Matematisk Seminar
Universitetet i Oslo

Nr. 11
Oktober 1964

NOTE ON LIE ALGEBRAS

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INTRODUCTION

All algebras considered will have a fixed field as scalars. By an <u>associative algebra</u> we will understand an algebra with the associative law of multiplication and a unit element 1. A homomorphism between associative algebras is supposed to map 1 into 1.

An associative algebra A may be considered as a Lie algebra with the commutator product $[x,y] = xy - yx$; $x,y \in A$. When we speak of a homomorphism $\mathfrak{g} \longrightarrow A$ from a Lie algebra $\mathfrak{g}$ into A, we mean into A with this Lie algebra structure.

It is a consequence of the Poincaré-Birkhoff-Witt theorem that any Lie algebra $\mathfrak{g}$ is isomorphic to a Lie subalgebra of some associative algebra A. This implies that $\mathfrak{g}$ has a faithful representation as a Lie algebra of linear transformations on a vector space, because the left regular representation of A, taking a A into the linear transformation $x \longmapsto ax$ of A, is faithful.

A <u>universal enveloping algebra</u> of $\mathfrak{g}$ is defined to be a pair (U,α) consisting of an associative algebra U and a homomorphism $\alpha : \mathfrak{g} \longrightarrow U$ such that the following universal property is satisfied. If A is an associative algebra and $f : \mathfrak{g} \longrightarrow A$ is a homomorphism, then there exists a unique homomorphism $f' : U \longrightarrow A$ such that the diagram

$$\begin{tikzcd} U \arrow[dr, "f'"] & \\ \mathfrak{g} \arrow[u, "\alpha"] \arrow[r, "f"] & A \end{tikzcd}$$

commutes.

A universal enveloping algebra (U,α) of $\mathfrak{g}$ is unique up to isomorphism. Furthermore it exists as a quotient of the tensor algebra T on the vector space $\mathfrak{g}$ with respect to the ideal I generated by elements of the form $gg' - g'g - [g,g']$, where $g,g' \in \mathfrak{g}$.

Let $(g_i)_{i\in I}$ be a basis for the vector space $\mathfrak{g}$, where I is an ordinal. By a <u>standard monomial</u> of degree p, $p \geq 0$, we mean a monomial $g_{i_1} \cdots g_{i_p} \in T$ of basis elements in $\mathfrak{g}$ such that $i_1 \leq \ldots \leq i_p$, if $p > 0$ and 1 if $p = 0$.

T h e o r e m . <u>Poincaré-Birkhoff-Witt. The cosets of the standard monomials in T modulo I form a basis for the vector space U.</u>

The theorem may be formulated without reference to a basis in $\mathfrak{g}$ by introducing the symmetric algebra on $\mathfrak{g}$. As a corollary of the theorem $\alpha : \mathfrak{g} \longrightarrow U$ is injective. We identify $\mathfrak{g}$ with the Lie subalgebra $\alpha(\mathfrak{g})$ of U by means of the isomorphism α.

I. A GENERALIZATION OF THE POINCARÉ-BIRKHOFF-WITT THEOREM

Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras with subalgebras $\mathfrak{g}_o$ and $\mathfrak{h}_o$ respectively, and $\varphi : \mathfrak{g}_o \longrightarrow \mathfrak{h}_o$ an isomorphism. We define <u>a free product of $\mathfrak{g}$ and $\mathfrak{h}$ with identification φ</u> to be a triple $(\mathfrak{k}, \alpha, \beta)$ consisting of a Lie algebra $\mathfrak{k}$ and homomorphisms $\alpha : \mathfrak{g} \longrightarrow \mathfrak{k}$ and $\beta : \mathfrak{h} \longrightarrow \mathfrak{k}$ such that the diagram

I

$$\begin{tikzcd}
 & \mathfrak{h}_o \arrow[r] & \mathfrak{h} \arrow[dd, "\beta"] \\
\mathfrak{g}_o \arrow[ur, "\varphi"] \arrow[d] & & \\
\mathfrak{g} \arrow[rr, "\alpha"] & & \mathfrak{k}
\end{tikzcd}$$

is commutative and such that the following universal property is satisfied. If $\mathfrak{k}'$ is a Lie algebra and $\alpha' : \mathfrak{g} \longrightarrow \mathfrak{k}'$ and $\beta' : \mathfrak{h} \longrightarrow \mathfrak{k}'$ are homomorphisms such that the corresponding diagram I' commutes, then there exists a unique homomorphism $f : \mathfrak{k} \longrightarrow \mathfrak{k}'$ such that the diagram

II

$$\begin{tikzcd}
\mathfrak{k} \arrow[dr, "f"] & \mathfrak{h} \arrow[l, "\beta"'] \arrow[d, "\beta'"] \\
\mathfrak{g} \arrow[u, "\alpha"] \arrow[r, "\alpha'"] & \mathfrak{k}'
\end{tikzcd}$$

commutes.

