Matematisk Seminar
Universitetet i Oslo

Mars 1964
No 4.

Jordan homomorphisms of operator algebras.

by

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A Jordan homomorphism of a ring into another ring is a linear map with the two multiplicative properties i) $\emptyset(a^2) = \emptyset(a)^2$, ii) $\emptyset(aba) = \emptyset(a)\emptyset(b)\emptyset(a)$ for all a,b in the ring. Jacobson and Rickart [2] studied such maps and showed in particular that a Jordan homomorphism of an $n \times n$ matrix ring D_n $(n \geq 2)$ over an arbitrary ring D with an identity is the sum of a homomorphism and an anti-homomorphism. In operator theory one makes the following assumptions on a Jordan homomorphism $\emptyset$, 1) $\emptyset$ is a linear map over the complex numbers, 2) $\emptyset$ is self-adjoint i.e. $\emptyset(A^*) = \emptyset(A)^*$, 3) $\emptyset(A^2) = \emptyset(A)^2$, with A a self-adjoint operator. Such maps are also called C*-homomorphisms, and have been studied by Kadison [3] and [4]. He showed in particular that a C*-homomorphism of a C*-algebra is a Jordan homomorphism and by use of von Neumann algebra techniques and Jacobson and Richart's result that a C*-homomorphism of a von Neumann algebra is the sum of a homomorphism and an anti-homomorphism. In [8] it was shown that a C*-homomorphism of a C*-algebra onto another C*-algebra is "locally" either homomorphism or an anti-homomorphism (see also [9]). It is the purpose of the present note to generalize this result (Theorem 1) and to show that for a large class of C*-algebras - GCR-algebras - every C*-homomorphism is "locally" either a homomorphism or an anti-homomorphism.

By a C*-algebra we mean a uniformly closed self-adjoint algebra of operators on a Hilbert space. A state of a self-adjoint family of operators is a positive linear self-adjoint functional of norm 1. The states form a convex set the extreme points of which are called pure states. A vector state is a state of the form $\omega_x : A \longrightarrow (Ax,x)$. A representation of a C*-algebra is a self-adjoint homomorphism of it into another C*-algebra. An irreducible C*-algebra is one the commutant of which equals the scalar operators, i.e. those of the form λI with I the identity operator and λ a complex number. If f is a state of a C*-algebra $\mathfrak{A}$ then there exist a canonical representation φ_f of $\mathfrak{A}$ and a vector state ω_x of $\varphi_f(\mathfrak{A})$ such that $f = \omega_x \varphi_f$. f is a pure state if and only if φ_f is irreducible [7].

<u>Definition</u>. Let $\emptyset$ be a C*-homomorphism of a C*-algebra into another C*-algebra. We denote by $(\emptyset(\mathfrak{A}))$ the C*-algebra generated by $\emptyset(\mathfrak{A})$. We say $\emptyset$ is <u>regular</u> if for each irreducible representation ψ of $(\emptyset(\mathfrak{A}))$ each pure state of $\psi(\emptyset(\mathfrak{A}))$ has a unique pure state extension to $\psi((\emptyset(\mathfrak{A})))$.

<u>Remark</u>: A C*-homomorphism of a C*-algebra onto another C*-algebra is regular.

<u>Theorem 1</u>. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras and $\emptyset$ a C*-homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$. Suppose $\mathfrak{B} = (\emptyset(\mathfrak{A}))$. Then the following three conditions are equivalent:

i) $\emptyset$ is regular

ii) If ψ is an irreducible representation of $\mathfrak{B}$ then $\psi \circ \emptyset$ is either a homomorphism or an anti-homomorphism.

iii) There exists a closed (not necessarily proper) ideal N in $\mathfrak{A}$ such that $\mathfrak{A}/N$ is abelian and $\emptyset$ restricted to N is regular and the sum of a homomorphism and an anti-homomorphism.

A couple of consequences of Theorem 1 are worth mentioning.

Corollary 1. Let ϕ be a regular C*-homomorphism of a C*-algebra $\mathfrak{A}$ into another C*-algebra. If $\mathfrak{B}$ is a C*-algebra contained in $\mathfrak{A}$ then ϕ restricted to $\mathfrak{B}$ is regular.

