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FROM THE THEORY OF JACOBIAN VARIETIES

By

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1. INTRODUCTION

The theory of Jacobian varieties began with Riemann's recognition of the deep relations between a Riemann surface and the associated theta-functions. Our aim here is to review some of the principal results obtained by Riemann and to give an application which permits a generalization of Torelli's theorem.

We confine ourselves to the classical case, although many of the results can be obtained for Jacobian varieties over an arbitrary groundfield. In particular, the results of section 2 are clearly quite independent of the groundfield.

We work with a closed Riemann surface of genus $g > 1$. If $\alpha^1, \ldots, \alpha^g$ is a basis for the Abelian differentials of the first kind on X, and $A_1, \ldots, A_g, B_1, \ldots, B_g$ is a canonical homology basis, we form a matrix $\Omega = (\omega^i_j)$ by setting

$$\omega^i_j = \int_{A_j} \alpha^i \qquad\qquad \omega^i_{j+g} = \int_{B_j} \alpha^i$$

$i, j = 1, \ldots, g$. Ω is known as a <u>period matrix</u> of X. It is often convenient to write $\Omega = (\Omega_1, \Omega_2)$ where Ω_i is a $g \times g$ matrix. A standard result in the theory of Riemann surfaces is that $\Omega_1^{-1}\Omega_2$ is symmetric and has positive definite imaginary part.

The column vectors of Ω are linearly independent over the real numbers, and generate a properly discontinuous group of translations on $\mathbb{C}^g$, which we denote by Ω^*. The quotient space $\mathbb{C}^g/\Omega^* = J(X)$ is a representation of the <u>Jacobian variety</u> of X. There exists an imbedding $X \to J(X)$ given by

$$u^i(Q) = \int_P^Q \alpha^i$$

where P is a fixed reference point in X. It follows from Abel's theorem that this is an imbedding, and we denote by W^1 the image of X under this map.

We denote by W^r the set of points representable as a sum of $\leq r$ points in W^1. An r-tuple of points in W^1 may also be identified with a divisor of degree r on X, and Abel's theorem asserts that two divisors of degree r are linearly equivalent if and only if they determine the same point in W^r. The solvability of the Jacobi inversion problem implies that $W^g = J(X)$.

If A and B are subsets of $J(X)$, we define for $a \in J(X)$

$$A_a = \{u : u - a \in A\}$$

$$-A_a = \{u : -u \in A_a\}$$

$$A \oplus B = \{u : u = a + b, a \in A, b \in B\}$$

$$= \bigcup_{b \in B} A_b$$

$$A \ominus B = \bigcap_{b \in B} A_{-b} .$$

We note that $u \in A \ominus B$ if and only if $B_u \subset A$.

2. COMBINATORIAL FORMULAE

Our object is to study certain combinational relations between the sets W^r and to indicate some applications to the theory of linear series on X.

L e m m a 2.1 <u>Let $u \in J(X)$, $u \neq 0$. Then there is a unique positive integer $r \leq g$, such that $u \in W^r$, $u \notin W^t$ for $t < r$, and u is the image of a unique positive divisor of degree r.</u>

P r o o f : The first assertion is obvious. By Abel's theorem two positive divisors D_1 and D_2 of the same degree are linearly equivalent if and only if $\varphi(D_1) = \varphi(D_2)$. Suppose $u = \varphi(D_1) = \varphi(D_2)$, where $D_1 \neq D_2$, and D_1 and D_2 are of degree r. Then $D_1 \sim D_2$, and there is a positive divisor D of degree $r - 1$ such that $D_1 \sim D + P$. But then $u = \varphi(D) \in W^{r-1}$.

The result may also be stated by saying that a point $u \in J(X)$ has a <u>unique</u> <u>minimal</u> <u>representation</u> of the form $u = w_1 + \dots + w_r$, where $w_i \in W^1$, and $r \leq g$.

We now introduce some notation. For subsets $A, B \subset J(X)$ we define A_u, $-A_u$, $A \oplus B$, $A \ominus B$ by

$$v \in A_u \iff v - u \in A ,$$

$$v \in -A_u \iff -v \in A_u ,$$

$$A \oplus B = \bigcup_{b \in B} A_b ,$$

and
$$A \ominus B = \bigcap_{b \in B} A_{-b} .$$

It is immediate from the definition that $u \in A \ominus B$ iff $B_u \subset A$. We also observe that $A_1 \subset A_2$ and $B_1 \supset B_2$ imply $A_1 \ominus B_1 \subset A_2 \ominus B_2$.

L e m m a 2.2 <u>$W^{g-1} = -W^{g-1}_{-K}$, where $K = \varphi(Z)$, Z being a canonical devisor on X.</u>

P r o o f : Given any positive divisor D of degree $g - 1$ there exists a divisor D' of degree $g - 1$ such that $D + D' \sim Z$. Hence

$\varphi(D) = -(\varphi(D') - \varphi(Z))$, and as the left hand side traverses W^{g-1} the right hand side traverses $-W^{g-1}_{-K}$ and conversely.

L e m m a 2.3 <u>Let $0 \leq r \leq t \leq g-1$. Then</u>

$$W^r_a \subset W^t_b \iff a \in W^{t-r}_b \quad .$$

P r o o f : The implication from right to left being trivial it suffices to prove the implication from left to right. The inclusion $W^r_a \subset W^t_b$ means that for every positive divisor D of degree $\leq r$ there is a positive divisor D' of degree $\leq t$ such that

$$\varphi(D) + a-b = \varphi(D') \quad .$$

Setting $D = P$, we have $(a - b) \in W^t$. Let A be the divisor of degree $s \leq t$ which corresponds to the unique minimal representation of $a - b$. Suppose $s > t - r$. Since A is unique, $l(A) = 1$, and by the Riemann-Roch theorem there is a divisor D of degree $t - s + 1 \leq r$ not containing P such that $l(A + D) = 1$. By assumption $\varphi(A + D) = \varphi(D)+a-b \in W^t$. Since $D + A$ is of degree $t + 1$, we must have $D + A \sim D' + P$, where D' is of degree t . But P does not occur on the left, hence $l(D+A) > 1$. This is a contradiction. It follows that $s \leq t - r$, i.e. $\varphi(A) = a - b \in W^{t-r}$, and the theorem follows.

