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Some necessary and sufficient conditions in the theory of Vitali coverings.

Erik M. Alfsen

The paper [1] was published after the seminar lecture, but before the final preparation of the notes. Complete proofs are given in [1], and so we shall restrict ourselves to present the following:

Summary of results. A covering $\mathcal{K}$ of a set A in a measure space is called a Vitali covering if it consists of measurable sets of finite positive measure and if for all finite collections $K_1,\dots,K_n$ from $\mathcal{K}$ the set of members of $\mathcal{K}$ not meeting $K_1 \cup \dots \cup K_n$ is a covering of $A \setminus K_1 \cup \dots \cup K_n$. Recall that in the classical case the space is required to be metric, the members of $\mathcal{K}$ are required to be closed, and every point is required to be in members with arbitrarily small diameter. This information, however, is used only through the simple consequence, which is taken as a general definition above.

A Vitali covering $\mathcal{K}$ of a set A is star-regular (with parameter $\lambda < \infty$) if for every disjoint sub-collection $\mathcal{R}$ of $\mathcal{K}$ with $R = \bigcup_{K\in\mathcal{R}} K$, there shall exist a set $Q \in \mathcal{K}$ disjoint from R such that the $\mathcal{R}$-star

$$S_{\mathcal{R}}(Q) = \bigcup\{K \mid K \in \mathcal{K},\quad K\cap Q \neq \emptyset,\quad K \cap R = \emptyset\},$$

satisfies

$$\mu^*(S_{\mathcal{R}}(Q)) \leq \lambda\,\mu(Q)$$

It is not hard to verify that the set of all closed sets (or even rectangles) in the plane is non-regular, whereas it becomes regular if there is imposed a lower bound on the ratio between diameter and volum.

Theorem 1. If $\mathcal{K}$ is a star-regular Vitali covering of a set A in totally finite measure space, then there exists a disjoint (hence countable) sub-collection covering A almost entirely.

Every collection $\mathcal{K}$ of measurable sets of positive finite measure determines a largest (possibly empty) set $A_{\mathcal{K}}$ such that $\mathcal{K}$ is a Vitali covering of that set.

<u>Theorem 2.</u> <u>A Vitali covering $\mathcal{K}$ of a set A in an arbitrary measure space admits a countable, disjoint sub-collection covering A almost entirely if and only if $\mathcal{K}$ is countably star-regular in the sense that there shall exist a sequence $\mathcal{K}_n$ of star-regular sub-collection (the parameters may tend to infinity) each of which confined to a set of finite measure such that $\{A_{\mathcal{K}_n}\}$ covers A almost entirely.</u>

Corollaries are the classical theorem of Vitali, Charathéodory and Banach; the theorem of A.P. Morse for metric spaces with a "halo operation" defined by an abstract "disentanglement function", and also a recent theorem of Comfort and Gordon on Vital coverings of homogeneous spaces.

The Theorems 1,2 may be rephrazed <u>modulo null sets</u> (i.e. within the measure algebra). This is particularly convenient for applications to martingales with directed index sets.

We recall that if $\{\mathcal{B}_\gamma\}_{\gamma\in\Gamma}$ is an ascending stocastic base on a probability space (Γ -directed), then $\{F_\gamma\}_{\gamma\in\Gamma}$ is a <u>fine covering</u> of a set A if

$$A \subset \text{ess.}\,\overline{\lim}_\gamma\ F_\gamma$$

We define a <u>sub-base</u> $\{\mathcal{S}_\gamma\}_{\gamma\in\Gamma}$ by requiring that every $F\in\mathcal{B}_\gamma$ be an essential union of members of $\mathcal{S}_\gamma$. Note that if $\{F_\gamma\}_{\gamma\in\Gamma}$ is a fine covering of A relatively to $\{\mathcal{B}_\gamma\}_{\gamma\in\Gamma}$ with sub-base $\{\mathcal{S}_\gamma\}_{\gamma\in\Gamma}$ then

$$\mathcal{K} = \bigcup_{\gamma\in\Gamma}\{K \mid K\in\mathcal{S}_\gamma\ ,\quad K\subset F_\gamma \quad (\text{mod.}\,\mu)\}$$

is a <u>Vitali-covering of A (mod μ)</u> .

This is how the Vitali coverings appear in the present context.

Now define a sub-base $\mathcal{S}_\gamma$ to be <u>regular</u> (with parameter λ) if for every $\{\mathfrak{M}_\gamma\}_{\gamma\in\Gamma}$, where $\mathfrak{M}_\gamma\subseteq\mathcal{S}_\gamma$ there shall exist a $Q\in\bigcup_{\gamma\geq\gamma_1}\mathfrak{M}_\gamma$ such that

$$\mu(K_1\cup\ldots\cup K_n)\leq\lambda\,\mu(Q)$$

for all $K_1,\ldots,K_n\in\bigcup_{\gamma\in\Gamma}\mathfrak{M}_\gamma$ or only for $K_1\ldots K_n\in\bigcup_{\gamma\geq\gamma_1}\mathfrak{M}_\gamma$, if

$\gamma_1\in\Gamma$ is some index such that $\mathfrak{M}_\gamma\neq\emptyset$ only for $\gamma\geq\gamma_1$.

The weakening permitted by the last addition is particularly usefull if Γ is linearly ordered. If moreover Γ is well ordered, then γ_1 may simply be the first index for which $\mathfrak{M}_\gamma \neq \emptyset$. If $\Gamma = \{0,1,2,\ldots\}$ and if $\mathcal{S}_\gamma$ are partitions, then the sub-base is automatically regular.

If on the other hand Γ is a tree , then the possibility of choosing a γ_1 indicated is worthless.

In between these two extremes is the case where the measure space is an infinite dimensional product, Γ is the directed set of finite sets of factor spaces, and $\mathcal{B}_\gamma$ is the corresponding Borel-field of cylinder sets.

The main theorem in this connection is:

Theorem 3. An ascending stochastic base generated by a regular sub-base enjoys the strong Vitali property of Kricheberg and Pauc.

Recall that $\{\mathcal{B}_\gamma\}_{\gamma\in\Gamma}$ enjoys the strong Vitali property if for every fine covering $\{F_\gamma\}_{\gamma\in\Gamma}$ of a set $A \subset \bigcup_{\gamma\in\Gamma}\mathcal{B}_\gamma$ and every $\varepsilon > 0$, there is a finite set $\{\gamma_1,\ldots,\gamma_n\}$ from Γ and an essentially disjoint sequence $\{N_1\ldots N_n\}$ such that $N_i \in \mathcal{B}_{\gamma_i}$, $N_i \subset F_{\gamma_i}$ (mod. μ) for $i=1,\ldots,n$, and such that

$$\mu(A \setminus N_1 \cup \ldots \cup N_n) < \varepsilon$$

This property is important since it implies essential convergence ("almost certain convergence") of the integral representation $\{f_\gamma\}_{\gamma\in\Gamma}$ of any martingale of bounded variation with base $\{\mathcal{B}_\gamma\}_{\gamma\in\Gamma}$

Reference:

E.M. Alfsen: Some covering theorems of Vitali type. Math.Ann.159 (1965) p.203-216.
