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Holomorphically convex sets and domains of holomorphy

by

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Introduction: The purpose of these seminar notes is to show that if U is a holomorphically convex open set of an arbitrary complex space, then every point on the boundary is the natural frontier of a holomorphic function of $\dot{U}$. This was proved for connected open sets in C^n by Cartan-Thullen in 1932 [1], and it turns out that the essential idea of their proof carries over to the general case. The question whether a holomorphically convex relativly compact open subset of a complex space is a domain of holomorphy, is of importance in study of Levi's problem. In the papers of Grauert and Narasimhan ([3] and [5]) the problem is to estabish the implications:

1. U is strongly pseudoconvex $\Longrightarrow U$ is holomorphically convex, and
2. U is holomorphically convex $\Longrightarrow U$ is a domain of holomorphy.

In these papers, special properties of U are used to construct a function that cannot be extended to a given boundary point, and th technique is not always applicable. The main theorem of this note shows that implication 2 is always true.

This note is highly preliminary. There is a step in the rea soning of Cartan-Thullen that I dont understand, and it seems probable that if there is not a lacuna in their proof, a stronger res and a simpler proof in the general case may also be obtained.

Part I: Notation and preliminaries.

In the sequel, we use the standard notation of set theory. $\mathbb{N}$ denotes the set of natural numbers, and $\mathbb{C}$ the field of complex numbers.

Let U be a set, and $(f_i)_{i\in I}$ a family of complex-valued functions on U. We write $N((f_i)_{i\in I}) = \{x \in U : f_i(x) = 0; \ i \in I\}$

To avoid ambignity, we sometimes add a subscript U, and write $N_U((f_i)_{i\in I})$.

Let x be a point inatopological space X. N_x shall denote the set of open neighbourhoods of x in X. If A is a subspace of X. If A is a subspace of X and $x \in A$, $N_x(A)$ is the set of open neighbourhoods of x in A.

U and V are open subsets of topological space X. We write $U \subset\subset V$ iff $\overline{U}$ is compact subset of V.

If E is a subset of a topological space X, we write $\dot{E} = \overline{E} - \overset{o}{E}$, i.e. the set of frontier points of E.

For the notions of analytic and locally analytic sets of a domain G in C^n, and the basic results on irreducibility and local decomposition in irreducible components we refer the reader to [2],§1 or to [6]. Notice that if B is a proper analytic subset of an irreducible analytic set A, then $A - B$ is connected, and B is nowhere dense in A.

By a complex space, we mean a (reduced) complex space in the sense of Serre. For simplicity, we always assume that a complex space satisfies the second axiom a countabity. If A is a locally analytic subset of X, the sheaf of traces of holomorphic functions on A induces a structure of complex space on A. (Remark that open subsets and irreducible components of X are locally analytic subsets.)

When we consider A as a complex space, it is always with its induced complex structure. By $H(X)$ we denote the vectorspace of holomorphic functions on X.

When E is a relativly compact subset of X, $f \longrightarrow \|f\|_E = \sup_{x \in E} |f(x)|$ is a seminorm on $H(X)$. $H(X)$, equipped with the topology generated by these seminorms, is a locally convex topological vectorspace.

We say that a sequence $(f_n)_{n \in N}$ in $H(X)$ converges u.e.c. (uniformly on every compact), if it converges in this topology. By [2] , Satz 28, $H(X)$ is complete.

Let K be a compact subset of a complex space X.

By $\hat{K}$ we denote $\{x \in X : |f(x)| \leq \|f\|_K \; ; \; f \in H(X)\}$.

We say that f is holomorphically convex, iff K is compact $\Rightarrow$ $\hat{K}$ is compact.

Part II: The theorem.

Theorem: Let X be a complex space, and U an open set in X.

U is holomorphically convex $\Rightarrow (\forall x \in \dot{U})(\exists f \in H(U)$ [f is not extendable to a holomorphic function in any neighbourhood of x].

Proof: We assume $\dot{U} \neq \emptyset$. If not, the theorem is trivially true.

Let $X = \bigcup_{i \in I} X_i$ be a decomposition into irreducible components and $(U_s)_{s \in S} = \bigcup_{i \in I} \{V : V$ is a connected component of $U \cap X_i\}$. *)

We may assume $S = N$, or $S = \{1 \ldots n_o\}$; $n_o \in N$.