Corollary 2. Let ϕ be a regular C*-homomorphism of a C*-algebra $\mathfrak{A}$ into another C*-algebra. Then the following identities hold:.

1) If $A_i \in \mathfrak{A}$, $i = 1,2,\ldots,n$ then,

$$\phi\left(\prod_{i=1}^{n} A_i + \prod_{i=n}^{1} A_i\right) = \prod_{i=1}^{n} \phi(A_i) + \prod_{i=n}^{1} \phi(A_i).$$

2) If A,B,C,D are in then

$$(\phi(AB)-\phi(A)\phi(B))(\phi(CD)-\phi(D)\phi(C)) = 0.$$

Corollary 2 is immediate from Theorem 1 ii) and the fact that the irreducible representations separate points of a C*-algebra.

The difficult part of Theorem 1 is i) $\Rightarrow$ ii). It is trivial that ii) $\Rightarrow$ i).

Lemma 1. If ϕ is a C*-homomorphism of a C*-algebra $\mathfrak{A}$ into a C*-algebra $\mathfrak{B}$ then the nullspace $\mathcal{N}$ of ϕ is a closed two-sided ideal in $\mathfrak{A}$.

Proof: If f is a state of $\mathfrak{A}$ then $\mathcal{J}_f = \{A \in \mathfrak{A} : f(A^*A) = 0\}$ is a left ideal in $\mathfrak{A}$. Thus $\mathcal{J} = \{A \in \mathfrak{A} : \phi(A^*A) = 0\} = \bigcap \mathcal{J}_{f\circ\phi}$ where f ranges through the states of $\mathfrak{A}$. Hence $\mathcal{J}$ is a left ideal in $\mathfrak{A}$. If A is self-adjoint in $\mathcal{N}$ then $\phi(A^2) = \phi(A)^2 = 0$ so $A \in \mathcal{J}$. Thus with $B \in \mathfrak{A}$, $0 = \phi(B^*A) = \phi(B^*A)^* = \phi((B^*A)^*) = \phi(AB)$. Since $\mathcal{N}$ is self-adjoint, $\mathcal{N}$ is a twosided ideal.

The key lemma is the next, which is a restatement of [8, Theorem 5 and Corollary 5.9]. See also [9].

<u>Lemma 2</u>. Let $\emptyset$ be a positive linear map of a C*-algebra with identity I into the bounded operators on a Hilbert space $\mathcal{H}$ such that $\emptyset(I)$ is the identity operator I on $\mathcal{H}$. Suppose that for each unit vector x in $\mathcal{H}$ the state $A \longrightarrow (\emptyset(A)x,x)$ is pure on $\mathfrak{A}$. Then either $\emptyset$ is of the form $A \longrightarrow f(A)I$ with f a pure state of $\mathfrak{A}$ or there exist an isometry V of $\mathcal{H}$ into a Hilbert space $\mathcal{K}$ and an irreducible homomorphism or anti-homomorphism ρ of $\mathfrak{A}$ on $\mathcal{K}$ such that $\emptyset = V^*\rho V$. Moreover, if $\emptyset$ is also a C*-homomorphism then $\emptyset$ is either a homomorphism or an anti-homomorphism.

In order to prove i) $\Longrightarrow$ ii) it is straightforward to show that we may assume $\mathfrak{A}$ has an identity I. Then $\emptyset(I)$ is the identity in $\mathcal{B} = (\emptyset(\mathfrak{A}))$.

<u>Lemma 3</u>. Let f be a state of $\mathcal{B}$ which is pure when restricted to $\emptyset(\mathfrak{A})$. Then $\varphi_f o\emptyset$ is either a homomorphism or an anti-homomorphism.

