \in \mathcal{R}(\mathfrak{A})$. But then $A \in \mathfrak{J} = H\mathfrak{A}$. Thus $A = GA = HA = 0$, $G\mathfrak{A}$ is totally non-reversible. The proof is complete.

C o r o l l a r y 2.5 . A JW-factor is either reversible or totally non-reversible.

## 3. ABELIAN PROJECTIONS

We shall discuss the relationship between Abelian projections in a reversible JW-algebra $\mathfrak{A}$ and its double commutant $\mathfrak{A}''$, which by the double commutant theorem equals the von Neumann algebra generated by $\mathfrak{A}$.

L e m m a 3.1. Let $\mathfrak{A}$ be a reversible JW-factor and $E$ an Abelian projection in $\mathfrak{A}$. Then every operator in $E\mathcal{R}(\mathfrak{A})E$ is normal.

P r o o f . Let $A \in E\mathcal{R}(\mathfrak{A})E$. Then $A^*A$ and $AA^*$ are in $E\mathfrak{A}E$. By ((6, Theorem 14)) $E\mathfrak{A}E = \mathcal{C}E$, where $\mathcal{C}$ is the center of $\mathfrak{A}$, hence $A^*A = aE$, $AA^* = bE$ with $a$ and $b$ non-negative real numbers.

Thus $a^2E = (A^*A)^2 = A^*(AA^*)A = (A^*A)(AA^*) = A(A^*A)A^* = (AA^*)^2 = b^2E$ . Thus $a = b$ , $A$ is normal.

L e m m a 3.2. Let $\mathfrak{A}$ be a reversible JW-factor and $E$ an Abelian projection in $\mathfrak{A}$ . Let $A$ be self-adjoint in $E\,\mathfrak{A}''E$ . Then there exist two orthogonal projections $P$ and $Q$ with $P + Q = E$ and real numbers $u$ and $v$ such that

$$A = (u + v)P + (u - v)Q \quad .$$

P r o o f . We first assume $A = S + i\,T$ with $S$ and $T$ in $E\mathcal{R}(\mathfrak{A})E$ . Since $A$ is self-adjoint $A = \frac{1}{2}(A + A^*) = \frac{1}{2}(S + S^*) + i\,\frac{1}{2}(T - T^*)$ . $\frac{1}{2}(S + S^*) \in \mathfrak{A}$ since $\mathfrak{A}$ is reversible. As in Lemma 3.1 it follows that there exists a real number $u$ such that $\frac{1}{2}(S + S^*) = uE$ . Thus $(A - uE)^2 = (i\,\frac{1}{2}(T - T^*))^2 = -\frac{1}{4}(T - T^*)^2 \in \mathfrak{A}$ , hence equal to $v^2E$ with $v$ real. By spectral theory the $C^*$-algebra generated by $A$ and $E$ is isomorphic to $C(\sigma(A))$ - the continuous complex functions on the spectrum of $A$ - under a map which carries $A$ into the real function $\lambda \to \lambda$ , and $E$ into the constant function $\lambda \to 1$ . Thus $(\lambda - u)^2 = v^2$ for all $\lambda \in \sigma(A)$ . Thus $\sigma(A) = \{u + v, u - v\}$ . If $v = 0$ , $A = uE$ , if $v \neq 0$ let $P$ and $Q$ be the projections in the $C^*$-algebra generated by $A$ and $E$ corresponding to the characteristic functions for $u + v$ and $u - v$ respectively. Then $A = (u + v)P + (u - v)Q$ , and $P + Q = E$ .

Notice that $|u|, |v| \leq \frac{1}{2}(|u + v| + |u - v|) \leq \max\{|u + v|, |u - v|\} = \|A\|$ .