The first step of the proof is to construct a function $f \in H(U)$, such that:

i. $(\forall s \in S)[\, f|_{U_s} \not\equiv 0$, unless U_s is compact$]$.

ii. $\overline{N(f)} \supset \dot{U}$. (Closure taken in X).

* See correction at the end, 1.)

The principle of construction, which goes back to Cartan-Thullen [1] is as follows:

1. Choose a sequence $(x_i)_{i \in N}$ of points in U, with every point in $\dot{\bar{U}}$ as an accumulation point.

2. Find a sequence $(f_i)_{i \in N}$ in $H(U)$, such that:

a. $(\forall s \in S)\left[f_i|_{U_s} \not\equiv 0\text{, unless } U_s \text{ is compact}\right]$.

b. $f_i(x_i) = 0$.

c. $\prod_{i=1}^{\infty} f_i$ converges u.e.c. in U.

Then $N(\prod_{i=1}^{\infty} f_i) = \bigcup_{i=1}^{\infty} N(f_i)$, so $f = \prod_{i=1}^{\infty} f_i$ satisfies i and ii

Let $(V_i)_{i \in N}$ be an increasing sequence of open sets in U, with

$V_i \subset\subset U, \quad i \in N.$

$V_i \cap U_s \neq \emptyset$, for $i \leq s$.

$\bigcup_{i=1}^{\infty} V_i = U$ Set $\overline{V}_i = K_i$.

Choose a dense subsequence $(y_j)_{j \in N}$ of $\dot{U}$, and for each j, a fundamental system $(O_{j,k})_{k \in N}$ of neighbourhoods of y_j.

Let $(O_i)_{i \in N}$ be an ennumeration of $(O_{j,k})_{(j,k) \in N \times N}$.

Choose a sequence $(x_i)_{i \in N}$, such that $x_i \in (O_i \cap U) - \hat{K}_i$.

Obviously $\overline{\{x_i : i \in N\}} \supset \dot{U}$.

$$x_i \notin \hat{K}_i \Rightarrow (\exists\, h_i \in H(U))\left[\, |h_i(x_i)| < \|h_i\|_{K_i}\right.$$

$$(\exists\, e \in N)\left[\, \left\|\left(\frac{h_i}{h_i(x_i)}\right)^e\right\|_{K_i} < \frac{1}{2^i}\right]. \qquad \text{Set} \quad \left(\frac{h_i}{h_i(x_i)}\right)^e = g_{i,o}$$

and $f_{i,o} = 1 - g_{i,0}$. Notice that $f_{i,0}(x_i) = 0$.

If X is irreducible and U is connected, $f_{i,o}$ is the sought function, because $\sum_{i=1}^{n} \|f_{i,o} - 1\|_{K_1} = \sum_{i=1}^{n} \|g_{i,o}\|_{K_1}$ converges for

every $l \Longrightarrow \prod_{i=1}^{\infty} f_{i,o}$ converges u.e.c. on U. (Just as in [1]).
In the general case we must modify $f_{i,o}$ to get a. satisfied, without destroying property b and c.

We are going to construct inductivly two sequences $(g_{i,n})_{n \geqslant o}$ and $(f_{i,n})_{n \geqslant o}$ of functions in H(U), with $g_{i,o}$ as defined above, and

$$f_{i,n} = 1 - \sum_{k=o}^{n} g_{i,k} \,.$$

$(f_{i,n})_{n \geqslant o}$ and $(g_{i,n})_{n \geqslant o}$ shall satisfy

A. $f_{i,n} | U_s \not\equiv 0$ for $s \leqslant n+i$, unless U_s is compact.

B. $f_{i,n}(x_i) = 0$,

C. $\| g_{i,n} \|_{K_{i+n}} < \frac{1}{2^{i+n}}$

D. $\| g_{i,n+1} \|_{U_s \cap K_l}$ $\quad \frac{1}{2^{n+1}} \cdot \| f_{i,n} \|_{U_s \cap K_l}$ for $0 < s \leqslant l \leqslant i+n$,
and U_s noncompact.

Let $f_{i,o} \dots f_{i,n}$, $g_{i,p} \dots g_{i,n} \in H(U)$, satisfying A - D , be given.