<u>Proof</u>. $f = \omega_x \varphi_f$ with φ_f irreducible. The map $\varphi_f o\emptyset$ is a C*-homomorphism of $\mathfrak{A}$ with nullspace $\mathcal{J}$. By Lemma 1 $\mathcal{J}$ is a closed two-sided ideal in $\mathfrak{A}$. Replacing $\emptyset$ by $\varphi_f o\emptyset$ and $\mathfrak{A}$ by $\mathfrak{A}/\mathcal{J}$ we may assume $\emptyset$ is a C*-isomorphism, ω_x is pure on $\emptyset(\mathfrak{A})$, and $\mathcal{B}$ is irreducible on the Hilbert space $\mathcal{H}$. Then $\omega_x\emptyset$ is a pure state of $\mathfrak{A}$, hence of the form $\omega_y o \psi$, where ψ is an irreducible representation of $\mathfrak{A}$ on a Hilbert space $\mathcal{K}$ and y is a unit vector in $\mathcal{K}$ [7]. Making use of the fact that $\emptyset$ is regular and that an irreducible C*-algebra is algebraically irreducible [5] we now contruct a linear isometry W of $\mathcal{K}$ onto the subspace $[\emptyset(\mathfrak{A})x]$ (= the subspace of $\mathcal{H}$ generated by vector of form $\emptyset(A)x$ with $A \in \mathfrak{A}$) such that if z is a unit vector in $\emptyset(\mathfrak{A})x$, then

$$\omega_z \circ \emptyset = \omega_{z'} \circ \psi$$

for some z' in $\mathcal{K}$ with $Wz' = z$. Hence $\omega_z \circ \emptyset$ is a pure state of $\mathfrak{A}$, and the map $A \longrightarrow E\emptyset(A)E$, where E is the projection on the subspace $[\emptyset(\mathfrak{A})x]$ satisfies the conditions of Lemma 2, hence is of the form $V^*\rho V$. It is not difficult to show V is a unitary map and then to show $E = I$, hence that $\emptyset$ is unitarily equivalent to ρ. The proof is complete.

Every pure state of $\emptyset(\mathfrak{A})$ has a pure state extension to $\mathfrak{B}$ [7]. Hence states like f in Lemma 3 separate $\emptyset(\mathfrak{A})$, hence by Lemma 3 representations like φ_f separate $\emptyset(\mathfrak{A})$. Again using that $\emptyset$ is regular, we can now complete the proof of i) $\Rightarrow$ ii).

It remains to show ii) $\Leftrightarrow$ iii). Recall that the structure space of a C*-algebra is the set of primitive ideals (i.e. kernels of irreducible representations) equipped with the hull-kernel topology.

<u>Lemma 4</u>. Let $\mathfrak{A}$ and $\mathfrak{B}$ be C*-algebras. Suppose $\emptyset$ is a regular C*-homomorphism of $\mathfrak{A}$ into $\mathfrak{B}$ such that $\mathfrak{B} = (\emptyset(\mathfrak{A}))$. Let Z denote the structure space of $\mathfrak{B}$. Suppose the set of kernels of 1-dimensional representations of $\mathfrak{B}$ is open in Z. Then $\emptyset$ is the sum of a homomorphism and an anti-homomorphism. Moreover, if Z is connected then $\emptyset$ is either a homomorphism or an anti-homomorphism.

<u>Proof of ii) $\Rightarrow$ iii)</u>. $\mathfrak{B}$ has no 1-dimensional representations if $\mathfrak{A}$ has none, in which case iii) follows from Lemma 4 with $N = \mathfrak{A}$. Otherwise let N be the intersection of the kernels of 1-dimensional representations of $\mathfrak{A}$. Then $\mathfrak{A}/N$ is abelian and N has no 1-dimensional representation. By Corollary 1 (which is a consequence

of ii) and therefore applicable) ϕ restricted to N is regular. An application of Lemma 4 completes the proof.

The proof of iii) $\Longrightarrow$ ii) is an easy consequence of the next two lemmas, with $\mathcal{J}$ = N in Lemma 6, and the fact that a C*-homomorphism from an abelian C*-algebra is regular - in fact is a homomorphism.

Lemma 5. Let ϕ be a C*-homomorphism of the C*-algebra $\mathfrak{A}$ into the C*-algebra $\mathcal{B}$. Suppose $\mathcal{B} = (\phi(\mathfrak{A}))$. Let $\mathcal{J}$ be a closed two-sided ideal in $\mathfrak{A}$. Then $(\phi(\mathcal{J}))$ is a closed two-sided ideal in $\mathcal{B}$.