P r o p o s i t i o n 1 . <u>A free product of $\mathfrak{g}$ and $\mathfrak{h}$ with identification φ exists and is unique up to isomorphism.</u>

P r o o f . The uniqueness follows from the universal property. To prove existence we proceed as follows. Let T be the tensor algebra on the vector space $\mathfrak{g} + \mathfrak{h}$. Let I be the ideal in T generated by the elements of the form

$$gg' - g'g - [g,g'] \qquad g,g' \in \mathfrak{g}$$

$$hh' - h'h - [h,h'] \qquad h,h' \in \mathfrak{h}$$

$$g - \varphi(g) \qquad g \in \mathfrak{g}.$$

We obtain a quotient algebra $U = T/I$ and homomorphisms $\alpha : \mathfrak{g} \longrightarrow U$ and $\beta : \mathfrak{h} \longrightarrow U$. Let $\mathfrak{k}$ be the Lie subalgebra of U generated by $\alpha(\mathfrak{g}) \cup \beta(\mathfrak{h})$. We have then homomorphisms $\alpha : \mathfrak{g} \longrightarrow \mathfrak{k}$ and $\beta : \mathfrak{h} \longrightarrow \mathfrak{k}$ such that the diagram I commutes.

To prove that $(\mathfrak{k}, \alpha, \beta)$ satisfies the universal property of a free product with identification, suppose $\mathfrak{k}'$ is a Lie algebra and $\alpha' : \mathfrak{g} \longrightarrow \mathfrak{k}'$ and $\beta' : \mathfrak{h} \longrightarrow \mathfrak{k}'$ are homomorphisms making the corresponding diagram I' commutative. Let U' be the universal enveloping algebra of $\mathfrak{k}'$. $\alpha' : \mathfrak{g} \longrightarrow \mathfrak{k}'$ and $\beta' : \mathfrak{h} \longrightarrow \mathfrak{k}'$ extend to a linear map $\mathfrak{g} + \mathfrak{h} \longrightarrow \mathfrak{k}'$ and therefore to a homomorphism $T \longrightarrow U'$ annihilating I. This gives a homomorphism $f : U \longrightarrow U'$ such that the diagram

$$\begin{tikzcd} U \arrow[dr, "f"] & \mathfrak{h} \arrow[l, "\beta"'] \arrow[d, "\beta'"] \\ \mathfrak{g} \arrow[u, "\alpha"] \arrow[r, "\alpha'"] & U' \end{tikzcd}$$

commutes.

$f : U \longrightarrow U'$ is a homomorphism of Lie algebras as well. $f^{-1}(\mathfrak{k}')$ is a Lie subalgebra of U containing $\alpha(\mathfrak{g}) \cup \beta(\mathfrak{h})$, hence containing $\mathfrak{k}$. Therefore f gives by restriction a homomorphism $f : \mathfrak{k} \longrightarrow \mathfrak{k}'$ such that the diagram II commutes.

Since $\alpha(\mathfrak{g}) \cup \beta(\mathfrak{h})$ generates $\mathfrak{k}$ such an homomorphism f is unique. This proves the universal property of $(\mathfrak{k}, \alpha, \beta)$.

We write $\mathfrak{g} \circ_{\varphi} \mathfrak{h}$ for the free product of $\mathfrak{g}$ and $\mathfrak{h}$ with identification φ. If in particular $\mathfrak{g}_0 = \mathfrak{h}_0 = 0$, we get the free product $\mathfrak{g} \circ \mathfrak{h}$, and the diagram I reduces to a coproduct diagram.