Case 1. If $i+n+1 \notin S$, U_{i+n+1} is compact, or $f_{i,n} | U_{i+n+1} \not\equiv 0$, set $g_{i,n+1} = 0$, (and $f_{i,n+1} = f_{i,n}$). A - D are verified.

Case 2. $f_{i,n+1} \big|_{U_{i+n+1}} \equiv 0$. Let $Z_{i,n+1} \in U_{i+n+1} - \hat{K}_{i+n+1}$.
$(\exists\, h_{i,n+1} \in H(U)) [\, |h_{i,n+1}(Z_{i,n+1})| > \|h_{i,n+1}\|_{K_{i+n+1}} \,]$.

Thus $h_{i,n+1} - h_{i,n+1}(x_i)$ is nonconstant on U_{i+n+1}.

For each s, Us is an irreducible complex space. $f_{i,n} | Us$: $\Longrightarrow N(f_{i,n} | Us)$ is nowhere dense in Us. Therefore $f_{i,n} | Us \cap V_l$: if $n+i \geqslant l \geqslant s$, and $\| f_{i,n} \|_{Us \cap V_l} > 0$.

We may find a positive constant $r_{i,n+1}$, such that $\| r_{i,n+1} h_{i,n+1} - h_{i,n+1}(x_i) \|_{K_{i+n+1}} < \frac{1}{2^{i+n+1}}$ and

$$r_{i,n+1} \| h_{i,n+1} - h_{i,n+1}(x_i) \|_{K_l \cap Us} < \frac{1}{2^{n+1}} \cdot \| f_{i,n} \|_{K_l \cap Us}, \quad \text{for}$$

$$0 < s \leq l \leq i + n \quad \text{and} \quad Us \text{ noncompact.}$$

$g_{i,n+1} = r_{i,n+1}(h_{i,n+1} - h_{i,n+1}(x_i))$ satisfies <u>C</u> and <u>D</u>.

$$f_{i,n+1}(x_i) = f_{i,n}(x) - g_{i,n+1}(x_i) = 0$$

$$\| f_{i,n+1} \|_{Us \cap V_l} \geq \| f_{i,n} \|_{Us \cap V_l} - \| g_{i,n+1} \|_{Us \cap V_l} > (1 - \frac{1}{2^{n+1}}) \| f_{i,n} \|_{Us}$$

Thus $f_{i,n+1}$, $g_{i,n+1}$ satisfyes <u>A</u> - <u>D</u>, and there exists sequences satisfiing <u>A</u> - <u>D</u>.

For every compact K, we can find a K_{i+m} containing K. When $n > m$, $\| g_{i,n} \|_K \leq \| g_{i,n} \|_{K_{i+n}} < \frac{1}{2^{i+n}}$, so $\sum_{n=0}^{\infty} g_{i,n}$ converges u.e.c. on U against a $g_i \in H(U)$.

$\lim_{n \to \infty} f_{i,n} = 1 - g_i = f_i$. Evidently $f_i(x_i) = 0$ and $\| g_i \|_{K_i} < \frac{1}{2^{i-1}}$. Let $0 < s \leq l$, and Us noncompact. If $i+n \geq l$ $n \geq 0$, $\| f_{i,n+1} \|_{Us \cap K_l} \geq \| f_{i,n} \|_{Us \cap K_l} - \| g_{i,n+1} \|_{Us \cap K_l} > (1 - \frac{1}{2^{n+1}})$

$$\cdot \| f_{i,n} \|_{Us \cap K_l}$$

By inductien and going to the limit, we get $\| f_i \|_{Us \cap K_l} \geq$ $\| f_{i,n} \|_{Us \cap K_l} \cdot \prod_{s=1}^{\infty} (1 - \frac{1}{2^{n+s}})$. $\prod_{s=1}^{\infty} (1 - \frac{1}{2^{n+s}})$ is convergent with positive limit, so $f_i \not\equiv 0$ on Us. Now form $\prod_{i=1}^{\infty} f_i$. $\| f_i - 1 \|_{K_i} < \frac{1}{2^{i+1}} \Rightarrow \sum_{i=1}^{\infty} \| f_i - 1 \|_{K_l}$ converges for every l

so $\lim_{n\to\infty} \prod_{i=1}^{n} f_i$ converges u.e.c. against a function $f \in H(U)$.