C o r o l l a r y 1 . <u>Let $0 \leq r \leq t \leq g - 1$. Then</u>

$$W^t_a \ominus W^r_b = W^{t-r}_{a-b} \quad .$$

P r o o f : $u \in W^t_a \ominus W^r_b$ if and only if $W^r_{b+u} \subset W^t_a$, i.e. if and only if $u \in W^{t-r}_{a-b}$.

C o r o l l a r y 2 . *Let $0 \leq r \leq g-1$. Then*

$$-W_a^{g-1} \ominus W_b^r = W_{-a-b-K}^{g-1-r}$$

P r o o f : $-W_a^{g-1} = (-W^{g-1})_{-a} = W_{-a-K}^{g-1}$.

C o r o l l a r y 3 . *Let $0 \leq r \leq g-1$. Then*

$$W_a^{g-1} \ominus (-W_b^r) = -W_{-a-b-K}^{g-1-r}$$

P r o o f : $$W_a^{g-1} \ominus (-W_b^r) = \bigcap_{u \in W_b^r} W_{a+u}^{g-1}$$

$$= -\bigcap_{u \in W_b^r} (-W_a^{g-1})_{-u}$$

$$= -\left\{-W_a^{g-1} \ominus W_b^r\right\}$$

C o r o l l a r y 4 . *Let $0 \leq r \leq t \leq g-1$. Then*

$$-W_a^r \subset W_b^t \iff -W_{-b}^{g-1-t} \subset W_{-a-K}^{g-1-r}$$

P r o o f : If $-W_a^r \subset W_b^t$ then $W^{g-1} \ominus W_b^t \subset W^{g-1} \ominus (-W_a^r)$, and the corollary follows from corollaries 1 and 3.

Corollary 4 has an interesting interpretation. An inclusion of the form $-W_a^r \subset W^t$ means that for every positive divisor D of degree r there is a positive divisor D' of degree t such that $-\varphi(D) - a = \varphi(D')$, or

$$-a = \varphi(D + D') .$$

This again means that $-a$ is representable by a positive divisor of degree $r + t$ and dimension $r + 1$. The corollary then states that $K - (-a)$ is representable by a positive divisor of degree $2g - 2 - (r + t)$ and dimension $g - t$. This is an expression for the Brill-Nöther reciprocity theorem, which is equivalent to a restricted form of the Riemann-Roch theorem. Thus Lemma 3 may be regarded as a combinatorial version of the latter.

We now turn to some intersection properties of the sets W^r. Consider first an intersection $W_a^r \cap -W_b^t$. If u is a point of the intersection, there are positive divisors D and D' of degree r and t such that

$$u = \varphi(D) + a = -\varphi(D') - b .$$

Hence $\varphi(D + D') = -(a + b)$. The divisor $D + D'$ has degree $r + t$, and if we form a divisor D_1 by selecting any r points from $D + D'$, we [illegible]

$$u_1 = \varphi(D_1) + a = -\varphi(D_1') - b ,$$

where u_1 is another point in the intersection. Hence, if $-(a + b)$ is uniquely representable as the image of a divisor of degree $r + t$, then the intersection $W_a^r \cap -W_b^t$ contains $\binom{r+t}{r}$ points (counting multiplicities), $u_1, u_2, \ldots u_{\binom{r+t}{r}}$, and

$$\sum u_i = -\binom{r+t-1}{r-1}(a + b) .$$

If the representation of $-(a + b)$ as the image of a positive divisor of degree $r + t$ is not unique, then $l(D + D') > 1$, and for every $Q \in X$ there is a divisor D_1 of degree $r + t - 1$ such that $D + D' \sim Q + D_1$. In this case the intersection will contain sets of dimension ≥ 1.

Lemma 2.4 Let $1 \leq r \leq g-1$. Then the intersection $W_a^r \cap -W_b^{g-r}$ is non-empty, and, if proper, consists of a discrete set of points $u_1, \ldots, u_s$ with multiplicities $m_1, \ldots, m_s$ such that

$$\sum m_i = \binom{g}{r} \quad \text{and} \quad \sum m_i u_i = -\binom{g-1}{r-1}(a+b) .$$

The proof is immediate from the preceding considerations, observing that $-(a+b)$ can always be represented as the image of a divisor of degree g.

Lemma 2.5 Let $1 \leq r \leq g-1$. Then the intersection $W_a^1 \cap -W_b^r$ is either equal to W_a^1, or else consists of a discrete set of points $u_1, \ldots, u_s$ with multiplicities $m_1, \ldots, m_s$ such that $\sum m_i = r+1$, $\sum m_i u_i = -(a+b)$.

The proof is again immediate, except for the observation that if $-(a+b)$ is not uniquely representable as a divisor of degree $r+1$, then it is representable as a divisor of degree $r+1$ and dimension > 1. By the remarks following lemma 3, corollary 4, we then have $W_a^1 \subset -W_b^r$.

We next turn to intersections of the form $W_a^{r+1} \cap W_b^t$. These are in general difficult to get explicitly, but we can get some information in a special case. Suppose $1 \leq r < t \leq g-1$. If $W_a^{r+1} \not\subset W_b^t$, the intersection should be of dimension $\leq r$. We shall now assume that $W_c^r \subset W_a^{r+1} \cap W_b^t$, for some c. Then, by lemma 3, $c \in W_a^1 \cap W_b^{t-r}$, i.e. $c = a + x = b + y$, $x \in W^1$, $y \in W^{t-r}$.

Lemma 2.6 Let $x \in W^1$, $y \in W^{t-r}$. Then either $W_a^{r+1} \subset W_{a+x+y}^t$, or else

$$W_a^{r+1} \cap W_{a+x-y}^t = W_{a+x}^r \cup S$$

where $S = W_a^{r+1} \cap (W_{a-y}^t \ominus (-W^1))$.

It should be observed that S does not depend on x .

