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ON COMPACT SETS AND SIMPLEXES

IN INFINITE DIMENSIONAL SPACES

By

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Let K be a compact, convex subset of a locally convex, topological (Hausdorff) vectorspace E over the reals, and let $\mathcal{H}$ be the set of all K-restrictions of continuous, affine functionals on E. $\mathcal{H}$ is an Archimedian, partially ordered vectorspace with strong order unit, and these properties characterize function spaces $\mathcal{H}$. More specifically, it can be shown ((8)) that if $\mathcal{A}$ is an Archimedian, partially ordered vectorspace with strong order unit, and if K is taken to be the convex set of states (normalized, positive functionals on $\mathcal{A}$) and E is taken to be the space of order-bounded linear functionals on $\mathcal{A}$ (i.e. $E = \mathbb{R}^+K - \mathbb{R}^+K$), then E is locally convex and K is compact in the w^*-typology, and the mapping $a \longrightarrow \hat{a}$ defined by:

$$\hat{a}(p) = p(a) , \qquad p \in K , \tag{1}$$

is a linear isomorphism and an order isomorphism of $\mathcal{A}$ onto $\mathcal{H}$.

The term "state" is transferred from the case in which $\mathcal{A}$ is the self-adjoined part of a C^*-algebra. Another important special case is obtained if $\mathcal{A} \subset \mathcal{C}_R(\Omega)$ for some compact space Ω, and if:

(i) $\mathcal{A}$ is a vector-subspace of $\mathcal{C}_R(\Omega)$

(ii) $1 \in \mathcal{A}$

(iii) $\mathcal{A}$ separates points.

The classical example of a function space $\mathcal{A}$ satisfying (i) - (iii), is the space of harmonic functions on a disk. The conditions (i) - (iii), however, admit an abundancy of interesting special spaces, many of which are of the form $\mathcal{A} = \{ a \mid L a = 0 \}$ where L is a linear operator.

If $\mathcal{A}$ is a function-space satisfying (i) - (iii), then one may assign to every point ω in Ω a state $\hat{\omega}$ defined as the evaluation of the functions in $\mathcal{A}$ at the point ω. It is easily verified that $\omega \longrightarrow \hat{\omega}$ is an homéomorphism of Ω into K, and by definition:

$$(2) \qquad \hat{a}(\hat{\omega}) = \hat{\omega}(a) = a(\omega) .$$

Thus, if we think of $\omega \longrightarrow \hat{\omega}$ as an embedding of Ω into K, then the representation $a \longrightarrow \hat{a}$ will be a continuous and affine extension.

It should be noted that all extreme points of K belong to the image $\hat{\Omega}$ of Ω. The corresponding initial points in Ω are called Choquet points, and the set of Choquet points is called the Choquet boundary, $\partial_c \Omega$ of Ω with respect to $\mathcal{A}$. It admits various intrinsic characterizations.

It is worth mentioning that a similar embedding of Ω into K is possible in more general cases with non-compact Ω. Then the extreme points of K will belong to the closure of $\hat{\Omega}$ in K, and corresponding initial points may be adjoined as limit points to Ω, in which case one usually talks about Martin boundary rather than Choquet boundary.

A series of interesting results on function spaces $(\Omega, \mathcal{A})$ may be obtained by transfer of known properties of continuous affine functions on convex compact sets. For example, it is known that the closure of the extreme boundary, $\partial_e K$ (i.e. the set of extreme points), acts as Šilov boundary of K with respect to H. (This follows by the theorems of Krein-Milman, Milman, and Hahn-Banach.) Hence one may conclude that the closure of the Choquet boundary acts as Šilov boundary of Ω with respect to $\mathcal{A}$. In this way one obtains an existence proof of the Šilov boundary which does not invoke any multiplicative structure (nor the axiom of choice).

More interesting perhaps, is the transfer of Choquets integral theorem, stating that every point x in K can be expressed as the barycenter

$$(3) \qquad x = \int t \, d\mu_x(t) \qquad \text{(weak integral),}$$

of a (generally, not unique) positive, normalized boundary measure μ_x. In the metrizable case, $\partial_e K$ is measurable (in fact a G_δ), and a boundary measure is simply a measure μ such that

(4) $\qquad |\mu|(K - \partial_e K) = 0$.