Let Us be noncompact. $N_{Us}(f) = \bigcup_{i=1}^{\infty} N_{Us}(f_i)$. $N_{Us}(f_i)$ is nowhere dense in Us, so by Baire's category theorem $N_{Us}(f) \neq Us$ and a $f|_{Us} \not\equiv 0$. f is 0 on every x_i, $i \in N$, and therefore satisfies <u>ii</u>.

Now let $x \in \dot{U}$. x is element of finitaly many irreducible components of X, say $X_1 \dots X_n$, and has a neighbourhood 0 that does only meet $X_1 \dots X_n$. *

There are two possibilities:

<u>Case 1</u>. $(\exists i)(1 \leq i \leq n)(\forall W \in N_x)[W \cap X_i \cap U \neq \emptyset \; \& \; W \cap X_i \cap \mathring{\complement U} \neq \emptyset.]$

or Case 2. $(\forall_i)(1 \leq i \leq n)(\exists W \in N_x)[W \cap X_i \cap U = \emptyset \vee W \cap X_i \cap \mathring{\complement U} = \emptyset.]$

<u>Proof of the theorem in case 1</u>: Suppose f may be extended to a function $\overline{f} \in H(U \cup W)$, for some $W \in N_x$.

Let V be the connected component of x in $W \cap X_i$. V must meet some U_s, so $\overline{f}$ cannot be $\equiv 0$ on V. $N_V(\overline{f})$ is nowhere dense in V, so $V - N_V(f)$ must contain points of both U and $\mathring{\complement U}$. $N_V(f)$ is closed, so $V - N_V(f)$ is contained in the disjoint union of $V \cap U$ and $V \cap \mathring{\complement U}$, which are open in V. <u>Contradiction</u>.

<u>Proof of the theorem in case 2</u>. Suppose f may be extended to a function $\overline{f} \in H(U \cup W)$, for some $W \in N_x$.

$x \in \dot{U} \Rightarrow (\forall 0 \in N_x)(\forall_i)(1 \leq i \leq n)[0 \cap X_i \cap \mathring{\complement U} = \emptyset.]$

Thus by possibly shrinking W, we get that for some i, $1 \leq i \leq n$,

$W \cap X_i \cap U \neq \emptyset$ and $W \cap X_i \cap \mathring{\complement U} = \emptyset$, that is, $W \cap X_i \cap \complement U = W \cap X_i \cap \dot{U}$.

* See correction 2.)

Let V be the connected component of x in $W \cap X_i$. V must meet some U_s, so $\bar{f} \not\equiv 0$ on V. $N_V(\bar{f}) \supset V \cap \complement U$.
Now we want to construct a one-demensional locally analytic subset A of V, and an open set D on A, with compact boundary in A, such that $\bar{D} \cap \complement U = \{x\}$.
We make use of the Remmert-Stein local description theorem. A neighbourhood V' of x may be mapped by a biholomorphic mapping F onto an analytic subset of polycylinder $P_n = P_d \times P_{n-d}$ in $\mathbb{C}^n$. $F(x) = 0$. By choosing V' and P_n suitably, we may assume that there are n-d pseudopolynomials

$$p_i(Z_1 \ldots Z_d) = Z_i^{l_i} + a_1^{(i)}(Z_1 \ldots Z_d) : Z_i^{l_i - 1} + \ldots + a_{l_i}^{(i)}(Z_1 \ldots Z_d), \quad d+1 \leqslant i \leqslant n,$$

with $a_j^{(i)}(Z_1 \ldots Z_d)$ holomorphic functions in P_d, such that $F(V)$ is an irreducible component of $N(p_{d+1}, \ldots p_n)$.
We may also assume that the projection $\pi : N(p_{d+1} \ldots p_n) \longrightarrow P_d$ is surjective, proper, and with discrete fiber, and that $a_j^{(i)}(0 \ldots 0) = 0$, $d+1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant l_i$, such that $\pi^{-1}(0, \ldots, 0) = \{(0, \ldots, 0)\} = \{F(x)\}$. ([2] p.255).
$B = \pi F(N_{V'}(\bar{f}))$ is a lowerdimensional analytic subset of P_d, so there is an $y \in P_d - B$. Set $L = \{z \in \mathbb{C}^d : z = x + \lambda(y-x) : \lambda \in \mathbb{C}\}$.
$B = N(g_1 \ldots g_m)$ in a neighbourhood of 0. $L \not\subset B \Longrightarrow g_1|_L \ldots g_m|_L$ are not all identically 0, so for at least one g_k, the zeros are discr
Thus we can find $r > o$, such that when $L_r = \{z \in L : |\lambda| \geq r\}$, $\bar{L}_r \cap B = \{o\}$. Set $\pi^{-1}(L_r) = D'$. $\dot{D}'$ is compact, and $\bar{D}' \cap F(N_{V'}(\bar{f})) = \{0\}$. Define $D = F^{-1}(D')$, and $A = F^{-1}(\pi^{-1}(L \cap P_d))$
A, D has the properties we want.