Lemma 6. Let $\phi, \mathfrak{A}, \mathcal{B}, \mathcal{J}$ be as in Lemma 5. Let $\mathcal{K} = (\phi(\mathcal{J}))$. Then induce a C*-homomorphism $\Theta : \mathfrak{A}/\mathcal{J} \longrightarrow \mathcal{B}/\mathcal{K}$. If ϕ restricted to $\mathcal{J}$ & Θ are regular so is ϕ.

The nice C*-algebras are the so called GCR-algebras. A GCR-algebra is a C*-algebra all irreducible representations of which consists of completely continuous (compact) operators. A C*-algebra $\mathfrak{A}$ is GCR if it has a composition series $\{\mathcal{J}_\alpha\}_{\alpha \in I}$ of closed two-sided ideals $\mathcal{J}_\alpha$ with $\mathcal{J}_0 = (0)$ and $\bigcup_{\alpha \in I} \mathcal{J}_\alpha = \mathfrak{A}$ such that $\mathcal{J}_{\alpha+1}/\mathcal{J}_\alpha$ is CCR and if α is a limit ordinal then $\bigcup_{\beta<\alpha} \mathcal{J}_\beta$ is uniformly dense in $\mathcal{J}_\alpha$ [6]. If $\mathfrak{A}$ is separable then $\mathfrak{A}$ is GCR if and only if each irreducible representation of $\mathfrak{A}$ contains the compact operators [1].

Theorem 2. Every C*-homomorphism from a GCR-algebra into a C*-algebra is regular.

To prove this we make the following

Definition. A C*-homomorphism ϕ of a C*-algebra is said to be semi-regular if it satisfies the identity 1) of Corollary 2. A uniformly closed self-adjoint (complex) linear family $\mathfrak{A}$ of operators on a Hilbert space is a Jordan algebra if $A,B \in \mathfrak{A}$ implies $A^2 \in \mathfrak{A}$, $ABA \in \mathfrak{A}$. $\mathfrak{A}$ is <u>semi-regular</u> if $\prod_{i=1}^{n} A_i + \prod_{i=n}^{1} A_i \in \mathfrak{A}$ whenever $A_1,\ldots,A_n \in \mathfrak{A}$, $n \geq 1$. We then denote by <u>$\mathcal{R}(\mathfrak{A})$</u> the uniform closure of the set of operators $\sum_{i=1}^{n} \prod_{j=1}^{m_i} A_{ij}$ with A_{ij} self-adjoint in $\mathfrak{A}$. Then $\mathcal{R}(\mathfrak{A})$ is a real algebra and the self-adjoint operators in $\mathcal{R}(\mathfrak{A})$ are those in $\mathfrak{A}$ when $\mathfrak{A}$ is semi-regular. In fact, if $A = \sum_{i=1}^{n} \prod_{j=1}^{m_i} A_{ij}$ is self-adjoint in $\mathcal{R}(\mathfrak{A})$, $A = \frac{1}{2}(A+A^*) = \frac{1}{2} \sum_{i=1}^{n} (\prod_{j=1}^{m_i} A_{ij} + \prod_{j=m_i}^{1} A_{ij}) \in \mathfrak{A}$. A partial converse of Corollary 2 is obtained in <u>Lemma 7</u>. Let $\mathfrak{A}$ be a C*-algebra and $\emptyset$ a semi-regular C*-homomorphism of $\mathfrak{A}$ into the bounded operators on the Hilbert space $\mathcal{H}$. Let $\mathcal{B} = \emptyset(\mathfrak{A})$ and assume 1) $\mathcal{B}'$ is the scalar operators on $\mathcal{H}$ 2) $\mathcal{J} = \mathcal{R}(\mathcal{B}) \cap i\mathcal{R}(\quad) \neq \{0\}$. Then $\emptyset$ is either homomorphism or an anti-homomorphism.

<u>Proof</u>. $\mathcal{J}$ is a non zero ideal in $\mathcal{R}(\mathcal{B}) + i\mathcal{R}(\mathcal{B})$ & hence in $(\mathcal{B})$. Thus $\mathcal{J}$ is an irreducible C*-algebra. If A is self-adjoint operator in $\mathcal{J}$ then $A \in \mathcal{R}(\mathcal{B})$ hence in $\mathcal{B}$ since $\mathcal{B}$ is semi-regular. Thus $\mathcal{B} \supset \mathcal{J}$, and every vector state is pure on $\mathcal{B}$. An application of Lemma 2 completes the proof.