In the general case $A$ is self-adjoint in $E\,\mathfrak{A}''E$ . Now, $\mathcal{R}(\mathfrak{A}) + i\,\mathcal{R}(\mathfrak{A})$ is strongly dense in $\mathfrak{A}''$ , hence $E(\mathcal{R}(\mathfrak{A}) + i\,\mathcal{R}(\mathfrak{A}))E$ is strongly dense in $E\,\mathfrak{A}''E$ . By the Kaplansky density theorem ((1, Théorème 3, p. 46)) there exists a net $(A_\alpha)$ of self-adjoint operators $A_\alpha$ in $E(\mathcal{R}(\mathfrak{A}) + i\,\mathcal{R}(\mathfrak{A}))E$ such that $\|A_\alpha\| \leq \|A\|$ and $A_\alpha \to A$ strongly. From the first part of the proof there exist real numbers $v_\alpha$ , $u_\alpha$ with

$|u_\alpha|$, $|v_\alpha| \leq \|A_\alpha\| \leq \|A\|$ such that $(A_\alpha - u_\alpha E)^2 = 4v_\alpha^2 E$. Let $u$ be a limit point of the $u_\alpha$ and $v$ a limit point of the $v_\alpha$. Consider a subnet $(A_j)$ of $(A_\alpha)$ for which $u_j \longrightarrow u$, $v_j \longrightarrow v$. Since multiplication is strongly continuous on the unit ball $(A_j - u_jE)^2 \longrightarrow (A - uE)^2$ strongly. Also $(A_j - u_jE)^2 = 4v_j^2E \rightarrow 4v^2E$ strongly. Thus $(A - uE)^2 = 4v^2E$. As in the first part of the proof there exist orthogonal projections $P$ and $Q$ with sum $E$ such that $A = (u + v)P + (u - v)Q$. The proof is complete.

The key lemma follows.

L e m m a 3.3. Let $\mathfrak{A}$ be an irreducible, reversible JW-factor and $E$ an Abelian projection in $\mathfrak{A}$. If $x$ is a non-zero vector in $E$ then $E = [x]$.

= all operators on $\mathcal{H}$

P r o o f . In this case $\mathfrak{A}'' = \mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is the underlying Hilbert space. Thus, if $A$ is any self-adjoint operator on $\mathcal{H}$ then $EAE = (u + v)P + (u - v)Q$ where $u$, $v$, $P$, and $Q$ are as in Lemma 3.2. If $v = 0$ for all self-adjoint $A$ then $A = uE$ for all $A$, and $E = [x]$. Otherwise $E\mathcal{B}(\mathcal{H})E$ must be isomorphic to $\mathcal{M}_2$ - the complex $2 \times 2$ matrices - and $\dim E = 2$. Then $E\mathcal{R}(\mathfrak{A})E$ is isomorphic to a real algebra $\mathcal{R}_2$ in $\mathcal{M}_2$, and $\mathcal{R}_2 + i\mathcal{R}_2$ is weakly dense in $\mathcal{M}_2$. Thus $\mathcal{R}_2$ is either the set of all matrices with real coefficients relative to a basis, or $\mathcal{R}_2 = \mathcal{M}_2$. In either case $\mathcal{R}_2$ contains non-normal operators. This contradicts Lemma 3.1. Thus $E = [x]$.

L e m m a 3.4 . Let $\mathfrak{A}$ be a reversible JC-algebra - and $E$ an Abelian projection in $\mathfrak{A}$. Then $E$ is Abelian in $(\mathfrak{A})$.

P r o o f . Let $\mathcal{B} = (\mathfrak{A})$ and let $\varphi$ be an irreducible representation of $\mathcal{B}$. If $\varphi(E) = 0$ then $\varphi(E\mathcal{B}E) = 0$ is Abelian. If $F = \varphi(E) \neq 0$ then $F$ is Abelian in $\varphi(\mathfrak{A})$, hence by continuity Abelian in $\varphi(\mathfrak{A})^-$, an irreducible, reversible JW-factor (it is reversible by the Kaplansky density theorem and the fact that multiplication is strongly continuous on the unit ball). By Lemma 3.3 $F = [x]$, with $x \in F$.