P r o o f : It is clear from lemma 3 that W_{a+x}^r is contained in the intersection. Suppose now that $u \in W_a^{r+1} \cap W_{a+x-y}^t$. We may then write

$$u = w_1 + a = w_2 + a + x - y$$

with $w_1 \in W^{r+1}$, $w_2 \in W^t$ and

$$w_1 + y = w_2 + x .$$

Each side of the last equation may be represented as a sum of $t + 1$ points in W^1 . If this representation is unique, we find that $x \in W^1$ must occur as a summand on the left. If x occurs in the representation of y, $y \in W_x^{t-r-1}$, whence $a \in W_{a+x-y}^{t-r-1}$ and $W_a^{r+1} \subset W_{a+x-y}^t$. If x occurs in the representation of w_1 , then $w_1 \in W_x^r$ and $u \in W_{a+x}^r$.

Hence, if $W_a^{r+1} \not\subset W_{a+x-y}^t$, and if $u \in W_a^{r+1} \cap W_{a+x-y}^t$, $u \notin W_{a+x}^r$, then the representation of $w_1 + y$ as the image of a divisor of degree $t + 1$ is not unique. This means that $-W_{-w_1-y}^1 \subset W^t$ or $w_1 + y \in W^t \ominus (-W^1)$, whence

$$u = w_1 + y - y + a \in W_{a-y}^t \ominus (-W^1) .$$

On the other hand it is easily seen that $W_{a-y}^t \ominus (-W^1) \subset W_{a+x-y}^t$.

F. Severi ((7 , p. 380)) shows that a special linear series of degree n and dimension $r + 1$ "depends on $(r + 1)(n - r) - rg$ parameters". We may interpret this geometrically by observing that a is the image of a divisor of degree $r + t$ and dimension $r + 1$ if and only if $-W_{-a}^r \subset W^t$, or, equivalently, if and only if $a \in W^t \ominus (-W^r)$. Thus the set $W^t \ominus (-W^r)$

represents the set of linear series of degree $r + t$ and dimension $r + 1$.

Consider first the case $r = 1$. By lemma 6 we have for $u, v \in W^1$

$$W^t_u \cap W^t_v = W^{t-1}_{u+v} \cup \{W^t \ominus (-W^1)\} .$$

Clearly, $W^t \ominus (-W^1) \not\subset W^{t-1}_{u+v}$ for arbitrary u, $v \in W^1$, and it follows that it occurs as separate components of $W^t_u \cap W^t_v$. Hence it has dimension $\geq 2t - g$. This number agrees with Severi's formula.

Next, consider the intersection

$$\{W^t \ominus (-W^r)\} \cap W^t_w$$

with $w \in W^{r+1}$. Let u be a point of the intersection. Then

$$u = w_1 + w = w_2 + v_1 + \ldots + v_r$$

where the v_i are arbitrarily chosen points in W^1, w_1 and w_2 are points in W^t.

Since $w \in W^{r+1}$, we can select the v_i such that $w = v_1 + \ldots + v_r + x$, $x \in W^1$. Then

$$w_1 + x = w_2 .$$

Hence, either x occurs in the representation of w_2 as a sum of t points in W^1, or else $w_2 \in W^t \ominus (-W^1)$. In the former case we have $w_1 \in W^{t-1}$. Hence, if $w_1 \notin W^{t-1}$, then $w_2 \in W^t \ominus (-W^1)$, and $u-(v_1+\ldots+v_r)$ $W^t \ominus (-W^1)$ for all r-tuples of points $v_i \in W^1$. Hence $u \in W^t \ominus (-W^{r+1})$. In other words, if $u \in \{W^t \ominus (-W^r)\} \cap W^t_w$ then either $u \in W^{t-1}_w$ or else $u \in W^t \ominus (-W^{r+1})$. However, $W^t \ominus (-W^{r+1})$ cannot be included in W^{t-1}_w for arbitrary $w \in W^{r+1}$. Hence $W^t \ominus (-W^{r+1})$ must occur in separate components of $\{W^t \ominus (-W^r)\} \cap W^t_w$. Hence its dimension must be greater than

$$\dim(W^t \ominus (-W^r)) + t - g \quad ,$$

and, inductively,

$$\dim(W^t \ominus (-W^r)) \geq (r + 1)t - rg \quad ,$$

provided $W^t \ominus (-W^r) \neq \emptyset$. This agrees with Severi's formula, and the result is not restricted to characteristic zero. We have of course not established that $W^t \ominus (-W^r) \neq \emptyset$ when $(r + 1)t - rg \geq 0$. A result of this kind is apparently established for the classical case in a paper by Meis.

3. MULTIPLICATIVE FUNCTIONS

Let S be a subset of $J(X)$, and let $\Omega = (\omega_k^j)$ be a period matrix of $J(X)$ formed with a canonical homology basis of X . Let $\Lambda = (\lambda_k^j)$ be a $g \times 2g$ matrix, and let γ be a column vector with components $\gamma^1, \ldots, \gamma^{2g}$.

A holomorphic function, φ , on $\mathbb{C}^g$ will be said to be <u>multiplicative of type</u> (Λ, γ) <u>over</u> S <u>relative to</u> Ω provided

$$(3.1) \quad \varphi(u + \omega_h) = \varphi(u)\exp(2\pi i({}^t\lambda_h u + \gamma^h))$$

for every u lying over S . From this definition it is clear that if vanishes at some point lying over a point $s \in S$, then it vanishes at all points lying over s .

L e m m a 3.1 Let φ be multiplicative of type (Λ, γ) over $J(X)$. Let $\partial_1, \ldots, \partial_r$ be first order partial differential operators on $\mathbb{C}^g$. Then $\partial_1 \ldots \partial_r \varphi$ is multiplicative of type (Λ, γ) over the (projection of the) common zeros of $\varphi, \partial_j\varphi, \partial_j\partial_k\varphi, \ldots, \partial_1 \ldots \hat{\partial}_j \ldots \partial_r \varphi$. (Here $\hat{}$ means that the operator ∂_j is to be deleted, as usual.)

P r o o f : This follows immediately upon differentiation of the defining relation.

