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Convex ideals in ordered group algebras

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1. Introduction. The main object of study in abstract harmonic analysis is the algebra $L^1_C(G)$ consisting of all the complex-valued Haar-integrable functions on a locally compact abelian group G. Along with this algebra one also considers the larger algebra $M_C(G)$ of all complex-valued bounded Radon measures on G. Much less attention has been paid to the real counterparts of these algebras: The subalgebra $L^1_R(G)$ consisting of all the real functions in $L^1_C(G)$ and $M_R(G)$ consisting of all the real measures in $M_C(G)$. These two algebras acquires the additional structure of an ordered ring via the concepts of a positive function and a positive measure. The building stones of these algebras will accordingly be the convex ideals since these are exactly the kernels of order preserving homomorphisms. This gives rise to the following general question: What can be said about convex ideals of $L^1_R(G)$ and $M_R(G)$ in comparison with well-known results from the ideal theory of $L^1_C(G)$ and $M_C(G)$?

We showed in [3] that there is only one convex ideal among the regular maximal ideals of $L^1_R(G)$ - namely the kernel of the Haar measure. We here restate this result in Theorem 1 as well as a more striking formulation of it in terms of order-preserving homomorphisms. This latter formulation shows that the theorem contains more than what can immediately be deduced from the uniqueness of the Haar measure. The main result of the present note is Theorem 2 which represents a considerable strengthening of Theorem 1. In fact, this theorem shows that the "ordered" versions of spectral analysis (Wiener's Tauberian theorem) and spectral synthesis are of a rather trivial nature because of the scarcity of closed convex ideals in $L^1_R(G)$. We also offer a quite elementary approach to Theorem 1 in case of a rather wide class of groups which does not seem to be defined in the literature and which may prove to be of independent interest. In conclusion we give some remarks which seem to indicate that there are no easy counterparts to the theorems 1 and 2 in case of $M_R(G)$.

2. Convex maximal ideals in $L_R^1(G)$. We shall first give the relevant definitions. As remarked in the introduction L_R^1 is the ordered group algebra of all real-valued integrable functions on G under the ordering $f \geqslant g$ whenever $f(x) \geqslant g(x)$ a.e. on G. L_C^1 shall denote the usual group algebra of all complex-valued integrable functions on G. We recall that an ideal $\mathfrak{A}$ in a commutative ring R is called regular whenever $R/\mathfrak{A}$ has an identity. The ideal $\mathfrak{A} \subseteq L_R^1$ is said to be convex if $f, g \in \mathfrak{A}$ and $f \geqslant h \geqslant g$ implies $h \in \mathfrak{A}$. When dealing with maximal ideals one should carefully distinguish between the following two statements

A. $\mathfrak{A}$ is a maximal ideal having the property P

B. $\mathfrak{A}$ is maximal among the ideals having the property P.

It is clear that $A \Rightarrow B$. By means of Zorn's lemma it is trivial that $A \Leftrightarrow B$ in case P stand for "regular". It will be an immediate consequence of our main theorem (Theorem 2) that this equivalence also holds for the property "closed and convex".

The solution to the problem of finding all regular and convex maximal ideals is given by the following

Theorem 1. The only regular and convex maximal ideal in L_R^1 is the maximal ideal consisting of all functions in L_R^1 with zero integral. Otherwise expressed: If μ is an order-preserving homomorphism of L_R^1 onto an ordered field F then F is isomorphic to the field of real numbers and μ is the Haar measure of G.

The proof is accomplished by using standard techniques from Fourier analysis. For details the reader is referred to [3]. See also the proof of Theorem 2.

3. Convex closed ideals in $L_R^1(G)$. We shall now prove a condirerably stronger result than Theorem 1. The essential step in the proof is the following lemma which may also have some independent interest.

Lemma A Let $f \in L^1_R$ with $\int_G f(x)dx \neq 0$. Then there exists a $g \in L^1_R$ such that $f*g > 0$.

Proof: Since $\int_G f(x)dx \neq 0$ we have $\hat{f}(\alpha) \neq 0$ for all α in a certain compact neighbourhood K of the identity in $\hat{G}$. Let $\hat{p}$ be a non-zero positive definite function on $\hat{G}$ with support contained in K. The function

$$p(x) = \int_{\hat{G}} (x, \alpha)\hat{p}(\alpha)d\alpha$$

will by Bochner's theorem be a non-zero positive function in L^1_R. By a well-known theorem of Wiener (see Godement [2] Théorème A) we can further determine a function $F \in L^1_C$ such that

$$(1) \qquad \hat{F}(\alpha) = \frac{1}{\hat{f}(\alpha)}$$

for all $\alpha \in K$. We now put $F * p = g + ih$ and get

$$(2) \qquad \widehat{f * (g + ih)} = \hat{f} \,.\, \hat{F}.\hat{p}$$

Inserting (1) in (2) we obtain $\hat{f}\,\hat{F}\,\hat{p} = \hat{p}$ on K and since $\hat{p}(\alpha) = 0$ for $\alpha \notin K$ this shows that $\hat{f}\,\hat{F}\,\hat{p} = \hat{p}$ holds for all $\alpha \in \hat{G}$. By inversion we thus have

$$f * (g + ih) = f * g + if * h = p$$

Since f, g, h and p all belong to L^1_R this gives the desired result

$$f * g = p > 0$$

We are now ready to prove

<u>Theorem 2.</u> <u>Any proper closed convex ideal in</u> L^1_R <u>is contained in the kernel of the Haar measure.</u>