Hence $\varphi(E\mathcal{B}E) = [x]\varphi(\mathcal{B})[x] = \mathbb{C}[x]$ is Abelian. Since the irreducible representations separate $\mathcal{B}$ they separate $E\mathcal{B}E$, which is thus Abelian.

T h e o r e m 3.5. Let $\mathfrak{A}$ be a reversible JW-algebra and E an Abelian projection in $\mathfrak{A}$. Then E is Abelian in $\mathfrak{A}''$.

P r o o f. E is Abelian in $(\mathfrak{A})$ by Lemma 3.4, hence, by strong continuity, in $(\mathfrak{A})^-$, hence in $\mathfrak{A}''$ ((1, p. 44)).

We shall now use Theorem 3.5 in order to show that if $\mathfrak{A}$ is of type I then $\mathfrak{A}''$ is of type I. The proof of the next lemma is a direct copy of the proof of ((1, Théorème 1, (iii) $\Rightarrow$ (v), p. 123)), and will therefore be omitted. Recall that the central carrier of a projection is the least central projection greater than or equal to it.

L e m m a 3.6. If $\mathfrak{A}$ is a (not necessarily reversible) JW-algebra of type I then there exists an Abelian projection in $\mathfrak{A}$ with central carrier I.

L e m m a 3.7. Let $\mathfrak{A}$ be a JW-algebra and E a projection in $\mathfrak{A}$. Then $[\mathfrak{A}E] = [\mathfrak{A}''E]$ is the central carrier for E with respect to both $\mathfrak{A}$ and $\mathfrak{A}''$.

P r o o f. By ((1, Corollaire 1, p. 7)) $[\mathcal{B}E] = C_E$ is the central carrier for E in $\mathfrak{A}'' = \mathcal{B}$. Clearly $[\mathfrak{A}E] \leq [\mathcal{B}E]$. Now $[\mathfrak{A}E] \in \mathfrak{A}'$. In fact, if $x \in E$, A,B $\in \mathfrak{A}$ then BAEx = (BAE + EAB)x - EABx $\in [\mathfrak{A}x] \vee E \leq [\mathfrak{A}E]$. Thus B leaves $[\mathfrak{A}E]$ in variant, $[\mathfrak{A}E] \in \mathfrak{A}'$. Moreover, $[\mathfrak{A}E] \in \mathfrak{A}$. In fact, if $A \in \mathfrak{A}$ then if $\mathcal{R}(B)$ denotes the range projection of an operator B, then $\mathcal{R}(AE) = \mathcal{R}(AE(AE)^*) = \mathcal{R}(AEA) \in \mathfrak{A}$, by spectral theory and the fact that $\mathfrak{A}$ is weakly closed. Thus $[\mathfrak{A}E] = \bigvee_{A\in\mathfrak{A}} \mathcal{R}(AE) \in \mathfrak{A}$, as asserted. Thus $[\mathfrak{A}E]$ belongs to the center of $\mathfrak{A}$, which in turn is contained in the center of $\mathfrak{A}''$. Since $[\mathfrak{A}E]E = E$, $[\mathfrak{A}E] \geq C_E$, hence

$[\mathfrak{A} E] = [\mathfrak{B} E] = C_E$ . The proof is complete.

T h e o r e m 3.8 . If $\mathfrak{A}$ is a reversible JW-algebra of type I then $\mathfrak{A}''$ is a von Neumann algebra of type I.

P r o o f . By Lemma 3.6 there exists an Abelian projection $E$ in $\mathfrak{A}$ with central carrier $C_E$ equal to $I$ relative to $\mathfrak{A}$ . By Lemma 3.7 $[\mathfrak{A} E] = C_E$ , and the central carrier of $E$ relative to $\mathfrak{A}''$ equals $I$ . By Theorem 3.5 $E$ is Abelian in $\mathfrak{A}''$ . Thus $\mathfrak{A}''$ has an Abelian projection with central carrier $I$ . By ((1, Théorème 1, p. 123)) $\mathfrak{A}''$ is of type I . The proof is complete.