P r o p o s i t i o n 2. <u>U is the universal enveloping algebra of its Lie subalgebra</u> $\mathfrak{g} \circ_{\varphi} \mathfrak{h}$

P r o o f . $\mathfrak{k} = \mathfrak{g} \circ_{\varphi} \mathfrak{h}$ is a Lie subalgebra of $U = T/I$. Suppose A is an associative algebra and $f : \mathfrak{k} \longrightarrow A$ a homomorphism. The homomorphisms $f\alpha : \mathfrak{g} \longrightarrow A$ and $f\beta : \mathfrak{h} \longrightarrow A$ extend to a linear map $\mathfrak{g} + \mathfrak{h} \longrightarrow A$ and from there to a homomorphism $T \longrightarrow A$ annihilating I.

This gives a homomorphism $f' : U \to A$ such that the diagram

$$\begin{tikzcd} U \arrow[drr, "f'"] & & \mathfrak{h} \arrow[ll, "\beta"'] \arrow[d, "f\beta"] \\ & & \\ \mathfrak{g} \arrow[uu, "\alpha"] \arrow[rr, "f\alpha"'] & & A \end{tikzcd}$$

commutes. From $f'\alpha = f\alpha$ and $f'\beta = f\beta$ follows that $f' = f$ on $\alpha(\mathfrak{g}) \cup \beta(\mathfrak{h})$, hence on $\mathfrak{R}$. Thus the diagram

$$\begin{tikzcd} U \arrow[dr, "f'"] & \\ \mathfrak{R} \arrow[u] \arrow[r, "f"'] & A \end{tikzcd}$$

commutes. $\mathfrak{g} \cup \mathfrak{h}$ generates T. Hence $\alpha(\mathfrak{g}) \cup \beta(\mathfrak{h})$ generates U. Since $\mathfrak{R} \supseteq \alpha(\mathfrak{g}) \cup \beta(\mathfrak{h})$, $\mathfrak{R}$ generates U and f' is uniquely determined by the fact that the last diagram is commutative.

Let $(g_i)_{i \in I}$ and $(h_j)_{j \in J}$ be bases for the vector spaces $\mathfrak{g}$ and $\mathfrak{h}$ respectively with the following property. I and J are ordinals, and there exists an ordinal $K \leq I, J$ such that $(g_i)_{i \in K}$ is a basis for $\mathfrak{g}_0$, $(h_j)_{j \in K}$ is a basis for $\mathfrak{h}_0$ and $h_i = \varphi(g_i)$ for all $i \in K$.

By a <u>monomial</u> of degree p, $p \geq 0$, we shall understand an element $\prod_{n=1}^{p} x_n \in T$ where each x_n is equal to one of the basis elements in $\mathfrak{g}$ or $\mathfrak{h}$ if $p > 0$, and $1 \in T$ if $p = 0$. The monomials form a basis for the vector space T.

The <u>index</u> ζ of a monomial $\prod_{n=1}^{p} x_n$ is defined as follows. If m,n is given with $1 \leq m \leq n \leq p$ we define $\zeta_{mn} = 1$ if one of the following cases occurs

a) $x_m = g_i, h_i$ and $x_n = g_j, h_j$ with $j < i$, $j \in K$.

b) $x_m = g_i$ and $x_n = g_j$ with $j < i$, and for all l such that $m \leq l \leq n$ we have $x_l \in \mathfrak{g} \cup \mathfrak{h}_o$.

c) $x_m = h_i$ and $x_n = h_j$ with $j < i$, and for all l such that $m \leq l \leq n$ we have $x_l \in \mathfrak{g}_o \cup \mathfrak{h}$.

Otherwise we define $\zeta_{mn} = 0$. Then $\zeta = \sum_{m \leq n} \zeta_{mn}$. This goes for $p > 0$. For $p = 0$ we put $\zeta = 0$.

A monomial $\prod_{n=1}^{p} x_n$ is called <u>standard</u> if $x_n \notin \mathfrak{h}_o$ for all n and $\zeta = 0$.

T h e o r e m 3 . <u>The cosets of the standard monomials in T modulo I form a basis for the vector space U .</u>

In the case $\mathfrak{h} = 0$ this is the Poincaré-Birkhoff-Witt theorem for $\mathfrak{g}$.

L e m m a 4 . <u>Every element in T is equal modulo I to a linear combination of standard monomials.</u>

P r o o f . Let T_p be the subspace of T spanned by monomials of degree p , and T_{ps} the subspace of T spanned by monomials of degree p and index $\leq s$. Then we have a direct sum $T = \coprod_{p \geq 0} T_p$ and for each p a finite filtration $0 \subseteq T_{po} \subseteq T_{p1} \cdots \subseteq T_p$.

