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ON THE EXISTENCE AND UNIQUENESS OF HAAR MEASURE

By

Erik M. Alfsen

More than 20 years have passed since H. Cartan gave his constructive proof ((5)) avoiding the axiom of choice and proving existence and uniqueness simultaneously. In spite of this, his proof has not been generally adopted in subsequent presentations of the subject. It is considered to be more complicated and less intuitive than the traditional proofs going back to A. Haar ((6)) and A. Weil ((20)). (Cf. e.g. ((13, p. 113))). The aim of this lecture is to sketch a version of the constructive proof which is as simple and intuitive as the traditional non-constructive proofs, if not equally short.

The papers ((1)) ((2)) appeared after the presentation of this lecture at the Seminar, but before the preparation of the notes. Some slight modifications have been made in the original manuscript in this connection.

In the sequel, G is an arbitrary locally compact group, L is the class of continuous real valued function with compact support on G and for every member V of the neighbourhood filter $\mathcal{V}$ of the identity e, L_V is the class of all $f \in L$ vanishing off V. For every $f \in L$ the *conjugate* function f^* is defined by $f^*(x) = f(x^{-1})$. For every $f \in L$ and $s \in G$, the *left* and *right translates* f_s and f^s are defined by $f_s(x) = f(s^{-1}x)$ and $f^s(x) = f(xs)$. (These are the conventions of ((17))). For $f, \varphi \in L^+$ and $\varphi \neq 0$, we define:

$$(1) \qquad \overline{(f : \varphi)} = \inf\left\{ \sum_{i=1}^{n} \alpha_i \;\middle|\; f \leq \sum_{i=1}^{n} \alpha_i \varphi_{s_i} \right\}$$

$$(2) \qquad \underline{(f : \varphi)} = \sup\left\{ \sum_{j=1}^{m} \beta_j \;\middle|\; \sum_{j=1}^{m} \beta_j \varphi_{t_j} \leq f \right\}.$$

By the local compactness, these expressions are well defined (and finite). If $f \neq 0$, then $(\overline{f : \varphi}) > 0$ for all φ, and $(\underline{f : \varphi}) > 0$ for all φ in some L_V^+, $V \in \mathcal{V}$.

It is an elementary fact that for a fixed φ, the mapping $f \to (\overline{f : \varphi})$ is i s o t o n e , s u b - l i n e a r , s u b - t r a n s i t i v e (i.e. $(\overline{f : \varphi}) \leq (\overline{f : \psi})(\overline{\psi : \varphi})$), and l e f t - i n v a r i a n t (i.e. $(\overline{f_s : \varphi}) = (\overline{f : \varphi})$). It is "nearly additive" when φ is "concentrated around e". Specifically, if $f_1, \dots, f_n \in L$ and $\lambda > 1$, then there exists a $V \in \mathcal{V}$, such that

$$(3) \qquad \sum_{i=1}^{n} (\overline{f_i : \varphi}) \leq \lambda (\overline{\textstyle\sum_{i=1}^{n} f_i : \varphi}) , \qquad \text{for all } \varphi \in L_V^+$$

For proof cf. e.g. ((13, p. 114)).

It is easy to prove ((1)) that if $f_\gamma \in L$, $f_\gamma \to f$ uniformly, and $\operatorname{spt}(f) \subset K$ where K is compact, then $(\overline{f_\gamma : \varphi})/(\overline{f : \varphi}) \to 1$ uniformly in φ. By the Dini Lemma, this entails that if $f_\gamma \in L^+$, $f_\gamma \uparrow f$ L, then $(\overline{f_\gamma : \varphi})/(\overline{f : \varphi}) \uparrow 1$ uniformly in φ. (In the theory of integration in completely regular spaces such properties are referred to as "uniform tightness" and "uniform τ-continuity", respectively.)

It should also be noticed that from the property (3) one can easily deduce that $(\underline{f : \varphi}) \leq (\overline{f : \varphi})$ for all $\varphi, f \in L^+$, $\varphi \neq 0$ ((1)).

The existence and uniqueness of left Haar measure will follow from the existence and uniqueness, up to a positive factor, of a non-trivial, left-invariant, positive linear functional on L. In this connection we observe that such a functional I is determined up to a positive factor by its a s s o c i a t e d p r e - o r d e r i n g :

(4) $\qquad f \leqq g \pmod{I} \iff I(f) \leqq I(g)$

In fact one has the following, somewhat stronger statement:

P r o p o s i t i o n 1 . Two non-trivial left-invariant, positive linear functionals on L with comparable (finer-coarser) pre-orderings differ only by some positive factor.

The p r o o f is a simple calculation based on homogeneity and strict positivity (i.e. $f > 0 \implies I(f) > 0$) which follows from non-triviality by a compactness argument ((1)) .