L e m m a 3.2 Let W^1 be the canonically imbedded image of X in $J(X)$. To every point $w \in W^1$ there corresponds a first order partial differential operator, ∂_w, on $\mathbb{C}^g$ with the following properties:

1. If φ vanishes identically over W^1_a, and $\partial_w \varphi$ is multiplicative of type (Λ, γ) over W^1_a, then $\partial_w \varphi$ has a zero over $a + w$.

2. In any neighborhood of any point of W^1 there exist g points, $w_1, \ldots, w_g$, of W^1 such that the operators $\{\partial_{w_j}\}$ form a basis for the first order partial differential operators on $\mathbb{C}^g$.

P r o o f : Define $\partial_w = \sum \frac{du^j}{dz}(w)\frac{\partial}{\partial u^j}$, where $\frac{du^j}{dz}(w)$ is the value of the derivative of the jth component of the imbedding function, taken with respect to a local coordinate, z, on W^1, and evaluated at w. This defines ∂_w up to a constant factor depending on the local coordinate.

If φ vanishes over W^1_a, then its derivative with respect to a local coordinate on W^1_a vanishes, i.e.

$$\sum \frac{\partial \varphi}{\partial u^j}\frac{du^j}{dz} = 0 .$$

But at the point $w + a \in W^1_a$ this condition simply says that $\partial_w \varphi$ vanishes. This establishes (1).

To obtain (2) we observe that $\{\partial_{w_j}\}$ will form a basis for the partial differential operators of the first order whenever the matrix $(\frac{du^j}{dz}(w_k))$ is non-singular. Since the du^j form a basis for the Abelian differentials of the first kind on W^1, this matrix will be non-singular for almost all g-tuples of points on W^1. This completes the proof of lemma 3.2.

We shall now assume that φ is a given holomorphic function, multiplicative of type (Λ, γ) over a set $S \subset J(X)$. Our object is to study the zeros induced by φ on some $W_a^1 \subset S$. We first show that if φ does not vanish identically over W_a^1, then it induces q zeros on W_a^1, counting multiplicities, where q depends only on Λ and Ω.

To that end we represent X by a fundamental polygon with sides (in order) $A_1, B_1, -A_1, -B_1, \dots, A_g, B_g, -A_g, -B_g$ where $(A_1, \dots, A_g, B_1, \dots, B_g)$ forms a canonical homology basis on X.

Using the canonical mapping $X \to W_a^1$, we can pull a single-valued branch of φ back on the polygon and study its zeros. The number of zeros is obtained by evaluating the integral of the logarithmic derivative of φ around the polygon. We observe that as we traverse A_h the point u goes into $u + \omega_h$, and as we traverse B_h the point u goes into $u + \omega_{g+h}$. Hence, for the number of zeros of φ over W we get

$$q = \frac{1}{2\pi i} \sum_h \left\{ \int_{A_h} \left(\frac{d\varphi}{\varphi}(u) - \frac{d\varphi}{\varphi}(u+\omega_g+h)\right) + \int_{B_h} \left(\frac{d\varphi}{\varphi}(u+\omega_h) - \frac{d\varphi}{\varphi}(u)\right) \right\}$$

Observing that $\frac{d\varphi}{\varphi}(u+\omega_h) = \frac{d\varphi}{\varphi}(u) + 2\pi i\, {}^t\lambda_h\, du$, we get

$$q = \frac{1}{2\pi i} \sum_h \left\{ -\int_{A_h} 2\pi i\, {}^t\lambda_{g+h} du + \int_{B_h} 2\pi i\, {}^t\lambda_h du \right\} = \sum_h \left\{ {}^t\lambda_h \omega_{g+h} - {}^t\lambda_{g+h} \omega_h \right\} .$$

If we write $\Omega = (\Omega_1, \Omega_2)$ and $\Lambda = (\Lambda_1, \Lambda_2)$, where Ω_j and Λ_j are $g \times g$ matrices, we may write

$$q = \text{Trace}({}^t\Lambda_1 \Omega_2 - {}^t\Lambda_2 \Omega_1)$$

This is also expressible in terms of the so-called <u>characteristic</u> matrix of φ, defined by $N = ({}^t\Omega \Lambda - {}^t\Lambda \Omega)$. To see the significance of this matrix, we consider the expression $\varphi(u + \omega_h + \omega_k)$ which, by the defining relation (3.1), may be expanded in two different ways. Since the resulting expressions must yield the same function, it follows by an easy calculation that $({}^t\lambda_k \omega_h - {}^t\lambda_h \omega_k)$ must be an integer. Since ${}^t\lambda_k \omega_h = {}^t\omega_h \lambda_k$, we find that N must be a skew-symmetric matrix with integral entries. The reader may now verify that the formula for q may be written

$$q = \tfrac{1}{2} \text{Trace } JN ,$$

where

$$J = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} ,$$

E is the $g \times g$ unit matrix, and 0 is the $g \times g$ null-matrix.

Consider next two holomorphic functions, φ_1 and φ_2, which are multiplicative of type (Λ, γ_1) and (Λ, γ_2) respectively, over S. We form the quotient $\beta(u) = \varphi_1(u)/\varphi_2(u)$, and evaluate the integral $\int u \, d\beta(u)/\beta(u)$ around the polygon. This integral should be equal to the sum

$$2\pi i \sum (u(Q_j) - u(R_j))$$

where the Q_j are the zeros of φ_1 and the R_j are the zeros of φ_2.