As an induction hypothesis suppose the lemma is true for elements in $\coprod_{r<p} T_r + T_{p,s-1}$. Consider a monomial $\prod_{n=1}^{p} x_n$ in T_{ps} . We shall prove

that $\prod_{n=1}^{p} x_n$ is equal modulo I to a linear combination of standard monomials. Since $h_i = g_i$ modulo I if $i \in K$, we may assume $x_n \notin \mathfrak{h}_0$ for all n.

If $s = 0$, $\prod_{n=1}^{p} x_n$ is standard and we are finished. Suppose $s > 0$. Then there exists a q with $1 \leq q < p$ such that one of the following cases occurs.

A) $x_q = h_i$, $x_{q+1} = g_j$ with $j < i$, $j \in K$.

B) $x_q = g_i$, $x_{q+1} = g_j$ with $j < i$ or
$x_q = h_i$, $x_{q+1} = h_j$ with $j < i$.

If we introduce the commutator product $\{x, x'\} = xx' - x'x$ in T, we can write

$$\prod_{n=1}^{p} x_n = x_1 \cdots x_{q+1} x_q \cdots x_p + x_1 \cdots \{x_q, x_{q+1}\} \cdots x_p$$

The first term on the right hand side is in $T_{p,s-1}$, while the second term is equal modulo I to a linear combination of monomials in T_{p-1}. Therefore $\prod_{n=1}^{p} x_n$ is equal modulo I to a linear combination of standard monomials. Hence the lemma holds for elements in $\coprod_{r<p} T_r + T_{ps}$, and by induction the lemma follows.

Let S be the subspace of T spanned by standard monomials.

L e m m a 5. <u>There exists a linear map $f : T \to S$ which is the identity map on S and annihilates I.</u>

P r o o f. I is the subspace of T spanned by elements of the form

1) $\quad x_1 \cdots x_p - x_1 \cdots x_{n+1} x_n \cdots x_p - x_1 \cdots \{x_n, x_{n+1}\} \cdots x_p$

where each x_k is a basis element in $\mathfrak{g}$ or $\mathfrak{h}$ and either $x_n, x_{n+1} \in \mathfrak{g}$ or $x_n, x_{n+1} \in \mathfrak{h}$, and by elements of the form

2) $$x_1 \dots x_p - x_1 \dots x_n' \dots x_p$$

where each x_k is a basis element in $\mathfrak{g}$ or $\mathfrak{h}$ and $x_n = g_i$, $x_n' = h_i$ with $i \in K$.

Assume as induction hypothesis that we have defined a linear map $f : \coprod_r \coprod_p T_r + T_{p,s-1} \longrightarrow S$ such that f is the identity map on any standard monomial where f is defined, and f annihilates any element 1) or 2) where f is defined. Let $\prod_{n=1}^{p} x_n$ be a monomial in T_{ps} . Assume first $x_n \notin \mathfrak{h}_0$ for all n . If $s = 0$, the monomial is standard, and we define $f(\prod_{n=1}^{p} x_n) = \prod_{n=1}^{p} x_n$. Suppose $s > 0$. Then there exists a q with $1 \leq q < p$ such that one of the cases A) or B) in the proof of Lemma 4, occurs.

In the case A) we have $x_q \in \mathfrak{h}$, $x_{q+1} \in \mathfrak{g}_0$ and we define

$$f_q(\prod_{n=1}^{p} x_n) = f(x_1 \dots x_{q+1}\, x_q \dots x_p) + f(x_1 \dots [x_q, \varphi(x_{q+1})] \dots x_p)$$

In the case B) we have $x_q, x_{q+1} \in \mathfrak{g}$ or $x_q, x_{q+1} \in \mathfrak{h}$, and we define

$$f_q(\prod_{n=1}^{p} x_n) = f(x_1 \dots x_{q+1}\, x_q \dots x_p) + f(x_1 \dots [x_q, x_{q+1}] \dots x_p)$$

The two terms to which f is applied are in $T_{p,s-1}$ and T_{p-1} in both cases.

We prove next that $f_q(\prod_{n=1}^{p} x_n)$ is independent of the choice of q .