Proposition 1 is also easily obtained from the fact that the kernels of positive linear functionals on L are the maximal o r d e r i d e a l s . Following Bonsall ((3)) one may apply Zorn's Lemma to yield a maximal order ideal containing all differences $f - f_s$, $f \in L$, $s \in G$. (The elementary properties of the mapping $f \rightarrow \overline{(f : \varphi)}$ are then used to prove that there e x i s t s any proper order ideal containing these differences.) It is also possible to proceed along these lines obtaining a simultaneous proof of the existence and uniqueness without the axiom of choice. In fact, one may prove that the s t r o n g c l o s u r e $\bar{J}$ of the order ideal J spanned by the differences $f - f_s$, is maximal. (The strong topology on L is the inductive limit of the uniform norm topologies on L_K , for K compact.) Now the existence is immediate, and the uniqueness follows by the strong continuity of Radon measures. However, the proof that $\bar{J}$ is a proper maximal ideal, depends on the Cartan Lemma or the closely related separation axiom (S) of this lecture; and so the suggested procedure will hardly be any simpler than ours. On the other hand it is worth mentioning that the above setting admits an interesting improvement of the uniqueness theorem. K.E. Aubert has proved by the methods of abstract harmonic analysis ((4)) that the kernel of Haar measure is not only the unique maximal order ideal in L containing all differences $f_s - f$, but it is in fact the only maximal, regular, convex ideal occurring in $L^1(G)$. It would be of some interest to know if this result could be obtained directly from the separation property (S) of the present lecture.

Proposition 1 shows the importance of the pre-orderings associated with non-trivial, left-invariant, positive linear functionals and proves it sufficient to find one such relation c o m p a r a b l e with any other. (It is not difficult to characterize such pre-orderings axiomatically, the crucial axiom being that of a l i n e a r pre-ordering. However, since our construction yields the numerical value of the functional together with the pre-ordering, there will be no need for such a characterization.)

There is a natural definition of "relative size" on L^+ (which is easily extended to L):

$$(5) \qquad f \subseteqq g \iff \underline{(f : \varphi)} \leqq \overline{(f : \varphi)} \qquad \text{for all} \quad \varphi \in L^+, \varphi \neq 0$$

This type of definition is of course not new. In principle it is identical with Eudoxos' definition of relative size for incommensurable proportions. (Properly speaking, the latter relates to the negation of $f \subseteqq g$, which may be pronounced " g is of strictly smaller size than f ", and defined by " $\overline{(g : \varphi)} < \underline{(f : \varphi)}$ for some φ ".)

The definition (5) is equivalent to G. Bredon's definition of $f \lesssim g$ in ((2)), and his exposition is closely related to our presentation in ((1)). Similar notions of "relative size" for functions have been studied in an extremely abstract setting by H. Hadwiger, A. Kirsch and W. Nef ((7)) ((8)) ((12)) ((16)).

In the present context, the definition (5) gains importance by virtue of the following

P r o p o s i t i o n 2 . <u>The relation (5) is coarser than the pre-</u>

<u>ordering associated with any non-trivial, left-invariant, positive linear functional on L .</u>

The proof is not difficult and is given in ((1)) .

Now it is sufficient to prove that the relation (5) is itself a pre-ordering associated with some non-trivial, left-invariant, positive linear functional. To this end we claim that for all non-zero $f, g \in L^+$

$$(6) \qquad \inf_{\varphi} \frac{(g : \varphi)}{(f : \varphi)} = \lim_{\varphi} \frac{(g : \varphi)}{(f : \varphi)} = \lim \frac{\overline{(g : \varphi)}}{\overline{(f : \varphi)}}$$

where the indices $\varphi \in L^+$, $\varphi \neq 0$ are ordered by inclusion of the sets $\{x \mid \varphi(x) \neq 0\}$. By the elementary properties of the mapping $g \to \overline{(g : \varphi)}$, the limit at the right hand side of (6) would define a functional with the desired properties, and by the alternative expression at the left hand side of (6) , its associated pre-ordering would be exactly the one defined by (5) . Hence it suffices to prove the claim (6) .

Proposition 3 . <u>The claim (6) follows from the following separation property</u>

$$(S) \qquad f, g \in L^+ , \ f(x) < g(x) \quad \text{for} \quad x \in \operatorname{spt}(f)$$

$$\Rightarrow \exists V \in \mathcal{V} , \forall \varphi \in L_V^+ , \exists s_1, \dots, s_n \in G ,$$

$$\exists \alpha_1, \dots, \alpha_n > 0 : f \leqq \sum_{i=1}^{n} \alpha_i \varphi_{s_i} \leqq g .$$

The proof is given in ((1)) .