The general definition is based on the following characterization of $\partial_e K$ due to M. Hervé (cf. ((5))):

(5) $\qquad \partial_e K = \bigcap \{ B_f \mid f \in \mathcal{C}_R(K) \}$

where

(6) $\qquad B_f = \{ x \mid f(x) = \underline{f}(x) \}$

and

(7) $\qquad \underline{f}(x) = \sup \{ h(x) \mid h \in \mathcal{H} , h \leq f \}$

The set B_f is called the <u>boundary set</u> determined by f , and $\underline{f}$ is the <u>lower semi-continuous convex envelope</u> of f (by Hahn-Banach's theorem in $E \times \mathbb{R}$) . In the duality theory of convex functions studied by Fenchel, Brönsted and Rockafellar ((6)) ((4)) , ((9)) , it is the second conjugate f'' . It is also worth noticing that transfer of (5) to the function space $(\Omega, \mathfrak{A})$ yields one of the intrinsic characterizations of the Choquet boundary of Ω with respect to $\mathfrak{A}$.

In the metrizable case one may replace the intersection (5) by the intersection of a <u>countable</u> subfamily of $\{ B_f \mid f \in \mathcal{C}(K) \}$ (and in fact by a single boundary set B_f , as shown by M. Hervé (cf. ((5))) . Hence in this case (4) equivalent to

(8) $\qquad |\mu|(K - B_f) = 0 \quad , \qquad\qquad$ for all $f \in \mathcal{C}(K)$.

In the general (non-metrizable) case (4) is meaningless, and (8) is taken as the <u>definition</u> of a boundary measure.

It is shown by Mokobodzki ((5)) that a positive measure is a boundary measure iff it is <u>maximal</u> in the sense of Bishop and de Leeuw, i.e. maximal in $\mathcal{M}^+(K)$ with respect to the partial ordering defined by the cone in $\mathcal{M}(K)$ which is <u>polar</u> to the cone $\mathcal{S}$ of continuous convex functions in

$\mathcal{G}(K)$. (The original definition of Bishop and de Leeuw only involves the set $\{h^2 \mid h \in \mathcal{H}\}$ <u>generating</u> the closed convex cone $\mathcal{S}$ ((2))). Now the proof of the integral theorem (3), is obtained by an application of Zorn's Lemma in which the inductivity of the set of measures with barycenter x (in the ordering of Bishop and de Leeuw) follows by w^*-compactness of the set of positive normalized measures.

This much being said about the general Choquet theorem, we return to the case of $(\Omega, \mathcal{A})$ with a <u>metrizable</u> Ω (hence with a separable $\mathcal{A}$ and a metrizable K). Writing μ_ω for the inverse image of $\mu_{\hat{\omega}}$ by the mapping $\omega \longrightarrow \hat{\omega}$, and using the definition of a weak integral, one obtains

$$(9) \qquad a(\omega) = \int_{\partial_c \Omega} a(\sigma)\, d\mu_\omega(\sigma) \, ,$$

for all $a \in \mathcal{A}$. In the terminology of H. Bauer ((1)), μ_ω is called a <u>harmonic</u> <u>measure</u> corresponding to the point ω (relatively to $\mathcal{A}$).

When is μ_x (resp. μ_ω) <u>unique</u> for every $x \in K$?

By Choquet's uniqueness theorem this is the case if and only if K is a <u>simplex</u>, and then the correspondence $x \longrightarrow \mu_x$ may be considered as a generalization of the unique barycentric coordinatization in finite dimensional simplexes.

But what is an infinite dimensional simplex?

Choquet's original definition is a transfer of a rather exotic characterization of simplexes in Enclidean space. They are exactly those convex compacts for which the set of homothetic images is closed under finite intersections. A different characterization of Choquet simplexes is obtained (Choquet, [illegible] ((5))) if one assumes (without lack of generality) that K spans E, and that K is located on a hyperplane not passing through the origin. Now, K is a simplex if and only if the generated cone $\bigcup_{\lambda \geq 0} \lambda K$ determines a <u>lattice</u> <u>ordering</u> in E. This property is the starting point for the proof of the uniqueness theorem which is based on a decomposition lemma for vector-lattices ((3, p. 19)) which in fact is a mere corollary to the general Schreier-Ore refinement theorem for modular lattices. In the present context it proves that the set of positive measures with barycenter x is directed in the ordering of Bishop and de Leeuw; hence there is a unique maximal measure majorizing all the others. Now, the "only if" part of the proof is fairly simple, and it is also possible to deduce a series of other interesting characterizations of simplexes. Among these we mention the fact, that K is a simplex if and only if $\underline{f}$ is affine (but not necessarily continuous) for every $f \in \mathcal{S}$ (i.e. f is continuous and convex). In 2-dimensional space this is the elementary fact that a chair with 3 legs stands firmly on the floor whereas chairs with more legs are unstable. Also it should be mentioned that K is a simplex if and only if the mapping $f \to \bar{f}$ is linear on $\mathcal{S}$, and that K is a simplex if and only if the set of continuous affine functions on K is a Riesz group in the sense of Fuchs ((7)).