In order to evaluate the integral we observe that $\beta(u+\omega_h) = \beta(u)\exp(2\pi i(\gamma_1^h - \gamma_2^h))$, whence

$$\frac{d\beta}{\beta}(u+\omega_h) = \frac{d\beta}{\beta}(u) .$$

We then get

$$\int u\frac{d\beta}{\beta}(u) = \sum_h \left\{ \int_{A_h} \left(u\frac{d\beta}{\beta}(u) - (u+\omega_{g+h})\frac{d\beta}{\beta}(u)\right) + \int_{B_h} \left((u+\omega_h)\frac{d\beta}{\beta}(u) - u\frac{d\beta}{\beta}(u)\right) \right\}$$

$$= \sum_h \left\{ - \int_{A_h} \omega_{g+h}\frac{d\beta}{\beta}(u) + \int_{B_h} \omega_h \frac{d\beta}{\beta}(u) \right.$$

Now, $\frac{d\beta}{\beta} = d\ \ln\beta$, and $\int_{A_h} d\ \ln\beta = 2\pi i(\gamma_1^h - \gamma_2^h)$, and $\int_{B_h} d\ \ln\beta = 2\pi i(\gamma_1^{g+h} - \gamma_2^{g+h})$, modulo multiples of $2\pi i$. Hence

$$\int u\frac{d\beta}{\beta}(u) = 2\pi i \sum \left\{ \omega_h(\gamma_1^h - \gamma_2^h) - \omega_{g+h}(\gamma_1^h - \gamma_2^h) \right\}$$

modulo a sum of the form $2\pi i \sum m_k\omega_k$. We can rewrite the right hand side as $(-\Omega_2, \Omega_1)(\gamma_1 - \gamma_2)$, and thus get, finally

$$\sum u(Q_j) - (-\Omega_2, \Omega_1)\gamma_1 = \sum u(R_j) - (-\Omega_2, \Omega_1)\gamma_2$$

modulo a linear combination of periods, i.e. the equation holds if the terms are interpreted as points in $J(X)$.

Our findings may be summarized as follows:

T h e o r e m 3.1 A <u>Let φ be a multiplicative holomorphic function over W_a^1 of type (Λ, γ). If φ does not vanish identically over W_a^1, it induces q zeros, $u_1, \ldots, u_q$ on W_a^1 counting multiplicities, such that</u>

$$q = \tfrac{1}{2} \text{ Trace } JN$$

<u>and</u>

$$\sum u_j = (-\Omega_2, \Omega_1)\gamma + z_o$$

<u>where N is the characteristic matrix of φ and z_o is a point of $J(X)$ which depends only on Ω, Λ, and the canonical imbedding $X \longrightarrow W_a^1$.</u>

This result can be given a different formulation of some interest. Given a function, φ, we define its translate by a, φ_a, by the relation $\varphi_a(u) = \varphi(u - a)$. If φ is multiplicative of type (Λ, γ) over $S \subset J(X)$, then φ_a is multiplicative of type $(\Lambda, \gamma - {}^t\Lambda a)$ over S_a. To see this, we use the defining relation, (3.1) and get

$$\varphi_a(u + \omega_h) = \varphi(u - a + \omega_h) = \varphi(u - a)\exp(2\pi i({}^t\lambda_h(u - a) + \gamma^h)$$

$$= \varphi_a(u)\exp(2\pi i({}^t\lambda_h u + \gamma^h - {}^t\lambda_h a) .$$

T h e o r e m 3.1 B <u>Let φ be a multiplicative holomorphic function of type (Λ, γ) over $S \subset J(X)$. Let $W_b^1 \subseteq S_a$. Then φ_a is multiplicative over W_b^1, and if it does not vanish identically over W_b^1, it induces q zeros, $b+u_1, \ldots, b+u_q$ on W_b^1, counting multiplicities, such that</u>

$$q = \tfrac{1}{2} \text{ Trace } JN$$

and

$$\sum u_j = T(a-b) + z_1$$

where T is an endomorphism of $J(X)$ represented by the matrix $(\Omega_2{}^t\Lambda_1 - \Omega_1{}^t\Lambda_2)$, and z_1 is a point of $J(X)$ depending only on , , and the canonical imbedding $X \longrightarrow W^1$. Moreover, $T\Omega = \Omega JN$, where N is the characteristic matrix of φ.

P r o o f : We first observe that $\varphi_{(a-b)}$ is multiplicative of type $(\Lambda, \gamma - {}^t\Lambda(a-b))$ over W^1. We apply Theorem 3.1 A and find that, if $\varphi_{(a-b)}$ does not vanish identically over W^1, then it induces q zeros, $u_1, \dots, u_q$, on W^1, counting multiplicities, where $q = \frac{1}{2}$ Trace JN, and

$$\sum u_j = (-\Omega_2, \Omega_1)(\gamma - {}^t\Lambda(a-b)) + z_0 .$$

Hence φ_a induces the zeros $b+u_1, \dots, b+u_q$ on W^1_b. We may also write $\sum u_j = (\Omega_2, -\Omega_1){}^t\Lambda(a-b) + z_1$, where $z_1 = z_0 + (-\Omega_2, \Omega_1)\gamma$. If φ is given, so is Λ, and γ, and hence z_1 is completely determined.

We now study the matrix $T = (\Omega_2, -\Omega_1){}^t\Lambda$. It is seen immediately that this may also be written as $T = (\Omega_2{}^t\Lambda_1 - \Omega_1{}^t\Lambda_2)$. To show that T is an endomorphism, we investigate its action on the periods by forming the matrix $T\Omega$.

It has been assumed that Ω was formed with a canonical homology basis of X. Hence Ω_1 is non-singular, and $\Omega_1^{-1}\Omega_2$ is symmetric, i.e. $\Omega_1^{-1}\Omega_2 = {}^t\Omega_2{}^t\Omega_1^{-1}$, or $\Omega_2{}^t\Omega_1 = \Omega_1{}^t\Omega_2$. Hence $(\Omega_2{}^t\Omega_1 - \Omega_1{}^t\Omega_2)\Lambda = (\Omega_2, -\Omega_1){}^t\Omega\Lambda = 0$, and we may write $T\Omega = (-\Omega_2, \Omega_1)({}^t\Omega\Lambda - {}^t\Lambda\Omega)$. From the relation $\Omega J = (-\Omega_2, \Omega_1)$ we finally get

$$T\Omega = \Omega JN ,$$

where N is the characteristic matrix of φ .

We showed earlier that N has integral entries, and hence it takes periods into periods. By the above relation, so does T . This shows that T is an endomorphism of $J(X)$, and completes the proof of Theorem 3.1 B. The explicit formula for $T\Omega$, however, enables us to obtain some additional information.