Suppose r is another possible choice in the sense that $1 \leq r < p$ and one of the cases A) or B) with r instead of q occurs. We may suppose $r > q$. Assume first $r > q+1$. Suppose we have the case B) both for q and r. Then

$$f_q(\prod_{n=1}^{p} x_n) = f(x_1 \dots x_{q+1} x_q \dots x_p) + f(x_1 \dots [x_q, x_{q+1}] \dots x_p) =$$

$$f(x_1 \dots x_{q+1} x_q \dots x_{r+1} x_r \dots x_p) + f(x_1 \dots x_{q+1} x_q \dots [x_r, x_{r+1}] \dots x_p) +$$

$$f(x_1 \dots [x_q, x_{q+1}] \dots x_{r+1} x_r \dots x_p) + f(x_1 \dots [x_q, x_{q+1}] \dots [x_r, x_{r+1}] \dots x_p) =$$

$$f(x_1 \dots x_{r+1} x_r \dots x_p) + f(x_1 \dots [x_r, x_{r+1}] \dots x_p) = f_r(\prod_{n=1}^{p} x_n)$$

since f is linear and annihilates the elements 1). If we have the case A) for q or r, then $f_q(\prod_{n=1}^{p} x_n) = f_r(\prod_{n=1}^{p} x_n)$ follows from what we just proved and the fact that f annihilates the elements 2).

Assume $r = q+1$. Suppose x_q, x_{q+1}, x_{q+2} are all in $\mathfrak{g}$ or all in $\mathfrak{h}$. Then

$$f_q(\prod_{n=1}^{p} x_n) = f(x_1 \dots x_{q+1}\ x_q\ x_{q+2} \dots x_p) + f(x_1 \dots [x_q, x_{q+1}] x_{q+2} \dots x_p)$$

$$= f(x_1 \dots x_{q+2}\ x_{q+1}\ x_q \dots x_p) + f(x_1 \dots [x_q, x_{q+1}] x_{q+2} \dots x_p)$$

$$+ f(x_1 \dots x_{q+1} [x_q, x_{q+2}] \dots x_p) + f(x_1 \dots [x_{q+1}, x_{q+2}] x_q \dots x_p)$$

and

$$f_r(\prod_{n=1}^{p} x_n) = f(x_1 \dots x_q\ x_{q+2}\ x_{q+1} \dots x_p) + f(x_1 \dots x_q [x_{q+1}, x_{q+2}] \dots x_p)$$

$$= f(x_1 \dots x_{q+2}\ x_{q+1}\ x_q \dots x_p) + f(x_1 \dots x_q [x_{q+1}, x_{q+2}] \dots x_p)$$

$$f(x_1 \dots \{x_q, x_{q+2}\} x_{q+1} \dots x_p) + f(x_1 \dots x_{q+2} \cdot \{x_q, x_{q+1}\} \dots x_p)$$

Therefore

$$f_q(\prod_{n=1}^{p} x_n) - f_r(\prod_{n=1}^{p} x_n) = f(x_1 \dots \{ \{x_{q+1}, x_{q+2}\}, x_q\} \dots x_p)$$

$$+ f(x_1 \dots \{x_{q+1}, \{x_q, x_{q+2}\}\} \dots x_p) + f(x_1 \dots \{\{x_q, x_{q+1}\}, x_{q+2}\} \dots x_p)$$

$$= f(x_1 \dots [\{x_{q+1}, x_{q+2}\}, x_q] \dots x_p) + f(x_1 \dots [x_{q+1}, \{x_q, x_{q+2}\}] \dots x_p)$$

$$+ f(x_1 \dots [\{x_q, x_{q+1}\}, x_{q+2}] \dots x_p) = 0$$

Hence $f_q(\prod_{n=1}^{p} x_n) = f_r(\prod_{n=1}^{p} x_n)$. The other possible cases are $x_q \in \mathfrak{h}$, $x_{q+1}, x_{q+2} \in \mathfrak{g}_0$ or $x_q, x_{q+1} \in \mathfrak{h}$, $x_{q+2} \in \mathfrak{g}_0$. Then we use the fact that f annihilates the elements 2), and get the same conclusion. This proves that $f_q(\prod_{n=1}^{p} x_n)$ is independent of q, and we define $f(\prod_{n=1}^{p} x_n) = f_q(\prod_{n=1}^{p} x_n)$.

If $\prod_{n=1}^{p} x_n$ is any monomial in T_{ps} we let $y_n = \varphi^{-1}(x_n)$ if $x_n \in \mathfrak{h}_0$ and $y_n = x_n$ otherwise. Then $\prod_{n=1}^{p} y_n$ is a monomial in T_{ps} and $y_n \notin \mathfrak{h}_0$ for all n. We define $f(\prod_{n=1}^{p} x_n) = f(\prod_{n=1}^{p} y_n)$. Then extending by linearity we have defined f on T_{ps}, and therefore a linear map $f : \coprod_{r<p} T_r + T_{ps} \longrightarrow S$.