An immediate consequence of (S) is that if $f, g \in L^+$, $f(x) < g(x)$ for $x \in \mathrm{spt}(f)$, then for every $\varphi \neq 0$ in some L_V^+ :

$$(7) \qquad \overline{(f : \varphi)} < \underline{(g : \varphi)}$$

(One may state this as follows: If $f, g \in L^+$, $f(x) < g(x)$ for $x \in \mathrm{spt}(f)$, then f is of "strictly smaller size" than g .)

Our motivation for introducing the lower estimates $\underline{(f : \varphi)}$ and not confine our attention to $\overline{(f : \varphi)}$, is the possibility to define the relation (5) and state the crucial claim (6) .

We are now approaching the hard part of the proof, the verification of (S) . The latter is easily transformed to a uniform approximation of some h between f and g by functions $\sum_{i=1}^{n} \alpha_i \varphi_{s_i}$. This in turn could be obtained by means of the Haar integral. The existence of (right) approximate identities in the convolution algebra would yield an approximation of $h \in L^+$ by $I(\varphi)^{-1} h * \varphi$, where φ is "sufficiently concentrated around e ". In the next step one should write $h = \sum_{i=1}^{n} h_i$ where each h_i is sufficiently concentrated around some point s_i (decomposition of unity). Now the existence of (left) approximate identities yields an approximation of $I(\varphi)^{-1} h * \varphi$ by $\sum_{i=1}^{n} I(h_i)/I(\varphi)\, \varphi_{s_i}$, and we are through.

The obvious defect of this procedure is its dependence on the Haar integral. This defect, however, is not so severe as it may appear at first sight. What is required, is an approximation theorem, and so it is natural to expect that an approximative Haar integral would suffice in the proof. This in fact, is the underlying idea in either of the two known proofs.

One of these two (historically the first one valid in the completely general case) is the proof of H. Cartan, who sketched it in a brief note in

1938 ((5)) . It is written out in greater detail in ((17)) (as far as we can see, this presentation is somewhat obscure at one point), and there is a complete presentation in ((11)) . The viewpoints stated above, are expressed very explicitly in our recent note ((1)) . The "approximate integral" in question is simply $f \to \overline{(f : \varphi)}$ for some φ "sufficiently concentrated around e". The convolution relatively to φ is defined on L^+ by:

$$(8) \qquad [f * g]_\varphi(x) = \overline{(f(s)g\,(s^{-1}x) : \varphi(s))}$$

It is easily verified that the functions $[f * g]_\varphi$ are continuous (when multiplied by a suitable "normalization factor", they become equicontinuous in the parameter φ ((1))). The relative convolution is approximatively additive in the sense that if $f_1, \ldots, f_n \in L^+$, $f = \sum_{i=1}^{n} f_i$, $g \in L^+$ and $\varepsilon > 0$, then there exists a $V \in \mathcal{V}$ such that for all $\varphi \in L^+_V$:

$$(9) \qquad [f * g]_\varphi \leqq \sum_{i=1}^{n} [f_i * g]_\varphi \leqq [f * g]_\varphi + \overline{(f : \varphi)}\,\varepsilon$$

Now the remaining proof proceeds in a few steps. (For detailed proofs and also for the proof of (9) one may consult ((1)) .)

Proposition 4. (Existence of approximate identities.) Let $g \in L^+$ and $\varepsilon > 0$ be arbitrary. Then there exists a $U \in \mathcal{V}$ such that

$$(10) \qquad \left\| [h * g]_\varphi - \overline{(h : \varphi)}\,g_t \right\|_\infty \leqq \overline{(h : \varphi)}\,\varepsilon$$

whenever $t \in G$, $h \in L^+_{tU}$, $\varphi \neq 0$.

An analogous expression is obtained for the reversed convolution product $[g * h]_\varphi$ when $h \in L_{Ut^{-1}}$. It involves the r i g h t translate g^t, and the c o n j u g a t e function h^* appears at the right hand side. The result (10) is used in the proof of Proposition 5, and the corresponding reversed formula (specialized to $t = e$) is used in conjunction with Proposition 5 to prove Proposition 6.

P r o p o s i t i o n 5. (Approximation of $[f * g]_\varphi$ by left translates of g.) <u>Let $f, g \in L^+$ and $\varepsilon > 0$. Then there exists a $V \in \mathcal{V}$, such that for every $\varphi \in L_V^+$, $\varphi \neq 0$ and suitable $t_1, \ldots, t_n \in spt(f)$, and $\alpha_1, \ldots, \alpha_n \geq 0$:</u>

$$(11) \quad \left\| [f * g]_\varphi - \sum_{i=1}^{n} \alpha_i g_{t_i} \right\|_\infty \leqq (\overline{f : \varphi})\varepsilon$$

P r o p o s i t i o n 6. (Cartan.) <u>For every $f \in L^+$ and $\varepsilon > 0$, there is a $V \in \mathcal{V}$ such that for every $g \in L_V^+$, there exist group elements $t_1, \ldots, t_n \in spt(f)$ and positive numbers $\alpha_1, \ldots, \alpha_n$ such that:</u>

$$(12) \quad \left\| f - \sum_{i=1}^{n} \alpha_i g_{t_i} \right\|_\infty < \varepsilon .$$

C o r o l l a r y. <u>The separation property (S) is valid in any locally compact group.</u>

In virtue of the previous results, the above Corollary completes the proof of the existence and uniqueness of Haar measure.