We leave the converse of the above theorem as an open question.

We refer the reader to ((6)) for the definition and properties of finite JW-algebras.

T h e o r e m 3.9 . Let $\mathfrak{A}$ be a finite, infinite dimensional JW-factor of type I (for the existence of such an $\mathfrak{A}$ see ((6)) ). Then $\mathfrak{A}$ is totally non-reversible.

P r o o f . By Corollary 2.5 $\mathfrak{A}$ is either reversible or totally non-reversible. Assume $\mathfrak{A}$ is reversible. Since $\mathfrak{A}$ is a finite JW-factor of type I it is clear that $\mathfrak{A}$ is simple (since all Abelian projections in $\mathfrak{A}$ are equivalent ((6)) ) . Let $\varphi$ be an irreducible representation of $(\mathfrak{A})$ . Then $\varphi | \mathfrak{A} \neq 0$ , hence is an isomorphism, and $\varphi(\mathfrak{A})$ is irreducible. Let $E_1 , \dots , E_n$ be orthogonal Abelian projections in $\mathfrak{A}$ with $\sum_{i=1}^{n} E_i = I$ . Then $\varphi(E_i)$ is Abelian in $\varphi(\mathfrak{A})$ , hence in $\varphi(\mathfrak{A})^-$ . Since $\varphi(\mathfrak{A})$ is reversible, so is $\varphi(\mathfrak{A})^-$ , hence $\varphi(E_i) = [x_i]$ by Lemma 3.3. Thus $\varphi(\mathfrak{A})^-$ is of finite dimension, hence $\mathfrak{A}$ is of finite dimension, a contradiction.

## 4. IRREDUCIBLE JW-ALGEBRAS

We shall now characterize the irreducible, reversible JW-algebras of type I.

L e m m a 4.1 . Let $\mathfrak{A}$ be an irreducible, reversible JW-algebra of type I. If $\mathcal{R}(\mathfrak{A}) \cap i\mathcal{R}(\mathfrak{A}) = \{0\}$ then there exists an orthonormal basis $(x_e)_{e \in I}$ for $\mathcal{H}$ such that for all pairs $x_e$, $x_k$ in the basis $(A x_e, x_k)$ is real for all $A \in \mathcal{R}(\mathfrak{A})$.

P r o o f . Let $E$ be an Abelian projection in $\mathfrak{A}$. Then $E = [x_1]$ with $x_1$ a unit vector in $\mathcal{H}$, by Lemma 3.3. If $A \in \mathcal{R}(\mathfrak{A})$ then $E A E = \lambda E$ with $\lambda$ real. In fact, if $\lambda = u + iv$ with $u$ and $v$ real and $v \neq 0$ then $iE = \frac{1}{v}(E A E - u E) \in \mathcal{R}(\mathfrak{A})$, contradicting the fact that $\mathcal{R}(\mathfrak{A}) \cap i\mathcal{R}(\mathfrak{A}) = 0$. In particular, $(A x_1, x_1) = ([x_1] A [x_1] x_1, x_1) = \lambda \|x_1\|^2$ is real for all $A$ in $\mathcal{R}(\mathfrak{A})$. Let $\mathcal{J} = \{x_1, x_2, \dots \}$ be a maximal set of orthonormal vectors $x_e$ in $\mathcal{H}$ such that

1) $[x_e] \in \mathfrak{A}$, $x_e \in \mathcal{J}$

2) For each $x_e \in \mathcal{J}$ there exists $A_e \in \mathfrak{A}$ with $x_e = A_e x_1$.