By definition f is still the identity map on standard monomials where it is defined, and it is clear that f annihilates elements 2) for which f is defined. Consider an element 1) for which f is defined. If $x_n = x_{n+1}$ the element is 0. If $x_n \neq x_{n+1}$ the two monomials $x_1 \dots x_p$

and $x_1 \dots x_{n+1} x_n \dots x_p$ have different indices. If the indices are both $< s$, f annihilates the element 1) by the induction hypothesis. We may therefore suppose that $x_1 \dots x_p \in T_{ps}$ and $x_1 \dots x_{n+1} x_n \dots x_p \in T_{ps-1}$. Then

$$f(x_1 \dots x_p) = f_n(x_1 \dots x_p) = f(x_1 \dots x_{n+1} x_n \dots x_p) +$$

$$f(x_1 \dots \{x_n, x_{n+1}\} \dots x_p)$$

hence f annihilates the element 1) . Therefore f annihilates all elements 1) or 2) for which f is defined.

By induction the lemma follows.

P r o o f o f t h e o r e m 3 . The linear map $f : T \to S$ of lemma 5 induces a linear map $f' : U \to S$, taking the coset modulo I of a standard monomial into the same standard monomial. The map $g' : S \to U$ taking a standard monomial into its coset modulo I is surjective by lemma 4. Furthermore $f' g' = 1$, hence g' is injective. Thus g' is an isomorphism with inverse f' , and g' maps the basis in S of standard monomials onto the basis in U of cosets modulo I of standard monomials.

C o r o l l a r y 6 . <u>Let $(\mathfrak{k}, \alpha, \beta)$ be a free product of $\mathfrak{g}$ and $\mathfrak{h}$ with identification $\varphi : \mathfrak{g}_0 \to \mathfrak{h}_0$. Then α and β are injective, and $\alpha(g) = \beta(h)$ if and only if $g \in \mathfrak{g}_0$, $h \in \mathfrak{h}_0$ and $h = \varphi(g)$</u> .

P r o o f . Let $g = \sum_{i \in I} m_i g_i \in \mathfrak{g}$ and $h = \sum_{j \in J} n_j h_j \in \mathfrak{h}$

Then

$$\alpha(g) = \sum_{i \in I} m_i (g_i + I)$$

$$\beta(h) = \sum_{j \in K} n_j (g_j + I) + \sum_{j \in J-K} n_j (h_j + I)$$

By theorem 3 we have therefore $\alpha(g) = \beta(h)$ if and only if $m_i = 0$ for $i \in I - K$, $n_j = m_j$ for $j \in K$, and $n_j = 0$ for $j \in J - K$, which is if and only if $g \in \mathfrak{g}_0$, $h \in \mathfrak{h}_0$ and $\varphi(g) = h$. In particular α and β are injective.

By the isomorphisms α and β we identify $\mathfrak{g}$ and $\mathfrak{h}$ with the subalgebras $\alpha(\mathfrak{g})$ and $\beta(\mathfrak{h})$ of $\mathfrak{k}$.

C o r o l l a r y 7. $\mathfrak{g} \underset{\varphi}{\circ} \mathfrak{h}$ _is infinite dimensional if_ $\mathfrak{g}_0 \neq \mathfrak{g}$ _and_ $\mathfrak{h}_0 \neq \mathfrak{h}$.

P r o o f. We let g be a basis element from $\mathfrak{g} - \mathfrak{g}_0$ and h a basis element from $\mathfrak{h} - \mathfrak{h}_0$. Then $g, h \in \mathfrak{g} \underset{\varphi}{\circ} \mathfrak{h}$ and consequently $(\mathrm{ad}g)^n(h) \in \mathfrak{g} \underset{\varphi}{\circ} \mathfrak{h}$. But we have

$$(\mathrm{ad}g)^n(h) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} g^{n-k} h g^k$$

where each $g^{n-k}hg^k$ is a standard monomial of degree $n+1$. If we let U_p be the subspace of U spanned by cosets modulo I of standard monomials of degree p, then we have proved $\mathfrak{g} \underset{\varphi}{\circ} \mathfrak{h} \cap U_p \neq 0$ for all p. Hence $\mathfrak{g} \underset{\varphi}{\circ} \mathfrak{h}$ is infinite dimensional.