Assume that every $h \in H(U)$ may be extended to a holomorphic function $\bar{h}$ in $U \cup V_h$, where V_h is some neighbourhood of x, depending on h. $\bar{h}$ induces an holomorphic function on $\bar{D}$, while $x \in V_h$. The maximum principle is valid for every complex space (Grauert-Remmert [2], so $(\forall h \in H(U)) \left[\|\bar{h}\|_{\bar{D}} = \|h\|_{\dot{D}} \right]$. This means that $D - \{x\} \subset \hat{D}$, so $\hat{D}$ cannot be compact.

Contratiction. Q.E.D.

Part III: Scetch of an alternative proof of the theorem.

We conserve the notation of part II. In essentially the same way construct a sequence of points of U $(x_i)_{i \in I}$ and functions of $H(U)$ $(f_i)_{i \in I}$, such that $\overline{\{x_i : x_i \in X_k\}} \supset \dot{U} \cap X_k$,

$$f_i(x_i) = 0 \quad , \quad (\forall_s \in S) \left[f_i|_{U_s} \not\equiv 0 \right.$$

$$x_i \notin N(f_1 \dots f_{i-1}) \quad , \quad \prod_{i=1}^{\infty} f_i \text{ converges u.e.c. on } U.$$

Set $f = \prod_{i=1}^{\infty} f_i$.

We also construct sequences $(x_i')_{i \in N}$ and $(f_i')_{i \in N}$, with the same properties, and such that $x_i' \notin N(f)$, $i \in N$. Set $f' = \prod_{i=1}^{\infty} f_i'$. Evidently f and f' cannot be extended to boundary points of type 1. Let x be a boundary point of type 2. We may suppose that W is a neighbourhood of x, such that f and f' may be extended to $\bar{f}$ and $\bar{f}'$, holomorphic in $U \cup W$, $W \cap X_i \cap \complement U \subset \dot{U}$, and that $N(\bar{f}) \cap W$ and $N(\bar{f}') \cap W$ consist of finitely many irreducible components. Let $A_1 \dots A_n$ be the irreducible components of $N(\bar{f}) \cap W \cap X_i$ and $B_1 \dots B_r$ the irreducible components of $N(\bar{f}') \cap W \cap X_i$ different from $A_1 \dots A_n$.

Then $\dot{U} \cap W \cap X_i \subset \bigcup_{K=1}^{n} \left(\bigcup_{l=1}^{r} (A_K \cap B_l) \right)$.

But $\bigcup_{l=1}^{r} (A_k \cap B_l)$ is nowhere dense and does not disconnect A_k.

$A_k - \dot{U}$ is therefore connected, and $N_U(f) \cap X_i \cap W$ has the irreducible components $A_1 - \dot{U}$, ..., $A_n - \dot{U}$.

On the other hand $A_k - \dot{U}$ must consist of infinitely many irreducible components, for if $x_i \in W \cap X_i$, $x_i \notin N(\prod_{j<i} f_j)$.

<u>Contradiction</u>.

In fact, we have proved a slightly stronger result:

For every $x \in \dot{U}$, either f or f' may not be extended to a function holomorphic in a neighbourhood of x.

<u>Part IV. Final remarks</u>.

We begin by mentioning two immediate consequences of the theorem.

Let X be complex space. We say that X is K-convex, if for every $x \in X$, we can find $f_1 \dots f_p \in H(X)$, such that if $F = (f_1 \dots f_p) : X \longrightarrow C^p$, x is an isolated point of $F^{-1}(F(x))$.

Suppose $D \subset\subset X$, X complex space. D is (strongly) pseudoconvex $\Leftrightarrow (\forall x \in \dot{D})(\exists V \in \mathcal{N}_x)(\exists \varphi : V \longrightarrow \mathbb{R})[\varphi$ is (strongly) pseudoconvex and continuous, and $V \cap D = \varphi^{-1}(]-\infty , 0[ )]$.