Finally we mention another characterization of simplexes which is non-essentially different from the uniqueness property, and which is particularly well suited for the subsequent investigations of compact convex sets. It is based on the following definition: We shall say that $\partial_e K$ is affinely independent if the zero measure is the only signed boundary measure μ for which:

$$(10) \qquad \mu(K) = 0 \quad , \qquad \int t \, d\mu(t) = 0 \quad .$$

Now, K is a simplex if and only if $\partial_e K$ is affinely independent.

In spite of all the pleasant properties listed above, the simplexes may still exhibit some rather odd features. In 1959, E.Th. Poulsen gave an example of a simplex whose extreme boundary is dense. It is also known that a continuous function on the extreme boundary cannot always be extended to a continuous affine function on the entire simplex, as in the finite dimensional case. It was shown in 1960 by H. Bauer ((1)) that the simplexes with closed extreme boundary are exactly those convex compacts K for which any continuous function on $\partial_e K$ can be extended to a continuous affine function on K. He also showed that they are exactly those convex compacts for which the set of continuous affine functions is lattice ordered (and not only a Riesz group).

In the sequel we shall refer to simplexes with closed extreme boundary as r-simplexes. The letter r signifies "resolutive" as Bauer's theorem guarantees the solvability of a certain natural Dirichlet problem for K with respect to $\partial_e K$. (A warning: This Dirichlet problem does not correspond to the Dirichlet problem for Ω with respect to $\partial_e \Omega$ for the function spaces $\mathcal{A}$ and $\mathcal{C}(\partial_c \Omega)$, but for $\tilde{\mathcal{A}}$ and $\mathcal{C}(\partial_c \Omega)$, where $\tilde{\mathcal{A}}$ consists of Bauer's "$\mathcal{A}$-harmonic functions", i.e. those f for which $f = \underline{f} = \bar{f}$. The passage from $\mathcal{A}$ to a larger space $\tilde{\mathcal{A}}$ corresponds to the passage from $\mathcal{H}$ to the set $\tilde{\mathcal{H}}$ of functions continuous and affine on K. Note that $\tilde{\tilde{\mathcal{A}}} = \tilde{\mathcal{A}}$, and in many important cases $\tilde{\mathcal{A}} = \mathcal{A}$.)

Bauer's theorem yields a "concrete" representation of r-simplexes, they are of the form $\mathcal{M}_1^+(X)$, where X is an arbitrary compact space, and the quantifier $\mathcal{M}_1^+$ assigns to X the w^*-compact set of positive normalized measures on X. The extreme boundary of $\mathcal{M}_1^+(X)$ is homéomorphic to X, as is well known.

After this brief introduction of simplexes and r-simplexes we return to the general case. We know that a function in $\mathcal{H}$ is determined by its values on $\partial_e K$; but how far is it determined by its values on a subset A of $\partial_e K$?

It follows by Hahn-Banach's theorem that it is determined throughout the closed affine span of A, and not any farther. In the terminology of differential equations we may say that $\overline{\text{aff}(A)} \cap K$ is the set of determinance by A.

More interesting perhaps, is a reverse problem. If we fix a point $x \in K$, then the values $h(x)$ for $h \in \mathcal{H}$ are determined by the values of h at $\partial_e K$; but is it really necessary to know the values of h at the whole of $\partial_e K$? - can it be that a certain part of $\partial_e K$ (depending on x) is actually <u>irrelevant</u> for the determination of the values at the point x ?

This question can be rendered precise through the definition of a stable subset. we shall say that a closed subset A of K is <u>stable</u> if it supports any positive normalized measure on K with barycenter in A .

Now the following theorem furnishes an answer to the question.

T h e o r e m 1 . <u>A closed subset of K is stable if and only if it is a union of closed faces.</u>

The proof of this theorem is fairly simple and will be omitted. However, we recall the definition of a face. Briefly they are obtained by <u>relativization of supporting affine spaces</u>. Specifically, a subset F of K is a face if and only if it is of the form

$$F = K \cap M \quad , \tag{11}$$

where M is a supporting affine space.

For the sake of completeness we recall that an affine space M <u>supports</u> K if and only if $M \cap K \neq \emptyset$ and $K - M$ is <u>convex</u>.

Very little is known about the "facial structure" of convex compact sets. In the present lecture we shall point out that it may be quite irregular even in the case of simplexes.