<u>Definition</u>. If $\mathfrak{A}$ is a Jordan algebra and E is a projection $\neq 0$ in $\mathfrak{A}$ we say E is <u>abelian</u> if $E\mathfrak{A}E$ is an abelian C*-algebra.

<u>Lemma 8</u>. Let $\mathfrak{A}$ be a Jordan algebra over the Hilbert space $\mathcal{H}$ such that $\mathfrak{A}'$ is the scalars. Let E be a projection in and x a unit vector in E. Then the following are equivalent:

i) E is abelian.

ii) ω_x is pure on $\mathfrak{A}$ and $E \leq I - [\mathfrak{A}x] + [x]$.

If $\mathfrak{A}$ is semi-regular the inequality in ii) is equality.

As the proof of this lemma has nothing to do with C*-homomorphism I will not go into it. By approximation and an application of [2] we obtain

Lemma 9. Let $\mathcal{C}(\mathcal{H})$ denote the compact operators on the Hilbert space $\mathcal{H}$. Let ϕ be a C*-homomorphism of $\mathcal{C}(\mathcal{H})$ into a C*-algebra $\mathcal{B}$. Then ϕ is semi-regular.

Lemma 10. Let ϕ be a C*-homomorphism of $\mathcal{C}(\mathcal{H})$ into the bounded operators on Hilbert space. Let $\mathfrak{A} = \phi(\mathcal{C}(\mathcal{H}))$, and assume $\mathfrak{A}'$ is the scalars. Then ϕ is either a homomorphism or an anti-homomorphism.

Proof. We show ϕ satisfies the conditions of Lemma 7. By Lemma 9 ϕ is semi-regular. Hence $\mathfrak{A}$ is a semi-regular Jordan algebra. Let F be a 1-dimensional projection in $\mathcal{C}(\mathcal{H})$. Then F is abelian, hence $E = \phi(F)$ is abelian in $\mathfrak{A}$. Let G be a 2-dimensional projection in $\mathcal{C}(\mathcal{H})$ containing F. Then $G\mathcal{C}(\mathcal{H})G \cong$ 2 x 2 matrices. Thus $\phi | G\mathcal{C}(\mathcal{H})G$ is the sum of a homomorphism ϕ_1, and en anti-homomorphism ϕ_2[2]. In order to show $\mathcal{R}(\mathfrak{A}) \cap i\mathcal{R}(\mathfrak{A}) \neq \{0\}$ it suffices to show $\phi_1 = 0$ or $\phi_2 = 0$. This follows from an application of Lemma 8 to the projection E.

Proof of Theorem 2. Let ψ be an irreducible representation of $\mathcal{B} = (\phi(\mathfrak{A}))$. Replacing $\mathcal{B}$ by $\psi(\mathcal{B})$, ϕ by $\psi \circ \phi$, and factoring out the kernel of $\psi \circ \phi$, we may assume ϕ is a C*-isomorphism and $\mathcal{B}$ irreducible. Then $\mathcal{B}$ has no ideal divisors of zero [6, Lemma 2.5], hence $\mathfrak{A}$ has no ideal divisors of zero, as follows from

Lemma 5. Since the homomorphic image of a GCR-algebra is GCR [6, Thm. 7.4], and a GCR-algebra with no ideal divisors of zero is isomorphic to an irreducible GCR-algebra [6, Lemma 7.4] we may assume $\mathfrak{A}$ is irreducible over the Hilbert space $\mathcal{H}$. This argument together with Lemma 10 shows incidentally that a C*-homomorphism of a GCR-algebra is regular. $\mathfrak{A}$ has a composition series $\{\mathcal{J}_\alpha\}_{\alpha \in I}$ with $\mathcal{J}_0 = (0)$ and $\mathcal{J}_1 = \mathcal{C}(\mathcal{H})$. To complete the proof we now use transfinite induction and all our available techniques.

Corollary. Let $\mathfrak{A}$ be a GCR-algebra and ϕ a C*-homomorphism of $\mathfrak{A}$ into a C*-algebra.
Then $(\phi(\mathfrak{A}))$ is a GCR-algebra.

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