C o r o l l a r y 8. _Every subalgebra_ $\mathfrak{g}_0$ _of a Lie algebra_ $\mathfrak{g}$ _is the difference kernel of two homomorphisms of_ $\mathfrak{g}$.

P r o o f. We have only to put $\mathfrak{h} = \mathfrak{g}$ and $\mathfrak{h}_0 = \mathfrak{g}_0$ with $\varphi : \mathfrak{g}_0 \longrightarrow \mathfrak{h}_0$ the identity. Then $\alpha, \beta : \mathfrak{g} \longrightarrow \mathfrak{g} \underset{\varphi}{\circ} \mathfrak{g}$ are two homomorphisms such that $\{ g \in \mathfrak{g} : \alpha(g) = \beta(g) \} = \mathfrak{g}_0$, that is $\ker(\alpha - \beta) = \mathfrak{g}_0$.

This means that the homomorphisms which are epic in the category of Lie algebras, that means cancels on the right in products of homomorphisms, are precisely the surjective homomorphisms. It is also true, but trivial, that the monic homomorphisms in the category are precisely the injective homomorphisms.

II. NON EXISTENCE OF INJECTIVE LIE ALGEBRAS

L e m m a 1 . There exist simple Lie algebras of dimension greater than any given cardinal.

P r o o f . Let V be a vector space with $\dim V > 1$ and let $\mathfrak{g}$ be the Lie algebra of all endomorphisms of V with finite dimensional range and trace 0. Then $\dim \mathfrak{g} \geq \dim V$. It is well known that $\mathfrak{g}$ is simple if $\dim V$ is finite.

We shall prove that $\mathfrak{g}$ is simple also if $\mathfrak{g}$ is infinite. Let $x \in \mathfrak{g}$, $x \neq 0$. Let I be the ideal in $\mathfrak{g}$ generated by x. Suppose $I \neq \mathfrak{g}$. We can then choose $y \in \mathfrak{g} - I$. $\operatorname{im} x + \operatorname{im} y$ is of finite dimension and has a suplementary subspace U of finite codimension in V. $\ker x$ and $\ker y$ are of finite codimension. Hence $W = \ker x \cap \ker y \cap U$ is of finite codimension and W has a suplementary subspace V_o of finite dimension such that $V_o \supseteq \operatorname{im} x + \operatorname{im} y$. We have then a direct sum decomposition $V = V_o + W$. V_o is invariant under x and y, and $\dim V_o > 1$ since y is not a scalar multiple of x.

If Z is a set of endomorphisms of V we define $Z_o = \{z/V_o : z \in Z,\ z(V_o) \subseteq V_o \text{ and } z(W) = 0\}$ Then $\mathfrak{g}_o$ is the Lie algebra of all endomorphisms of V_o of trace 0. I_o is an ideal in $\mathfrak{g}$, $x/V_o \in I_o$ and $y/V_o \notin I_o$. But this is a contradiction since $x/V_o \neq 0$ and $\mathfrak{g}_o$ is known to be simple. Hence $I = \mathfrak{g}$ and $\mathfrak{g}$ is simple. This proves the lemma.

P r o p o s i t i o n 2 . The category of Lie algebras over a given field has no injective object except 0.

P r o o f . Suppose $\mathfrak{g}$ is a non zero injective object. Let $\mathfrak{h}$ be a simple Lie algebra with $\dim \mathfrak{h} > \dim \mathfrak{g}$. Let $\mathfrak{h}_o$ be any 1-dimensional subalgebra of $\mathfrak{h}$ and $\varphi : \mathfrak{h}_o \longrightarrow \mathfrak{g}$ a non zero homomorphism. Since $\mathfrak{g}$ is injective φ extends to a non zero homomorphism

$\psi : \mathfrak{h} \to \mathfrak{g}$. Since $\mathfrak{h}$ is simple $\ker \psi = 0$ and ψ is an isomorphism into. But this is a contradiction since $\dim \mathfrak{h} > \dim \mathfrak{g}$. Hence the proposition follows.

On the other hand it is easy to prove that the projective Lie algebras are the free Lie algebras and their factors or equivalently the subalgebras with supplementary ideals in free Lie algebras.

References

N. Jacobson. Lie algebras.