<u>Proof:</u> The proof is a repetition of the last part of the proof of Theorem 1 in [3]: Assume that $\mathfrak{A}$ is a proper closed convex ideal which is not contained in the kernel of the Haar measure. The ideal $\mathfrak{A}$ must then contain a function f such that $\int_G f(x)dx \neq 0$. By the above lemma $\mathfrak{A}$ must therefore also contain a non-zero positive function p. Being translation invariant $\mathfrak{A}$ will further contain a positive function h which is $> \varepsilon > 0$ on a neighbourhood of the identity element $e \in G$. Now nh also belongs to $\mathfrak{A}$ for any positive integer n and we can choose for any sufficiently small neighbourhood U of e a function h_U such that

$$0 < h_U < nh \quad \text{on} \quad U$$

for a suitable n and such that the h_U's constitute an approximate identity for L^1_R, i.e.

$$\lim_U (h_U * f) = \text{for any} \quad f \in L^1_R$$

Since $\mathfrak{A}$ is supposed to be convex we have $h_U \in \mathfrak{A}$ and since $\mathfrak{A}$ is closed we get

$$f = \lim_U (h_U * f) \in \mathfrak{A} \text{ for all } f \in L^1_R$$

contradicting that $\mathfrak{A}$ is proper.

4. Groups with flat integrable functions. The crucial step in the proof of Theorem 2 was Lemma A. The proof of this lemma was accomplished by using a couple of classical but rather deep results from harmonic analysis. The usual proof of Bochner's theorem relies among several other things on the uniqueness of the Haar measure and the Riesz representation theorem. It is therefore of some interest to remark that one can give an entirely elementary proof of a weaker form of Lemma A which nevertheless is sufficient to prove for instance Theorem 1 on the real line.

We shall say that G is a group with flat integrable functions if it has the following property: Given $\varepsilon > 0$ and a compact neighbourhood K of the origin in G we can find a function $f \in L^1_R(G)$ such that $f(x) > 0$ for all $x \in G$ and

$$1 - \varepsilon < \frac{f(x_1)}{f(x_2)} \leq 1 + \varepsilon$$

for all pairs $x_1, x_2 \in G$ such that $x_1 - x_2 \in K$. We call such an f a flat integrable function of type (ε, K). The real line is for instance a group with faat integrable functions. If K is contained in the interval $[-a, +a]$ the function $e^{-\frac{\varepsilon}{2a}|x|}$ will be a flat integrable function of type (ε, K).

We shall now give an elementary proof of the following weakened version of Lemma A.

Lemma B. If f is a function in $L^1_R(G)$ with compact supportt and non-vanishing integral on a group G with flat integrable functions then there exists a $g \in L^1_R(G)$ such that $f * g > 0$.

Proof: Let K be the support of f. We can suppose without loss of generality that

$$\int_G f(x)dx = \int_K f(x)dx = 1 \quad \text{and} \quad \int_K |f(x)|dx = M$$

Let g b· a flat integrable function of type (ε, K). Then

$$f * g(x) = \int_K g(x-y)f(y)dy = g(x) - \int_K (g(x) - g(x-y))f(y)dy$$

which gives

$$f * g(x) \geqslant g(x) - \int_K |g(x) - g(x-y)|\,|f(y)|\,dy$$

or

$$f * g(x) \geqslant g(x) - \int_K g(x)\left|1 - \frac{g(x-y)}{g(x)}\right|\,|f(y)|\,dy \geqslant g(x) - g(x)\varepsilon . M$$

By choosing g of type (ε, K) with $\varepsilon < \frac{1}{M}$ we therefore get the desired inequality $f * g > 0$.

Remarking that any maximal ideal in $L^1_R(G)$ contains a function with non-vanishing integral and compact support we obtain a new proof of Theorem 1 in case of groups with flat integrable functions. On the basis of Lemma B we can proceed in the same way as in the proof of Theorem 2.

5. Convex ideals in $M_R(G)$. It is well known that the ideal theory of $M_C(G)$ is quite a bit of a mystery. Even the maximal ideals of $M_C(G)$ have not been described in a satisfactory way. It seems that one has a similar increasing complexity when passing from $L^1_R(G)$ to $M_R(G)$. What corresponds to the kernel of the Haar measure in case of $M_R(G)$ is the convex ideal m_0 consisting of all measures μ with total mass equal to zero:

$$\mu(G) = \int_G d\mu = 0$$

But m_0 is not the only convex maximal ideal in $M_R(G)$. In fact using the Lebesgue decomposition

$$\mu = \mu_d + \mu_s + \mu_a$$

when μ_d is discrete, μ_s is singular and μ_a is absolutely continuous the set

$$\{\mu, \mu_d(G)\} = 0$$

will form a convex maximal ideal in $M_R(G)$ which is in general different from m_o. We therefore have no immediate counterpart to Theorem 1 in the case of $M_R(G)$. It is also easy on the basis of the Lebesgue decomposition to exhibit several closed non-maximal convex ideals of $M_R(G)$.

References.

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