T h e o r e m 2 . <u>There exists a (compact) simplex K with a face F for which $\bar{F}$ is no longer a face.</u>

The proofs of Theorem 2 and of Theorem 3 below are based on ideas communicated to us by V. Klee, combined with a general method to construct convex sets with pre-ascribed properties, which will be explained in connection with Theorem 4.

Also, it is natural to ask if any <u>closed</u> face is obtained by relativization of <u>closed</u> supporting spaces (in the sense of (11)). Closed faces of this kind may naturally be termed <u>supporting faces</u>. They play a distinguished role. In addition to being stable subsets, they are also equal to their own "set of determinance". Our next theorem states that the concept of a supporting face does not coalesce with the concept of a closed face.

T h e o r e m 3 . <u>There exists a (compact) simplex with a closed face which is not a supporting face.</u>

Finally we shall discuss an interesting problem arising in the non-simplicial case. Is it possible to obtain uniqueness of μ_x for a fixed x , by restricting μ_x to a subset of $\partial_e K$?

Geometrically one may restate the problem as follows: <u>Is it possible for every x in K to find a (closed) simplex A such that $x \in A \subset K$ and such that $\partial_e A \subset \partial_e K$?</u>

It is classical, but not entirely trivial, that this question has an affirmative answer in the finite dimensional case. A well known theorem of Charathéodory states that every point x in a compact convex set in $\mathbb{R}^n$ is a convex combination of at most $n+1$ extreme points. What is actually shown by Charathéodory, is that x is a convex combination of <u>affinely independent</u> extreme points, and so they span a simplex with the desired properties.

In the general case the answer is negative. In fact there exist non-simplexes which are <u>irreducible</u> at certain points, in the sense of the following:

T h e o r e m 4 . <u>There exists a (compact) non-simplex K with a point $x \in K$ such that no proper closed convex subset A of K satisfies</u>

(12) $\qquad x \in A \subset K , \qquad \partial_e A \subset \partial_e K$

The proofs of Theorems 2, 3, 4 are all fairly similar. They are based on a general method to construct convex compact sets with pre-ascribed affine dependences on $\partial_e K$, i.e. non-zero signed boundary measures satisfying (10) .

The method is in fact analogous to the definition of groups by means of generators and relations. The set of generators corresponds to the extreme boundary $\partial_e K$ which is a (completely regular) topological space and not only a set, and the relations corresponds to affine dependences on $\partial_e K$. A <u>free group</u> corresponds to a simplex (no affine dependences on $\partial_e K$), and for every <u>compact</u> set X , there is a unique simplex (in fact an r-simplex) with extreme boundary homéomorphic to X .

In the case of groups, one may introduce relations between generators by factoring out the free group with respect to the subgroups generated by the "words" defining the desired relations. In the present case we introduce affine dependencies on X by factoring out the simplex $\mathcal{M}_1^+(X)$ in $\mathcal{M}(X)$ with respect to subspaces generated by the measures μ defining the desired affine dependences.

We shall sketch how this technique can be applied to prove Theorem 4. An inspection of Charathéodory's proof shows that it is based on the fact that if x is convex combination, i.e.

$$x = \sum_{i=1}^{n} \lambda_i x_i \quad ; \quad \sum_{i=1}^{n} \lambda_i = 1 \; , \quad \lambda_i > 0 \quad ,$$

and the points $x_1, \ldots x_n$ are affinely dependent, i.e.

$$\sum_{i=1}^{n} \mu_i x_i = 0 \quad ; \quad \sum_{i=1}^{n} \mu_i = 0 \; , \quad \{\mu_1, \ldots \mu_n\} \neq \{0 , \ldots, 0\} ,$$

then x can be written as a convex combination of a <u>proper</u> subset of $\{x_1, \ldots , x_n\}$. In fact there are two (and usually just two) points which can be eliminated, namely the ones where λ_i/μ_i comes nearest to zero from the positive and from the negative side. (This is easily visualized by drawing a quadrilateral in the plane.)

To construct the desired counter example one has to start with a simplex such as $\mathcal{M}_1^+(\bar{N})$ where $\bar{N}$ is the one-point compactification of the natural numbers, and introduce an affine combination (a sequence $\{\mu_i\}$) such that for some point $\lambda = \{\lambda_i\} \in \mathcal{M}_1^+(\bar{N})$ the sequences $(\lambda_i/\mu_i)^+$, $(\lambda_i/\mu_i)^-$ both approach 0 as $i \longrightarrow \infty$.

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N.B.: Many of the most important results have been obtained by the group of French mathematicians working around Choquet, and they are "published" in seminar notes from the Paris Seminar on Potential Theory. These results are given in the survey article ((5)) .