D is globally (strongly) pseudoconvex $\Leftrightarrow (\exists V)$ $(\exists \varphi : V \longrightarrow \mathbb{R})$ $[V$ is an open neighbourhood of $\dot{U}$, φ is (strongly) pseudoconvex, and $V \cap D = \varphi^{-1}(]-\infty , 0[ )$.

By [5] , Theorem I, D is strongly pseudoconves $\Rightarrow$ D is holomorphically convex.

By [7] Theorem 2, X is K-convex and D is globally pseudoconvex $\Rightarrow$ D is holomorphically convex.

By the theorem, in these cases D is a domain of holomorphy.

This does not seem to be wellknown, and in these cases the usual proof of implication 2 breaks down.

In [4], B. Malgrange gives a much shorter way of proving the theorem, for X a complex manifold. He obtains the stronger result (just as Cartan-Thullen) that there is a $f \in H(U)$ that can not be extended to any boundary point of U. We may form the function

$F = \prod_{i=1}^{\infty} f_i^i$. F has a 0 of order i at x_i. If M is a C^{∞}-Manifold, and f a C^{∞}- function on M, then for every $n \in N$, $\{x \in M : x$ is a zero of order n of $f\}$ is closed in M. Suppose $x \in \dot{U}$, and f may be extended to $\overline{f} \in H(U \cup V)$, where $V \in N_x$. Then, for every $n \in N$, x is an accumultaion point of zeros of order n of f, thus i a zero of order n of $\overline{f}$. $\overline{f}$ must be identically zero in some neighbourhood of x. Contradiction.

Suppose x is element of a complex space X. By M_x we denote the maximal ideal of the local ring $\mathcal{O}_x$ of x. If for every $\{x \in X : f_x \in M_x^n\}$ is closed (f_x is the germ of f at x), Malgranges proof would carry over to the general case, because $\bigcap_{n=1}^{\infty} M_x^n = \{0\}$. But I dont know whether this is true - .

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Correction. October 1965.

In general there is no reason to believe an irreducible complex space is locally irreducible, or that an open connected subset of an irreducible complex space is irreducible. This was mistakenly assumed in these seminar notes, and therefore some details must be changed in order to get a correct proof.

1.) Let $(\mathfrak{U}_s)_{s \in S}$ be a decomposition of $\mathfrak{U}$ into irreducible components. Each $\mathfrak{U}_s$ is closed in $\mathfrak{U}$, so if K is a compact contained in $\mathfrak{U}$, $\mathfrak{U}_s \cap K$ is compact. Thus, if $\mathfrak{U}_s$ is noncompact, $\mathfrak{U}_s$ is not contained in any compact subset of $\mathfrak{U}$.

The construction of f is then to be made with $\{\mathfrak{U}_s\}_{s \in S}$ Defined above.

2.) Instead of the construction given in the notes, we use the following result (Abhyankar (44.29, p. 414):

Let a be a point in complex space X. Then exists a neighbourhood V' of a, and a fundament system of neighbourhoods $\{V_j\}_{j \in N}$ of a in V', such that:

i.) $X \cap V'$ has finitely many irreducible components $X_1' \ldots X_p'$, and $X'_{1_a} \ldots X'_{p_a}$ are the (distinct) irreducible components of X_a. (A_a denotes the germ of A at a, for any $A \subset X$).

ii.) $X_i' \cap V_j$ is irreducible for any i, j; $1 \leq i \leq p$, $j \in N$.

In the notes we may suppose that all neighbourhoods in question are contained in V', and substitute X_i' for X_i. The neighbourhood W may be chosen in V_j, and $V = W \quad X_i$. It is easily seen that a compact $\mathfrak{U}_s$ is also an irreducible component of X, and since it does not meet $\complement\mathfrak{U}$, we may suppose $V' \cap \mathfrak{U}_s = \emptyset$, for any compact $\mathfrak{U}_s$. Therefore $\mathfrak{U}_s \cap V \neq \emptyset$ implies that $f \not\equiv 0$ on $\mathfrak{U}_s \cap V$.

With these changes, the rest of the proof goes through.

Reference: S. Abhyankar: "Local Analytic Geometry." Academic Press 1964.