Since the column vectors of Ω form a basis for $\mathbb{C}^g$ over the reals, it follows that T is non-singular if and only if N is non-singular. In that case, the endomorphism is surjective. If, in particular, N is unimodular, then the column vectors of $T\Omega$ form a new basis for the periods, and hence T is an automorphism of $J(X)$. When N is non-singular, φ is said to be non-degenerate. Hence we have

C o r o l l a r y 1 <u>If φ is non-degenerate, T is surjective, and if N is unimodular, then T is an automorphism of $J(X)$.</u>

Unimodularity of N is found in a very important class of multiplicative functions, the thetafunctions, which will be studied in the next section.

C o r o l l a r y 2 <u>If φ is multiplicative over $J(X)$, and is non-trivial[1)], then $q > 0$. If φ is both non-trivial and non-degenerate,</u> then $q \geq g$.

P r o o f : Assume $q = 0$. Let $a \in J(X)$ be a point over which φ has a zero. Then φ has a zero over W_a^1 , and since $q = 0$ it follows that φ must vanish identically over W_a^1 . But by the same argument φ has a zero over W_w^1 for every $w \in W_a^1$, and hence vanishes identically over each. But then φ vanishes identically over W_a^2 . Continuing the

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1) i.e. does not vanish identically, and has zeros.

argument, we find that φ must vanish identically over $W_a^g = J(X)$.

Suppose, finally, that $0 < q < g$. If φ does not vanish over a, then $-Ta + Z_1 \in W^q$. But the set of such a must be dense in $J(X)$. Hence $q \geq g$, if T is non-singular.

We conclude this section with a proof of the following result:

L e m m a 3.3 Let φ be multiplicative of type (Λ, γ) over $S \subset J(X)$. Let $n \geq 1$ be an integer, and define $\varphi_n(u) = \varphi(nu)$. Then φ_n is multiplicative of type (Λ', γ') over $\frac{1}{n} S$, where

$$\frac{1}{n} S = \{u ; \ nu \in S\}$$

$$\Lambda' = n^2 \Lambda$$

$$(\gamma')^h = n\gamma^h + \tfrac{1}{2}n(n-1)^t\lambda_h\omega_h$$

P r o o f : We use induction over n to establish the formula
$$\varphi(u + n\omega_h) = \varphi(u)\exp(2\pi i({}^t\lambda_h(nu) + n\gamma^h + \tfrac{1}{2}n(n-1)^t\lambda_h\omega_h)) ,$$
whence the lemma follows upon substituting nu for u.

For $n = 1$ the formula is trivially verified, and by assumption

$$\varphi(u + n\omega_h) = \varphi(u + (n-1)\omega_h)\exp(2\pi i({}^t\lambda_h u + \gamma^h + (n-1)^t\lambda_h\omega_h)) .$$

The formula to be established is now easily derived using the induction hypothesis.

4. ON THE VANISHING OF THETAFUNCTIONS

Let $\Omega = (\pi i E, A)$ be a period matrix formed with a canonical homology basis of X, and define

$$\Theta(u;A) = \sum_{m\in Z^g} \exp({}^t m(Am + 2u)) \quad .$$

Since A has a negative definite real part, it is easy to show that the series on the right converges absolutely and uniformly on compact subsets. Hence $\Theta(u;A)$ is defined as a holomorphic function on $\mathbb{C}^g$. It can be shown that the function is non-trivial. By an elementary calculation one finds that Θ is multiplicative of type (Λ, γ) over $J(X)$ where

$$\Lambda = (0\ ,\ -\frac{1}{\pi i}E)$$

and

$$\gamma^1 = \dots = \gamma^g = 0 \ , \quad \gamma^{g+h} = -\frac{1}{2\pi i}\, a^h_h \qquad A = (a^j_k) \quad .$$

This section will be devoted to a proof of a fundamental result first obtained by Riemann, which characterizes the zeros of Θ in terms of the imbedded image of X in $J(X)$.

We say that a function vanishes of order r at a point provided the function and all of its partial derivatives of order $\leq r$ vanish at the point, while some partial derivative of order r does not.

T h e o r e m 4.1 (Riemann). <u>Let W^1 be a canonically imbedded image of X in $J(X)$. Then there exists a fixed point $k \in J(X)$ depending only on the canonical imbedding and on A, such that $\Theta(u;A)$ vanishes of order $r + 1$ over $b \in J(X)$ if and only if</u> $-W^r_{-b} \subset W^{g-1-r}_k$.

R e m a r k : The condition $-W^r_{-b} \subset W^{g-1-r}_k$ cannot be satisfied unless $2r \leq g - 1$. This inequality does not have to be assumed, however, less will be a consequence of the theorem. Hence, Θ cannot vanish of order greater than $\frac{1}{2}(g - 1) + 1$ at any point of $J(X)$.

To see the significance of the result, we first observe that for $r = 0$ the theorem gives the important special result that the zeros induced by Θ on $J(X)$ are precisely the set W_k^{g-1}. For $n \geq 1$ we first note that an inclusion $-W_{-b}^r \subset W_k^{g-1-r}$ means that $b - k$ is the image of a positive divisor of degree $(g - 1)$ and dimension $(r + 1)$. Thus the theorem asserts that the order of vanishing of $\Theta(u,A)$ over a point $b \in J(X)$ indicates the dimension of the complete linear series of degree $(g - 1)$ all of whose divisors map on $b - k$.

We shall first prove the theorem for $r = 0$, and then obtain the full theorem by induction. Since the proof is rather long, we present it in the form of a series of lemmas.

L e m m a 4.1 <u>If $\Theta(u - b)$ does not vanish identically over W^1, it induces g zeros, $u_1, \ldots, u_g$, on W^1, counting multiplicities, such that $\sum u_j = b - k$, where k is a point in $J(X)$ independent of b.</u>

P r o o f : Using Theorem 3.1 B and the given forms of Λ and we find that $T = E$, and $\frac{1}{2}$ Trace $JN = g$. This proves the lemma, and defines k. It will turn out that the definition of k is the correct one.