Then for each $A \in \mathcal{R}(\mathfrak{A})$, $x_e$, $x_k \in \mathcal{J}$, $(Ax_e, x_k) = (AA_e x_1, A_k x_1) = (A_k A A_e x_1, x_1)$ is real by the preceding. We show $[\mathcal{J}] = I$. Since $\mathfrak{A}$ is weakly closed and each $[x_e] \in \mathfrak{A}$, $[\mathcal{J}] \in \mathfrak{A}$. Assume $[\mathcal{J}] \neq I$. Then there exists an Abelian projection $E$ in $\mathfrak{A}$, $E \leq I - [\mathcal{J}]$. Let $y$ be a unit vector in $E$. By Lemma 3.3 $E = [y]$. Moreover, there exists a (self-adjoint) unitary operator $S$ in $\mathfrak{A}$ such that $S[y]S = [x_1]$ ((6, Corollary 17)). Then $z = [y] S = S [x_1] = [y] S [x_1] \neq 0$. Let $T = [y] S [x_1] + [x_1] S [y]$. Then $T \in \mathfrak{A}$ and $Tx_1 = z$, and $z \in [y] \leq I - [\mathcal{J}]$. This contradicts the maximality of $\mathcal{J}$, $\therefore$ $[\mathcal{J}] = I$, the proof is complete.

D e f i n i t i o n 4.2 . Let $\mathfrak{A}$ be an irreducible, reversible JW-algebra acting on a Hilbert space $\mathcal{H}$. We say $\mathfrak{A}$ is the <u>real symmetric operators in $\mathcal{B}(\mathcal{H})$ with respect to a basis</u> $(x_e)_{e \in I}$ for $\mathcal{H}$ if $\mathfrak{A}$ consists of all self-adjoint operators such that $(Ax_e, x_k)$ is real for all $x_e$, $x_k$ in the basis. ($\mathcal{B}(\mathcal{H})$ denotes all bounded operators on $\mathcal{H}$).

T h e o r e m 4.3. Let $\mathfrak{A}$ be an irreducible, reversible JW-algebra of type I acting on a Hilbert space $\mathcal{H}$. Then

1) If $\mathcal{R}(\mathfrak{A}) \cap i\,\mathcal{R}(\mathfrak{A}) \neq \{0\}$ then $\mathfrak{A} = \mathcal{B}(\mathcal{H})_{SA}$.

2) If $\mathcal{R}(\mathfrak{A}) \cap i\,\mathcal{R}(\mathfrak{A}) = \{0\}$ then $\mathfrak{A}$ is the real symmetric operators in $\mathcal{B}(\mathcal{H})$ with respect to a basis $(x_e)_{e \in I}$.

P r o o f. By Theorem 2.4 and Lemma 4.1 it remains to show that in case 2) $\mathfrak{A}$ consists of all self-adjoint operators such that $(Ax_e, x_k)$ is real for all $x_e$, $x_k$ in the basis. Let $A$ be a self-adjoint operator such that $(Ax_e, x_k)$ is real for all $x_e, x_k$ in the basis. Let $E$ be a projection of the form $\sum_{i=1}^{n} [x_{e_i}]$, $e_i \in I$. Then $E$ is an $n$-dimensional projection in $\mathfrak{A}$. Clearly $E \mathfrak{A} E \subset \mathfrak{A}$ and is isomorphic to the real $n \times n$ symmetric matrices with respect to the basis $(x_{e_i})_{i=1,\dots,n}$. Thus $EAE \in \mathfrak{A}$. Since the net $E_n$ of such projections converges strongly to $I$, $E_n A E_n \to A$ strongly. Since $\mathfrak{A}$ is strongly closed $A \in \mathfrak{A}$. The proof is complete.

The next result shows which 1-dimensional projections belong to $\mathfrak{A}$ with $\mathfrak{A}$ as in Theorem 4.3. It suffices to consider the case when $\mathfrak{A}$ is the real symmetric operators with respect to a basis. Recall that a vector state $\omega_x$ of $\mathfrak{A}$ is a state of the form $A \to (Ax, x)$.