The other existing proof goes back to J. von Neumann, who used it to prove the existence and uniqueness of Haar measure in c o m p a c t groups and applied the same methods to prove the existence and uniqueness of the mean value for almost periodic functions on a group ((18)) ((19)). As far as we know, G. Bredon was the first who succeeded in applying this technique to arbitrary l o c a l l y c o m p a c t groups ((2)). While the Cartan proof makes use of an approximate integral which is invariant, but only approximately additive, the other proof makes use of an integral which is additive, but only approximately invariant. It is merely a finite sum of function values $\sum_{i=1}^{n} f(x_i)$, where the approximate invariance is obtained by choosing the points $x_1, \dots, x_n$ (left) "equally spaced". This expression is rendered precise by the notion of a "minimal covering". A covering $\{U_i\}_{1 \leq i \leq n}$ of a compact set K by open sets U_i which are small of some order $V \in \mathcal{V}$ with respect to the left uniform structure on G , is said to be m i n i m a l (w.r. to V) if there exists no other covering of the same kind with a smaller number of constituents. In the compact case the "equally spaced points" $x_1, \dots, x_n$ are to form a s e t o f r e p r e s e n t a t i v e s (i.e. $x_i \in U_i$, $i = 1, \dots, n$) for a minimal covering $\{U_i\}_{1 \leq i \leq n}$ of G by open sets which are small of some order V . The crucial property of minimal coverings of compact groups, on which the approximative invariance is founded, is the fact that <u>any two minimal open coverings of the same order admit a common set of representatives</u>. This result follows from a combinatorial lemma of P. Hall and W. Maak ((9)) ((14)). A very short proof was given by P. Halmos and H. Vaugham, who introduced the term "marriage lemma" ((10)) (cf. also ((15))). In this context we prefer to state the lemma in the relevant mathematical form, and we leave it to the reader to find out what it has to do with marriages.

<u>Let $\mathfrak{A}$ and $\mathfrak{B}$ be two finite collections of subsets of a set S and suppose that for any subcollection $\{A_1, \dots, A_k\}$ of $\mathfrak{A}$, the number of sets $B \in \mathfrak{B}$ meeting $A_1 \cup \dots \cup A_k$ is greater than or equal to k ,</u>

then it is possible to establish a one-one correspondance between $\mathcal{A}$ and a subcollection $\mathcal{B}_o$ of $\mathcal{B}$ such that any pair of corresponding sets will meet.

G. Bredon has adopted the "marriage lemma" to minimal coverings of compact subsets of a l o c a l l y c o m p a c t group (Lemma 3 of ((2))). By means of this, he has established the following "almost invariance property":

P r o p o s i t i o n 7 . Let K be a compact subset of G , let $\varepsilon > 0$ and let $g \in L^+$, $g \neq 0$. Then there exist points $x_1, \dots, x_n \in G$ such that:

$$(13) \qquad \left| \sum_{i=1}^{n} g_x(x_i) - \sum_{i=1}^{n} g(x_i) \right| < \varepsilon \sum_{i=1}^{n} g(x_i)$$

for all $x \in K$. Moreover, the points x_i can be chosen so that the above inequality is simultaneously true for any finite number of given functions g .

Now the proof of the Cartan Lemma (our Proposition 6) is comparatively easy, since the "relative convolutions" in question are simply finite sums. (Cf. proof of Corollary 2 to Lemma 5 of ((2)) . Here the theorem is stated for right translates instead of left translates, and the proof is a simple application of Proposition 7, stated above.)

Finally we wish to mention a problem which turns out to be surprisingly difficult, namely to give a constructive proof based on Haar's original approximations $(C : K)$ for compact sets C, K $(K^o \neq \emptyset)$, rather than the similar approximations $\overline{(f : \varphi)}$ for functions. The difficulty comes from the fact that the c o n t e n t f u n c t i o n λ , obtained as limit of a subnet of $\{(C : K)/(C_o : K)\}_K$ (e.g. by using the axiom of choice in the form of the Tykhonov Theorem), is not outer regular. Thus the associated measure μ is not necessarily an extention of λ . Hence the uniqueness of μ does not entail the uniqueness of λ , and the original net need not be convergent. Thus the passage to a subnet seems to be necessary to get convergence in this case, and it would be of some interest to know if such a subnet can be described without the axiom of choice.

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