L e m m a 4.2 <u>If Θ vanishes over $-b \in J(X)$, then $b \in W_k^{g-1}$, where k is the constant of lemma 4.1.</u>

P r o o f : The argument is similar to that of Theorem 3.1 B, corollary 2. Consider the function $\Theta(u - b)$. If it does not vanish identically over W^1, it induces g zeros, $u_1, \ldots, u_g$, counting multiplicities, on W^1, such that $b = k + \sum u_j$. Since $u = 0$ must be a zero, the right hand side is a point in W_k^{g-1}.

Suppose now that Θ vanishes identically over W^1. Let t be the largest number such that $\Theta(u - w - b)$ vanishes identically over W^1 for all w in W^t. Then $t < g - 1$, and there is a dense subset of W^{t+1} such that $\Theta(u - w - b)$ does not vanish identically over W^1 for any w

in the subset. By lemma 4.1, $b = k - w + \sum u_j$, whence $b \in W_k^{g-1-t} \subset W_k^{g-1}$, since the points of w must occur among the u_j . This completes the proof of lemma 4.2.

L e m m a 4.3 <u>Θ vanishes identically over $-W_k^{g-1}$.</u>

P r o o f : Let $b \in W_k^{g-1}$, $b \notin W_k^{g-2}$. Then $b - k$ is uniquely representable as a sum of $g - 1$ points in W^1 . The corresponding divisor must have dimension 1 , and hence we can find a point $v_g \in W^1$ such that the divisor corresponding to $v_1 + \ldots + v_g$ is of dimension 1 , $v_1, \ldots, v_{g-1}$ being the original points. Consider the function $\Theta(u - b - v_g)$. If it does not vanish identically over W^1 , it induces g zeros, $u_1, \ldots, u_g$ on W^1 such that

$$\sum u_j = b - k + v_g = \sum v_j$$

Since the sum on the right is unique, v_g must appear in the sum on the left. Hence v_g is a zero of $\Theta(u - b - v_g)$, or $-b$ is a zero of Θ . If Θ vanishes identically over W^1 , this is a fortiori the case.

The set of b's considered is dense in W_k^{g-1} , and hence lemma 4.3 is established.

From the definition of $\Theta(u;A)$, it is easily seen that Θ is an even function. From this fact and lemmas 4.2 and 4.3, we now get Theorem 4.1 for the case $r = 0$, i.e.: Θ vanishes over $b \in J(X)$ if and only if $b \in W_k^{g-1}$.

L e m m a 4.4 <u>If $-W_{-b}^r \subset W_k^{g-1-r}$, then Θ vanishes of order $\geq s + 1$, $s < r$, over every point of W_a^{r-s} whenever $b \in W_a^{r-s}$.</u>

P r o o f : We proceed by induction over s . For $s = 0$, suppose $b \in W_a^r$. Then $a \in -W_{-b}^r \subset W_k^{g-1-r}$. Hence $W_a^r \subset W_k^{g-1}$, i.e. Θ vanishes over W_a^r .

Suppose now that the lemma holds for $s \leq s_o < r$. Then Θ and all of its partial derivatives of order $\leq s_o$ vanish over $W_a^{T-s_o}$ whenever $b \in W_a^{T-s_o}$. It follows that every partial derivative of Θ of order $s_o + 1$ is multiplicative of the same type as Θ over $W_a^{r-s_o}$ whenever $b \in W_a^{T-s_o}$.

Suppose $b \in W_a^{r-s_o-1}$. By lemma 2.2 $W_a^{r-s_o-1} \subset W_{a-w}^{r-s_o}$ for every $w \in W^1$. Select any $w \in W^1$. Then $b \in W_{a-w}^{r-s_o}$, and $\partial_w \partial^{s_o}\Theta$ is multiplicative over $W_{a-w}^{r-s_o}$, for any partial differential operator, ∂^{s_o} , of order s_o . By definition, $\partial_w \partial^{s_o}\Theta$ vanishes over $c + w$ on every $W_c^1 \subset W_{a-w}^{r-s_o}$. By lemma 2.2, $\partial_w \partial^{s_o}\Theta$ vanishes over $W_a^{r-s_o-1}$. Since w was chosen arbitrarily, it follows that every partial derivative of order $s_o + 1$ vanishes over $W_a^{r-s_o-1}$, and the induction is completed. This establishes lemma 4.4.

L e m m a 4.5 <u>If Θ vanishes of order $r + 1$ at $b \in W_k^{g-1}$, then</u> $-W_b^r \subset W_k^{g-1-r}$.

P r o o f : For $r = 0$ this is the result of lemma 4.2. Let $b \in W_a^1$, and let $s \leq r$ be the largest integer such that Θ vanishes of order $s + 1$ over every point of W_a^1 . Then for any $(s + 1)$-tuple of points, $w_o, \ldots , w_s$, in W^1 the partial derivative $\partial_{w_o} \ldots \partial_{w_s}\Theta$ is multiplicative of the same type as Θ over W_a^1 . By the assumption on s , and by lemma 3.2 (2) , the $(s + 1)$-tuples for which the derivative does not vanish identically over W_a^1 have sums which form a dense subset of W^{s+1} .

Let $w_o, \ldots , w_s$ be such an $(s + 1)$-tuple, and consider the zeros induced by $\partial_{w_o} \ldots \partial_{w_s}\Theta$ on W_a^1 . Let them be $a+u_1, \ldots , a+u_g$. Then, by lemma 4.1 and theorem 3.1 B

$$-a - k = \sum u_j .$$

Now, among the u_j we must find all of the w_j , by the definition of ∂_{w_j}.

Hence $s + 1 \leq g$. By assumption, $b \in W_a^1$ whence $a = b - w$, for some $w \in W^1$, and if $r > s$, w must occur among the u_j with multiplicity $(r - s)$. We can then write $-b + w - k = w_o + \ldots + w_s + (r-s)w + u_{r+2} + \ldots, + u_g$, after a suitable renumbering of the u (if necessary). If $r = s$, we get $-b + w - v \in W_k^{g-r-1}$, where $v = w_o + \ldots + w_s$ may be chosen arbitrarily from a dense subset of W^{s+1} . Hence the left hand side may be chosen arbitrarily from a dense subset of $-W_{b-w}^{s+1}$ which contains $-W_b^r$.