T h e o r e m 4.4. Let $\mathfrak{A}$ be the real symmetric operators in $\mathcal{B}(\mathcal{H})$ with respect to a basis $(x_e)_{e \in I}$. Let $x = \sum_{e \in I} \lambda_e x_e$ be a unit vector in $\mathcal{H}$. Then the following are equivalent.

1) $[x] \in \mathfrak{A}$

2) $\omega_x$ is a pure state on $\mathfrak{A}$

3) $\lambda_e \bar{\lambda}_k$ is real for all $e, k \in I$.

P r o o f . Clearly 1) $\Longrightarrow$ 2) . Also 3) $\Longrightarrow$ 1) since $([x]x_e, x_k) = \bar{\lambda}_e \lambda_k$ is real for all $e,k \in I$ implies $[x] \in \mathfrak{A}$ by Theorem 4.3.

Assume 2) . Let $\lambda_e = u_e + i\, v_e$ with $u_e, v_e$ real. Then $\sum_{e\in I} u_e^2 + \sum_{e\in I} v_e^2 = \sum_{e\in I} |\lambda_e|^2 = 1$ , so the vectors $y = \sum_{e\in I} u_e x_e$ and $z = \sum_{e\in I} v_e x_e$ have norms $\leq 1$ . Moreover, by 3) $\Longrightarrow$ 1) $[y]$, $[z] \in \mathfrak{A}$. Let $A \in \mathfrak{A}$ . Then $\omega_x(A) = (A(y + iz), y+iz) = \omega_y(A) + \omega_z(A) + i(Az,y) - i(Ay,z)$ . But $(Az,y) = (z,Ay)$ is real, hence $(Az,y) = (Ay,z)$, because

$$(Az,y) = (A \sum_{e\in I} u_e x_e \, , \sum_{k\in I} v_k x_k) = \sum_{e,k\in I} u_e v_k (A x_e \, , x_k)$$

is real. Thus $\omega_x = \omega_y + \omega_z$ on $\mathfrak{A}$ . Since $\omega_x$ is pure $\omega_y = k\omega_z$ , $k$ a positive real number, unless $\omega_z = 0$ or $\omega_y = 0$ , in which case we are through . Since $[y]$ , $[z] \in \mathfrak{A}$ it follows that $[y] = [z]$ , and $y = \lambda z$ with $\lambda$ a complex number. Thus $u_e = \lambda v_e$ , and

$$\lambda_e \bar{\lambda}_k = (\lambda v_e + i\, v_e)(\bar{\lambda} v_k - i\, v_k) = v_e v_k (\lambda + i)(\bar{\lambda} - i)$$

is real. Thus 2) $\Longrightarrow$ 3) . The proof is complete.

In particular we have shown

C o r o l l a r y 4.5 . Let $\mathfrak{A}$ be the real symmetric operators in $\mathcal{B}(\mathcal{H})$ with respect to a basis. Let $x \in \mathcal{H}$ . Then $\omega_x = \omega_y + \omega_z$ on $\mathfrak{A}$ , where $\omega_y$ and $\omega_z$ are pure states on $\mathfrak{A}$ . Moreover, $\|x\|^2 = \|y\|^2 + \|z\|^2$ .

As for von Neumann algebras it seems that a good understanding of the cyclic projections will solve many of the problems of JW-algebras. However, while cyclic projections of the form $[\mathfrak{A} x]$ with $\mathfrak{A}$ a von Neumann algebra belong to the commutant $\mathfrak{A}'$ of $\mathfrak{A}$ , this is not so for JW-algebras. This even fails for reversible JW-factors of type I . However, we have

T h e o r e m 4.6 . Let $\mathfrak{A}$ be an irreducible, reversible JW-algebra of type I acting on the Hilbert space $\mathcal{H}$ . If $x$ is a non-zero vector in $\mathcal{H}$ then $[\mathfrak{A} x] = I$ .

For the proof we shall need

L e m m a 4.7 . Let $\mathfrak{A}$ be a reversible JC-algebra. Let $E$ be a projection in $\mathfrak{A}$ and $x$ a vector in $I - E$ . If $[\mathcal{C}(\mathfrak{A})x] = I$ then $E \leq [\mathfrak{A}x]$ .

P r o o f . Denote by $\mathfrak{A}^n$ the uniformly closed self-adjoint linear space of operators generated by operators of the form $\prod_{i=1}^{n} A_i$ with $A_i \in \mathfrak{A}$ . Then $E[\mathfrak{A}^n x] = E[\mathfrak{A} x]$ for all $n = 1, 2, \dots$ . Indeed, since $Ex = 0$ , $E\prod_{i=1}^{n} A_i x = (E \prod_{i=1}^{n} A_i + \prod_{i=n}^{1} A_i E)x \in [\mathfrak{A}x] \leq [\mathfrak{A}^n x]$ . In particular, $E$ maps $[\mathfrak{A}^n x]$ into $[\mathfrak{A} x]$ . Since $[\mathcal{C}(\mathfrak{A})x] = I$ the projections $[\mathfrak{A}^n x]$ converge strongly to $I$ . Thus $E[\mathfrak{A}^n x] \to E$ strongly, and $E[\mathfrak{A} x] = \lim E[\mathfrak{A}^n x] = E$ , so $[\mathfrak{A} x] \geq E$ .

P r o o f o f T h e o r e m 4.6 . If $\mathfrak{A} = \mathcal{B}(\mathcal{H})_{SA}$ the theorem is clear. By Theorem 4.3 we may assume $\mathfrak{A}$ is the real symmetric operators in $\mathcal{B}(\mathcal{H})$ with respect to a basis $(x_e)_{e \in I}$ . From the proof of Lemma 4.1 there exists $A_e \in \mathfrak{A}$ such that $x_e = A_e x_1$ for all $e \in I$ . Thus $[\mathfrak{A} x_1] = I$ . Since $x_1$ was any vector such that $[x_1] \in \mathfrak{A}$ , it follows that $[\mathfrak{A} x] = I$ whenever $[x] \in \mathfrak{A}$ . Let now $x$ be any unit vector and assume $[x] \notin \mathfrak{A}$ . By Corollary 4.5 there exist vectors $y$ and $z$ with $[y]$ , $[z] \in \mathfrak{A}$ , $x = y + iz$ , $\|y\|^2 + \|z\|^2 = \|x\|^2 = 1$ , and $\omega_x = \omega_y + \omega_z$ on $\mathfrak{A}$ . Let $E = [y] + [z]$ . Then $\omega_x(E) = \omega_y(E) + \omega_z(E) = \|y\|^2 + \|z\|^2 = 1$ , and $x \in E$ . The theorem is easily proved in case $\dim \mathcal{H} = 2$ . Therefore $[E \mathfrak{A} E x] = E$ , hence $[\mathfrak{A} x] \geq E$ . Assume now $[\mathfrak{A} x] \neq I$ . Let $y \in I - [\mathfrak{A} x]$ . Then $y \in I - E$ . Hence, by Lemma 4.7 $E \leq [\mathfrak{A} y]$ . In particular $x \in [\mathfrak{A} y]$ . Thus there exists $A, B \in \mathfrak{A}$ such that $\|(A + iB)y - x\| < \frac{1}{2}$ .
Then

$\left|((A + i\,B)y,x) - 1\right| = \left|((A + i\,B)y - x,x)\right| \leq \|(A + i\,B)y - x\| < \frac{1}{2}$ ,
and $(y,(A - i\,B)x) = ((A + i\,B)y,x) \neq 0$ , contrary to the assumption that $y \in I - [\mathfrak{A}\,x]$ . Thus $[\mathfrak{A}\,x] = I$ , the proof is complete.

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