If $r - s = 1$, then $-b - v \in W_k^{g-r-1}$, or $-W_b^r \subset W_k^{g-r-1}$.

Finally, if $r - s \geq 2$ for all choices of w , then $-b - v - w \in W_k^{g-3-s}$, i.e. $-W_b^{s+2} \subset W_k^{g-3-s}$, which, by lemma 2.3, corollary 4 implies that $-W_{-b}^{s+2} \subset W_k^{g-3-s}$, which by lemma 4.4 implies that Θ vanishes of order $s + 2$ on every W_a^1 containing b . This contradicts the choice of s , and completes the proof of lemma 4.5.

The proof of Theorem 4.1 follows by observing that $-W_{-b}^r \subset W_k^{g-1-r}$ if and only if $-W_b^r \subset W_k^{g-1-r}$ by virtue of lemma 2.3, corollary 4.

5. AN EXTENDED TORELLI THEOREM

A theorem originally proved by Torelli asserts that the conformal structure of X is completely determined by any of its canonical period matrices. By Theorem 4.1 it can be seen that this result would follow from the assertion that the conformal structure of X is completely determined by $J(X)$ and the class of translates of W^{g-1} . The latter statement is also the natural version of Torelli's result for curves over an arbitrary field.

Over the field of complex numbers, it is possible to give a somewhat stronger theorem from which Torelli's result would follow as a special case:

Theorem 5.1 _Let X and Y be closed Riemann surfaces of genus $g > 1$, with a common Jacobian variety, $J(X) = J(Y)$. Let W^k_

<u>(respectively V^k) be the canonical image of the k-fold symmetric product of X (respectively Y) with itself in J(X) . If there is a point $a \in J(X)$ such that $W^t = V^t_a$, for some t , $1 \leq t \leq g - 1$, then X and Y are conformally equivalent.</u>

P r o o f : We assume $W^t = V^t_a$, and have $W^1 \subset V^t_a$. Let r be the smallest integer such that $W^1 \subset V^{r+1}_b$ for some $b \in J(X)$. If $W^1 \cap V^r_c$ contains two distinct points for any $c \in J(X)$, then $-W^1$ is contained in a translate of V^r , by lemma 2.6. We assume first that this does not happen. Consider the intersection $W^1 \cap V^{g-1}_{b+x-y}$, where $x \in V^1$, $y \in V^{g-1-r}$. Since $W^1 \subset V^{r+1}_b$, by assumption, the intersection may be written as $W^1 \cap (V^{g-1}_{b+x-y} \cap V^{r+1}_b)$. By lemma 2.6 the intersection in parenthesis is of the form $V^r_{b+x} \cup S$, where S is independent of the choice of x .

We now invoke Theorem 3.1 B. V^{g-1}_{b+x-y} is the divisor induced by a translate of the thetafunction formed with a canonical matrix of Y . Hence it is multiplicative of some type over J(X) , and if it does not vanish identically over W^1 , it induces q zeros, $u_1, \ldots, u_q$ on W^1 , counting multiplicities, such that

$$(5.1) \qquad \sum u_j = T(a + x - y) + z_1 ,$$

where z_1 is a constant independent of a , x , and y . The induced zeros are the points of the intersection $W^1 \cap V^{g-1}_{b+x-y}$.

Suppose that the thetafunction vanishes identically over W^1 for all choices of x and y . Keeping x fixed, this means that $W^1 \subset V^{g-1}_{b+x} \ominus V^{g-1-r} = V^r_{b+x}$, contrary to hypothesis. Hence there is a point $y \in V^{g-1-r}$ and a point $w \in W^1$ such that the function does not vanish. It follows that this must be the case for all x in a sufficiently small neighborhood of the original one.

Now keep y fixed, and let x vary over this neighborhood. Since the right hand side of (5.1) varies, so must the left. But the set S does not vary, and hence the variation on the left must come from a point in the intersection $W^1 \cap V^r_{b+x}$. By assumption, there cannot be two distinct points in this intersection, and hence the left hand side must vary over some translate of kW^1 , obtained from W^1 by multiplying each point with a multiplicity k . Thus T takes a neighborhood on V^1 into a translate of kW^1 . But T is an automorphism,[1)] and by the irreducibility of the sets involved, we find that $T(V^1) = (kW^1)_d$ for some $d \in J(X)$.

$T(V^1)$ is clearly conformally equivalent to V^1 and hence to Y . $(kW^1)_d$ is obtained from W^1 by a map which is bijective, except possibly on isolated points.[2)] Hence we have a holomorphic map from X to Y which fails to be bijective on at most a finite number of points. But every such map is a covering map, and it follows that it must be bijective.

To get rid of the assumption that $-W^1$ is not contained in a translate of V^r , we suppose now that this is the case, and define r to be the smallest integer for which an inclusion of the form $-W^1 \subset V^{r+1}_c$ occurs. Then $-W^1$ cannot have two distinct points in common with V^r_c for any $c \in J(X)$, and we can repeat the argument for $-W^1$.

This establishes Theorem 5.1.

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1) The characteristic matrix of a thetafunction is unimodular with respect to any period matrix.

2) $kw_1 = kw_2 \Longrightarrow k(w_1 - w_2) = 0$, and this has only a finite number of distinct solutions. If $w_1 - w_2 = w_3 - w_4 = w_5 - w_6$. Then

$$w_1 + w_4 = w_3 + w_2$$

$$w_1 + w_6 = w_5 + w_2$$

which is impossible unless $w_4 = w_6$, $w_3 = w_5$.

References

The material in section 2 was inspired by lemmas 1 and 3 of Weil's paper on the Torelli theorem ((6)) . Sections 3 and 4 are expositions of results obtained by Riemann in ((4)) and ((5)) , with some modifications. A very elegant treatment of the material in section 4 will be found in Mayer ((3)), where further references will be found. Another exposition was recently published by Lewittes ((1)) . I should like to say that the idea of using derivatives of thetafunctions was my own, but Mayer informs me that it belongs to Christoffel. The extended Torelli theorem of section 5 is new, and the proof was suggested by the methods of ((2)) .

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